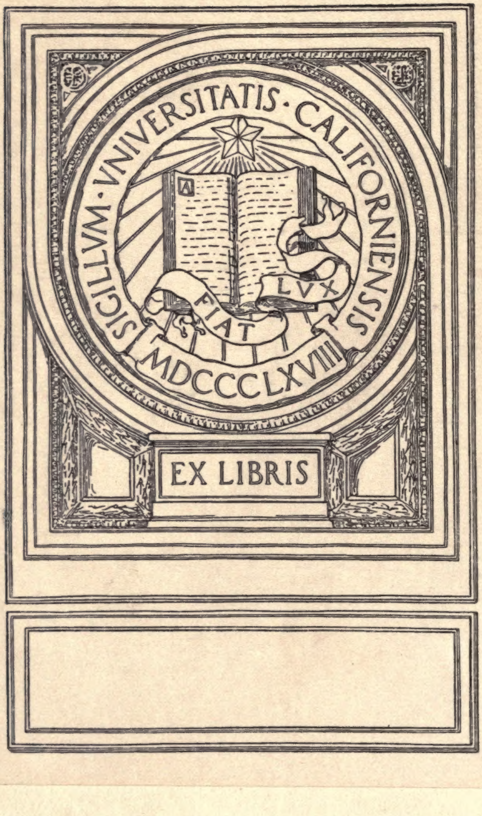

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ABSOLUTE MEASUREMENTS IN ELECTRICITY AND MAGNETISM

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ABSOLUTE MEASUREMENTS
IN
ELECTRICITY AND MAGNETISM

BY

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PREFACE.

In the interval since the publication of the First Edition of this book the subjects of physical study have changed enormously, and if it were not for the needs of Wireless Telegraphy, I question whether the theory and practice of absolute measurements would at the present time command serious attention. It has even been said that radioactivity and the phenomena of X rays are the only things worthy of the attention of physicists, and that the labours of applied mathematicians are relatively of little or no importance. Had it not been for the subject of the behaviour of aeroplanes, dynamics in its higher aspects, I believe, would long since have practically vanished from our curricula. Of the utter absurdity and danger of this attitude of mind the manifold problems of the late war provide a sufficient demonstration. As it is, we have now an army of students and others talking glibly of Einstein and of quantum theory, whose attention to the fundamentals of dynamics and physics has been wofully slight.

In deciding what part of my former work should be omitted and what part should be expanded, I have had a difficult task. It was not possible, without carrying the book to a most unwieldy length, to include the experimental treatment of electrical waves and the researches into atomic structure and theory. These are best studied in the special treatises which workers in that important line of research have published. I have therefore in this part of the subject confined my attention to constants of coils which play an important part in wireless telegraphy, renouncing, however, any attempt to make the book a treatise on the appliances of radio work. The discussion of coil constants is a difficult and somewhat thankless task, and has consumed much time and thought. I have, however, to acknowledge the assistance I have received from the writings of my friend Dr. Alexander Russell, and from those of Mr. Butterworth. These gentlemen generously placed their papers at my disposal, and very kindly looked over the proofs of the parts of the book in which their results and formulae appear.

The collected edition of the *B.A. Reports on Electrical Standards*, published in 1913 by Mr. F. E. Smith, has been of great service, and I am much indebted to Mr. Smith's account of the National Physical Laboratory current weigher, and of the determination of the ohm by

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an ingenious modification of the method of Lorenz which he carried out in 1912. I have given much more space to the determination of the horizontal component of the earth's magnetic force than I would have done if laboratories generally were provided with absolute current weighers.

To the late Mr. E. B. Rosa and his colleagues of the physical staff of the Bureau of Standards at Washington I am under many obligations. They tested and used my suggestions as to standards and formulae of inductance, and constructed an accurate electrodynamometer of the special dimensions recommended in my *Phil. Mag.* paper of 1892, and in the First Edition of this book. They have carried in their writings the investigation of the accuracy of formulae, and the allowance for errors of different kinds, to a high pitch of perfection. Their numerical comparison of formulae and their exact computation of the constants of apparatus have not been such as to attract popular attention and applause, but it has been none the less of enormous value to scientific progress. This merit it has in common with the whole work of the British Association Committee on Electrical Standards and of all those scientific men who have laboured at the subject of absolute electrical measurement. This task of founding a system of absolute measurements was forced upon science by applications to submarine telegraphy, and it is not too much to say that without such a system not one of the immense and varied range of practical applications could ever have been made.

I am indebted to my friend and former pupil, Professor E. Taylor Jones, who has made an elaborate theoretical and experimental study of the induction coil, and is an acknowledged authority on that subject, for the account of the action of such apparatus, which appears in Appendix I.

The Table of Contents and the Index have been constructed by my eldest daughter, to whose unfailing patience and ready help as amanuensis I have had continually to turn.

I wish finally to acknowledge the care of the readers and compositors of the University Press, which, by comparison of references, and otherwise, has saved the text from many inconsistencies, and increased its accuracy.

A. GRAY.

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ABSOLUTE MEASUREMENTS IN ELECTRICITY AND MAGNETISM

CHAPTER I.

UNITS AND DIMENSIONS OF PHYSICAL QUANTITIES.

1. Measure of a physical quantity. A physical quantity is expressed numerically in terms of some convenient magnitude of the same kind which has been taken as unit and compared with it by some process of measurement. The complete expression of the quantity may be described as consisting essentially of two factors, one a number, or, as we shall call it, a *numeric*, and another the unit with which the quantity is compared in the process of measurement. The numeric is then the measure of the ratio of the quantity determined to the quantity taken as unit for the comparison.

Thus when a certain distance is said to be 25 yards, what is meant is that the distance has by some proper process been compared with the length, under specified conditions, of a certain standard rod, which length is defined as a yard, and the ratio of the former to the latter has been found to be 25. We may therefore write

$$\text{Distance} = 25 \times \text{yard}.$$

Similarly in any other case of numerical reckoning of quantities (apart from direction) we may write

$$\text{Quantity} = N \times \text{unit}.$$

When the unit has been specified, it is thereafter (*e.g.* in algebraical calculations) as a rule understood, and the numeric alone is taken as representing the quantity.

The unit chosen may not be, as it is in the above example of a distance, a fundamental unit. According to the nature of the quantity expressed, it may be either a unit derived from a single fundamental unit, or a unit derived from a combination of fundamental units.

For example, a certain volume of earth or clay removed in digging may be expressed as so many, say N, cubic yards. In this case the unit of volume taken is the volume of a cube, the length of an edge of which is the yard. Or a certain volume of air may be expressed as so many, say N, cubic inches. In this case the unit of volume is the volume of a cube, the length of an edge of which is one inch.

2. Unit of length. Thus the unit of length chosen to define the unit of volume may be any multiple, or sub-multiple, denoted by the numeric l, say, of the fundamental unit of length which gives the fundamental unit cube, or, as we shall say, the *unit cube.* Hence we may indicate the magnitude of the unit chosen by the expression $l^3 \times$ *unit cube.* Thus we have

$$Volume = N \times l^3 \times unit\ cube.$$

The numeric N_1, expressing the volume in terms of the unit cube, is therefore given by the equation

$$N_1 = Nl^3 ;$$

and we have
$$N = N_1 \frac{1}{l^3} = N_1 n.$$

Thus the change from the numeric N_1, for the unit cube, to N, for the other unit chosen, is effected by dividing the former by the numeric l^3, or by multiplying it by the numeric $n = 1/l^3$. Calling N_1 the old, and N the new numeric, we see that the change from the old numeric to the new is effected by multiplying N_1 by the ratio of the old unit of volume to the new, or, as it is sometimes put, by "the number of times the old unit contains the new." It will be seen that this is the third power of the ratio of the old unit of length to the new. If we denote this ratio by $[L]$ and the third power of the ratio by $[L^3]$, we have

$$N = N_1 . [L^3].$$

3. Dimensional formulae. Change-ratios. The symbol $[L^3]$ is called a *dimensional formula.* It gives, by its value, the conversion-factor n, or *change-ratio* as we shall frequently call it, and, by its form, the manner in which the unit of volume depends on the unit of length.

We may apply to the example used above a method of specifying quantities which illustrates very clearly what has been stated, and gives the change-ratio at once as follows. A cubic inch is represented by the notation in^3, a cubic yard by yd^3, and we have the equation

$$N . \mathrm{in}^3 = N_1 . \mathrm{yd}^3,$$

which means that N cubic inches and N_1 cubic yards represent the same volume. We get then

$$N = N_1 \frac{\mathrm{yd}^3}{\mathrm{in}^3} = N_1 . \left(\frac{\mathrm{yd}}{\mathrm{in}}\right)^3.$$

The change-ratio is therefore 36^3, that is the l of 2 is 1/36, and $[L^3]$ or n is 36^3.

The change from N_1 to N cannot be made unless the change-ratio n is known. Each unit may have been arbitrarily chosen without reference to any other unit, and n determined by some process of measurement; or the units may have been derived from certain chosen fundamental units, and the change-ratio deduced from the relation of one set of fundamental units to the other. In the measurements described in this book the units employed are entirely of the second kind here referred to.

4. Derived units and fundamental units. Our task in this chapter is to determine the manner in which the various derived units involve the fundamental units, that is we have to determine for each quantity the change-ratio n in terms of the fundamental units. The formula which thus expresses n for a unit of measurement of any quantity, we call the dimensional formula of the quantity. To prevent the necessity for the constant repetition of these terms we shall denote the dimensional formula of any quantity, of which the numerical expression in terms of some chosen unit is denoted by any particular symbol, by the same symbol enclosed in square brackets, as we have indicated above. Thus the dimensional formula of volume may be written $[V]$, and when the manner in which the fundamental unit, that of length, is involved in the unit of volume, is to be exhibited write the formula in the form $[L^3]$.

Examples of dimensional formulae will be found in dealing with the various units to which we now proceed. We shall consider first the definitions and relations of the fundamental units in common use, and the derivation from them of the units of other physical quantities. In doing so we shall find the dimensional formula in each case and its numerical values for certain changes of units.

I. FUNDAMENTAL UNITS.

5. Unit of length. The standard unit of length in Great Britain is defined by Act of Parliament in the following terms: "The straight line or distance between the centres of the transverse lines in the two gold plugs in the bronze bar deposited in the Office of the Exchequer shall be the genuine standard of length at 62° F., and if lost it shall be replaced by its copies."

Authorized copies are preserved at the Royal Mint, the Royal Society of London, the Royal Observatory at Greenwich, and the New Palace of Westminster. The comparison of the standard with its copies has been effected with the utmost scientific accuracy, and formed a most elaborate and important scientific investigation.

The length of a simple pendulum which beats seconds has been determined for several places by means of very careful observations, and repeated pendulum experiments at these places would, in the event of the destruction of the standard and all its copies, give a means of accurately renewing them.

In France and in most Continental countries, including their colonies and offshoots in all parts of the world, the standard of length is the *Metre.* This is defined as the distance between the extremities of a certain platinum bar when the whole is at the temperature 0° of the Centigrade scale. This rod was made of platinum by the Chevalier Borda, and is preserved in the national archives of France. As in the case of the yard, authorized copies, whose lengths have been carefully compared with the standard, are preserved in various places.

The metre was constructed in accordance with a decree of the French Republic passed in 1795 (Loi du 18 germinal, an iii), which enacted, on the recommendation of a Committee of the French Academy of Sciences, consisting of Laplace, Delambre, Borda, and others, that the unit of length should be one ten-millionth part of the distance, measured along the meridian passing through Paris, from the Equator to the North Pole. The arc of that meridian extending from a point near Barcelona to a point near Dunkirk was measured by Delambre and Méchain, that is, was expressed in terms of the length of an arbitrary measuring rod by which the base-line of their triangulation was laid down. The relation of the length of the arc to that of the quadrant of the meridian between the equator and the pole was known from astronomical observations, and so the length of the quadrant was obtained in terms of the measuring rod. It was then a comparatively simple matter to construct a rod of platinum to represent the specified fraction of the length of the quadrant, and this was done with all possible care.

The metre, it is to be observed, is not now defined in relation to the earth's dimensions, and later and more exact results of geodesy have therefore not affected the length of the metre, but are themselves expressed in terms of the length which Borda's rod has at 0° C.

6. Prototype metre. An International Bureau of Weights and Measures has been established under the auspices of various governments, and has constructed copies of the metre and other standards. The metre of the Bureau is what is known as a line standard, that is the distance represented is taken between two cross-lines, as in the case of the British standard yard. It is called the International Prototype Metre. Copies of it have been issued to the governments who have contributed to the support of the Bureau: these are called "National Prototypes."

In the metric system the decimal mode of reckoning has been adopted for multiples and sub-multiples of all the units. Thus the metre is divided into ten equal parts each called a decimetre, the decimetre into ten equal parts each called a centimetre, and the centimetre into ten equal parts each called a millimetre.* Again, a length of ten metres is called a decametre, of one hundred metres a hectometre, and of one thousand metres a kilometre.

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* All these words, including *metre* itself, are now written in English without accents, and are pronounced as English words.

7. Units for scientific work. In accordance with the prevailing practice of scientific experimenters, who have adopted the suggestions of the British Committee on Standards, the centimetre has been very generally chosen as the unit of length for the expression of scientific results, and on it as unit of length the electric and magnetic units, approved by the International Congress of Electricians held at Paris in 1882, and now brought into universal use, have been founded. The reason for this choice will appear when we consider the unit of mass.

If we denote the numerical value of a length by L, the dimensional formula is $[L]$. The change-ratio is given by this when for $[L]$ is substituted the ratio of the old unit of length to the new. For example, if we wish to find the numeric for a length in terms of the metre as unit from the numeric for the same length in terms of the yard as unit, we have $[L] = \cdot 91439$, the ratio of the yard to the metre. Or we may put the matter thus. Let m stand for metre, yd for yard, N_m, N_y for the corresponding numerics. We then have

$$N_m \,.\, \mathrm{m} = N_y \,.\, \mathrm{yd},$$

or

$$N_m = N_y \frac{\mathrm{yd}}{\mathrm{m}}.$$

Similarly the value of $[L]$ for a change from the foot as unit of length to the centimetre as unit is $\mathrm{ft}/\mathrm{cm} = 30 \cdot 47945$.

8. Unit of mass. The legal standard of mass in Great Britain is the Imperial standard pound avoirdupois, a piece of platinum marked "P.S. 1844, 1 lb.," preserved in the Exchequer Office. In the Act of Parliament (the Act already referred to) which gives authority to the standard, it is called the "legal and genuine standard of weight"; and the act provides that if the standard is lost or destroyed it may be replaced by means of authorized copies, which are kept with the standards of length in certain national repositories. [See 5 above.]

The word "weight" which appears in the Act is constantly used in two senses: (1) as here, to signify a measure of the quantity of matter in a body; (2) to signify a measure of the downward force of gravity on the body. These two senses are distinct; but the context is, in general, sufficient to indicate in which sense the word is used. When a body is weighed its mass, or, as we often say, its weight, is determined by comparison of the force of gravity on it with the force of gravity on a standard piece, or an aggregate of standard pieces, of matter. At a given place the forces of gravity on different bodies are (as was proved by Newton's result that pendulums of the same length, but with bobs of different weights, vibrate in the same period) proportional to their masses; and thus a comparison of the weights (or gravities) of different bodies at the same place gives a direct comparison of their masses.

The pound has been much used in this country for the expression of dynamical results; but, in engineering and the arts, larger units, for example the ton, or mass of 2240 lb, and the hundred-weight, or mass of 112 lb, are frequently employed.

The French standard of mass is a piece of platinum called the *Kilogramme des Archives*, made also by Borda in accordance with the decree of the Republic mentioned above. It was connected with the standard of length by being made a mass as nearly as possible equal to that contained in a cubic decimetre of distilled water at the temperature of maximum density, or, very nearly 4° C. The comparison was of course made by weighing, and so far as this process was concerned it was certainly possible to obtain great accuracy; but the density of water is somewhat difficult to determine with exactness, and is still in a small degree uncertain. The relation of the standards is, however, so nearly that stated above, that, for practical purposes, the error may be neglected.

It is important, however, to remember that the standard is *defined* as the kilogramme made by Borda, and not as the mass of a cubic decimetre of distilled water at 4° C., which it approximately equals.

9. Relation between British and French standards of mass. A comparison between the French and British standards of mass made by the late Professor W. H. Miller gave the mass of the kilogramme as 15432·34874 grains. The pound avoirdupois contains exactly 7000 grains. Hence, according to Professor Miller's determination, a pound is equal to 0·45359265 kilogramme, and a kilogramme is 2·20462 pounds.

The Kilogramme des Archives has also been copied by the International Bureau of Weights and Measures, and one of the copies has been called the International Prototype Kilogramme. The other copies have been distributed in the same manner as the metre standards, and are called also National Prototypes.

The gramme, defined as 1/1000 of the kilogramme and approximately equal to the mass of one cubic centimetre of water at 4° C., was recommended by the British Association Committee in 1863 as the unit mass on which to base a system of absolute electric and magnetic units, and this choice has been ratified by the adoption of the gramme as the unit of mass for the expression of scientific results generally. The convenience of this unit lies in the fact that it is (when the centimetre is taken as unit of length) approximately the mass of unit volume of the substance—water at its temperature of maximum density—usually taken as standard of comparison in the estimation of specific gravities of bodies, which therefore become in this case the same numbers as those which express the densities of the bodies. The kilogramme and decimetre have the same advantage.

The multiples and sub-multiples of the gramme proceed decimally, and are distinguished by the same prefixes as are used for units of length derived from the metre.

If we denote the numerical value of a mass by M (or m), and hence write $[M]$ as its dimensional formula, the value of $[M]$, regarded as a change-ratio, or conversion factor, for a reduction from the pound as unit to the gramme as unit is 453·593 ; for reduction from the grain as unit to the gramme as unit it is 1/15·432.

10. Unit of time. The proper dynamical definition of equal intervals of time cannot be given here ; but that definition leads to the conclusion that to a very high degree of approximation the intervals of time in which the earth turns through equal angles about its axis of rotation are equal. These intervals correspond closely to equal intervals of time as these would be shown by a clock regulated to keep perfect dynamical time. Let it be supposed that such a clock is arranged to register a twenty-four hours' interval as the period of the earth's rotation about its axis (that is, the interval between two successive passages of a fixed star in the same direction across the meridian of any place, showing 0 h, 0 m, 0 s at each instant when the First Point of Aries crosses the meridian in the same direction) ; then the clock is said to show sidereal time.

Though sidereal time is used in astronomical observatories, it is more convenient in ordinary civil affairs to use solar time ; but as the actual solar day—the interval between two successive transits of the sun across the meridian of any place—varies in length during the year, the standard interval is a proper average of such intervals, and is called a mean solar day. On account of the orbital motion of the earth the mean solar day is about 3 m 55·9 s longer than the sidereal day.

The mean solar second, defined as 1/86400 part of the mean solar day —that is the interval in which the earth turns through 1/86400 part of the angle which it turns through in a mean solar day—is taken as the unit of time for the expression of all scientific results.

The unit of time based on the rotation of the earth is, there is some reason to believe (though the grounds for the belief have been questioned, and the matter is not yet finally settled), subject to a slow progressive lengthening, due to tidal retardation of the earth's rotation. It has been estimated that if the clock, referred to above as rated to keep perfect dynamical time, were arranged so as just to keep pace with the earth's rotation, and therefore show sidereal time, at the beginning of a century, it would at the end of the century be found to have outstripped the terrestrial time-keeper by about 22 seconds, or in other words—since the amount of gain if sensible must be proportional to the square of the time interval—it would at the end of a year from the beginning of the century be found to be ahead of the earth's rotation by about 22/10000 of a second.

11. The fundamental units are arbitrarily chosen. It will be seen that, whatever motives may have led to the choice of the various fundamental units now in use, the definitions of these units are entirely arbitrary, and that there is nothing in their nature which entitles them to be termed "absolute" ; and as a matter of definition, without reference in all cases to realization, many other standards might be suggested. Thus in Thomson and Tait's *Natural Philosophy* (Vol. I. Part I. p. 227, sec. ed.) it is stated that the period of vibration of a metallic spring, kept in a hermetically sealed exhausted chamber at a constant temperature, or the period of a particular mode of vibration of a quartz crystal (or other crystal of definite composition) would be theoretically preferable to the mean solar second, as fulfilling with a much nearer approach to perfection the condition of constancy. About this of course there may now be a difference of opinion.

It was suggested by Clerk Maxwell (*vide* the same reference) that the period of vibration of a gaseous atom of a widely diffused substance, easily procurable in a pure state, would be a more satisfactory unit of time. Modern results, however, proving the disintegration of atomic structure throw doubt on the legitimacy of this proposal.

We denote the numerical value of a time interval by T or t, and the dimensional formula for time by $[T]$.

II. DERIVED UNITS.

12. Dimensional formulae for derived units. In general the numerical expression N_1 of a measured physical quantity depends on the numerical values of certain lengths, masses, and times, so that we write for the present

$$N_1 = CL_1^{\lambda_1} M_1^{\mu_1} T_1^{\tau_1} \cdot L_2^{\lambda_2} M_2^{\mu_2} T_2^{\tau_2} \dots, \quad \dots\dots\dots\dots\dots\dots(1)$$

where L_1, M_1, T_1 are the numerics for a certain measured length, mass, and time, L_2, M_2, T_2 those for another set of these quantities, and so on, and C is a numerical coefficient which does not depend on the units, like the factor $\frac{1}{2}$ in the expression $\frac{1}{2}mv^2$.

Equation (1) is, as we shall see, sufficient for the calculation of the dimensional formula of the quantity, and obviously gives as its expression

$$[L^{\lambda_1+\lambda_2+\dots} M^{\mu_1+\mu_2+\dots} T^{\tau_1+\tau_2+\dots}];$$

or, if we write $\lambda = \lambda_1 + \lambda_2 + \dots$, $\mu = \mu_1 + \mu_2 + \dots$, $\tau = \tau_1 + \tau_2 + \dots$,

$$[L^\lambda M^\mu T^\tau],$$

so that the consideration of the case of

$$N = CL_1^\lambda M_1^\mu T_1^\tau \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

is sufficient.

The numeric of the quantity may however be given by a series of such terms: thus we may have

$$N = C_1 L_1^{\lambda_1} M_1^{\mu_1} T_1^{\tau_1} + C_2 L_2^{\lambda_2} M_2^{\mu_2} T^{\tau_2} + \dots, \quad \dots\dots\dots\dots\dots(3)$$

with the same units of length, mass, and time used in all the terms. Now it is clear that in these terms we must have

$$\lambda_1 = \lambda_2 = \dots, \quad \mu_1 = \mu_2 = \dots, \quad \tau_1 = \tau_2 = \dots,$$

as otherwise it would be possible by choosing a large or a small unit (of length, say) to make one term (or set of terms) or another predominate in the value of N. Equation (3) must in fact be homogeneous as regards each of the quantities L^λ, M^μ, T^τ which appear in its terms, that is, each term must be a contribution to the quantity measured, and all must be on the same footing as regards dimensions. Thus again we are brought back to

$$N = CL_1{}^\lambda M_1{}^\mu T_1{}^\tau,$$

and the dimensional formula is

$$[L^\lambda M^\mu T^\tau].$$

It has been assumed as obvious that the dimensional formula of the product of the factors $L_1{}^\lambda$, $M_1{}^\mu$, $T_1{}^\tau$ is the product of the dimensional formulae $[L^\lambda]$, $[M^\mu]$, $[T^\tau]$.

13. Formulae must be homogeneous in dimensions. The fact that in (3) every term must have the same dimensions in each unit affords a valuable check on the accuracy of algebraic work in physical mathematics. It is to be noted also, in this connection, that if we have an exponential multiplier, e^X say, X must be of zero dimensions in each unit, that is, in each term of X we must have $\lambda = 0$, $\mu = 0$, $\tau = 0$. If this were not so the relative values of the terms in the expansion of e^X would be changed by a change of units, whether of length, mass, or time (if all three units enter into the estimation of X), and therefore, since the first term of the expansion is 1, the exponents λ, μ, τ must each be zero.

We are now prepared to find the dimensional formulae of the various derived units. The process will consist in finding for each quantity the formula corresponding to the right-hand side of (3), and thence deriving the proper formula of dimensions. We shall consider first the units of Area, Volume, and Density; then the various dynamical units which are involved in those of electric and magnetic quantities.

14. Unit of area. The general formula for the area of any surface can be put in the form CL^2, where L is a numeric expressing a length, and C is a numeric which does not change with the units. Hence the formula of dimensions for area is $[L^2]$. Regarded as a change-ratio, the dimensional formula gives for a change from a foot to a centimetre as the unit of length the multiplier $[\text{ft}^2/\text{cm}^2] = 30{\cdot}48^2$.

Volume. Similarly the formula for the numeric of a volume is CL^3, and the formula of dimensions is $[L^3]$. For the change of units specified under Area, the change-ratio is $30{\cdot}48^3$.

15. Unit of density. The density of a body is measured numerically by the ratio of the numeric for the mass to the numeric for the volume. We denote the numeric for density by D.

If the body be of varying density from point to point, *the density at any point* is the limit towards which the ratio of the numeric, for the mass in an element of volume to the numeric for that volume, approaches as the element is taken smaller and smaller. It is to be understood that the diminution of volume cannot be pushed to the limit of the dimensions of the grained structure of matter; but it is certain that we approach a definite limit sufficiently closely for all practical purposes while still keeping the volume element large enough in every dimension to avoid the effect of molecular structure.

If then δV be the measurement of an element of volume, including a point at which the density is D, and δM be the measure of the mass of the element, we have

$$D = \text{Limit}\left(\frac{\delta M}{\delta V}\right) = \frac{dM}{dV}.$$

In either case we have CL^3 for the numerical expression of the volume taken, and for that of the mass contained in it some value M. Hence

$$[D] = [ML^{-3}].$$

The *specific gravity* of a body is the ratio of the density of the body to the density of the standard substance, and is therefore a numeric independent of the system of units adopted, that is, its dimensional formula is 1. If G denote the specific gravity of a body whose density is D, and D_s be the density of the standard substance,

$$D = G \,.\, D_s.$$

In the metric system of units, if either the kilogramme is taken as unit of mass and the decimetre as unit of length, or the gramme is taken as unit of mass and the centimetre as unit of length, D_s is unity, and we have

$$D = G.$$

This is one great convenience of the metric system; but it is to be remembered that density and specific gravity are essentially different ideas, and only in such cases as those cited coincide in numerical value. Density changes with the units adopted, specific gravity does not.

III. DYNAMICAL UNITS.

16. Dimensions of velocity and speed. The velocity of a body moving without rotation, or of a particle, is measured by the numeric of the length described per unit of time. The specification of velocity involves direction as well as magnitude; but in dealing with dimensions we are only concerned with the latter element, that is with the *speed.*

If the speed is uniform its numerical expression v is the ratio of the numeric L for the distance traversed in a time interval to the numeric T for that interval, that is

$$v = \frac{L}{T}.$$

If the *velocity* is variable the *speed* may or may not undergo change. If the speed is variable, its measure at any instant is obtained as follows.

Let δT be the numeric for an interval of time which includes the instant between its extremities, and δL that for the distance described in that time; and let δT be taken smaller and smaller without limit. Then the limit towards which the ratio $\delta L/\delta T$ continually approaches without limit of closeness as the interval δT is diminished, is the numeric v. That is

$$v=\frac{dL}{dt}=\dot{L},$$

where $\dot{L}$ denotes, in Newton's fluxional notation, the time-rate of variation of L.

We see that the numerical expression of a speed is always the ratio of a length-numeric to a time-numeric, and therefore we have

$$[\dot{L}]=[LT^{-1}].$$

As an example we take the change of v from mile-minute units to centimetre-second units. We have

$$v'\frac{\text{cm}}{\text{sec}}=v\frac{\text{mile}}{\text{min}},$$

and therefore

$$v'=v\frac{\text{mile}}{\text{cm}}\frac{\text{sec}}{\text{min}}$$

$$=v\frac{5280\times30{\cdot}4797}{1}\frac{1}{60},$$

or

$$v'=2682{\cdot}2136v.$$

Putting the values in the dimensional formula, we get

$$[LT^{-1}]=5280\times30{\cdot}4797\frac{1}{60},$$

the same result.

17. Dimensions of acceleration. The acceleration of a particle is the rate of change of velocity per unit of time.

Like velocity, acceleration involves in its signification the idea of direction as well as that of magnitude: a recognition of this fact removes the difficulty often felt by students in understanding acceleration in curvilinear motion, where there is always a component of acceleration in the direction towards the centre of curvature, that is at right angles to the direction of motion at the instant.

Let $\delta\dot{L}$ be the numeric for the velocity, given in direction and magnitude, which, compounded with the velocity, of numeric $\dot{L}$, which a particle possesses at the beginning of an interval of time δT, would give the velocity in direction and magnitude at the end of that interval; then $\delta\dot{L}/\delta T$ is the *average* acceleration during that interval. The limit towards which this ratio approaches as δT is made smaller and smaller is the true value of the acceleration at the instant which marks the beginning of the interval; that is we have

$$\text{Acceleration-numeric} = \frac{d\dot{L}}{dT} = \ddot{L},$$

where $\ddot{L}$ denotes in the fluxional notation the measure of the time-rate of $\dot{L}$.

The two dots above the L in $\ddot{L}$ serve to recall the double reference to time which is plainly involved in the notion of acceleration, and which should be clearly expressed in statements of amounts of acceleration. The statement that in this country the acceleration of a body falling freely under gravity "is about 32·2 feet, or 981 centimetres" (*B.A. Report on Electrical Standards*, 1863), is, as it stands, unmeaning, and requires the words "per second per second" to be understood after "feet" and after "centimetres."

The dimensional formula for acceleration is

$$[\ddot{L}] = [LT^{-2}].$$

The change from mile-minute units to centimetre-second units, exemplified above in the case of speed, affords an illustration of the use of the dimensional formula to give the necessary change-ratio. We have

$$[LT^{-2}] = 5280 \times 30{\cdot}4797 \, \frac{1}{60^2} = 44{\cdot}70356.$$

18. Dimensions of momentum. Momentum is also directed. Take the case of a rigid body moving without rotation, that is, so that all particles of the body (not the *ultimate parts*, the motions of which are unknown) have the same velocity at the same instant: the momentum of the body is expressed as the product of the numerics for the mass of the body and its speed. It is therefore expressed symbolically by $M\dot{L}$. The dimensional formula is therefore

$$[M\dot{L}] = [MLT^{-1}].$$

19. Time-rate of change of momentum. If the momentum of the body be not constant, then, since we suppose the mass constant, we must have for the measure of the time-rate of variation the expression $M\ddot{L}$, that is the product of the numerics for the mass and the acceleration. The dimensional formula is therefore

$$[M\ddot{L}] = [MLT^{-2}].$$

20. Dimensions of force (F). The time-rate of change of momentum is the measure of the force acting on a body and causing its acceleration. Hence the dimensional formula just found is that of force.

Unit force is thus that force which, acting for unit time on unit mass, produces unit change of velocity, or simply that which produces unit acceleration in unit mass. When the unit of length is one foot, the unit of time one second, and the unit of mass one pound, unit force is that force which, acting for one second on a pound of matter, generates a velocity of one foot per second. This unit has been called a *poundal.*

In the c.g.s. (centimetre-gramme-second) system, unit force is that force which, acting for one second on a gramme of matter, generates a velocity of one centimetre per second. This unit has been called a *dyne.*

For most purely scientific purposes, and especially in electricity and magnetism, the dyne is the unit of force employed; but for many practical applications of dynamics a gravitation unit of force is often used. This unit is the force of gravity on the chosen unit of mass, and therefore (unless the force of gravity at a particular place and at a particular level at that place is referred to) has different values at different places on the earth's surface, and at different vertical distances from the mean surface level. The poundal and the dyne however have no such dependence on the position of the body acted on with reference to the earth, nor are they connected in any way with the properties of instruments used for the measurement of force. They have therefore been called *absolute* units of force.

Units of force thus defined *absolutely* are sometimes called *Gaussian units*, from the fact that the mode of definition was suggested by Gauss as the dynamical foundation of a system of absolute units for the expression of electric and magnetic quantities.

21. Dimensions of work (W). Work is done by a force when the place of application (a point, or congeries of points, marked by particles of a body, to which the force is kept applied as the body moves) of the force receives a component displacement in the direction in which the force acts; and the work done is measured by the product of the numerics of the force and the component displacement referred to.

The work done in overcoming a resisting force through a certain displacement of a body is equal by this definition to the product of the resisting force and the distance through which it is overcome.

For many practical purposes the unit of work used in this country is one foot-pound, that is the amount of work done in lifting a pound vertically against gravity through a distance of one foot. This unit of work has the same variability as the gravitation unit of force; the difference due to position is however not very great, and causes but small inaccuracy or inconvenience.

In the c.g.s. system of units the unit of work is the work done in overcoming a force of one dyne through a distance of one centimetre, and is called one *centimetre-dyne* or one *erg*.

In practical electricity 10^7 ergs is frequently used as unit of work, and is called a *joule.*

If F denote the numerical measure of a force and L that of the space interval through which it has acted, the numeric for work done is FL. Hence we have

$$[W]=[FL]=[ML^2T^{-2}].$$

22. Dimensions of energy (E). A force acting on a body is one aspect of a stress which exists between the body and matter external to it. Action and reaction are the names given to the two aspects of the stress, and thus the stress is really a system of two equal and opposite forces. These forces are exerted in opposite directions across a cross-section of a tie or a strut in a structure; but though this is the case they, being applied to different things, do not cancel one another. So called "action at a distance" can be similarly dealt with if we suppose that the different bodies really exist in a medium by which the stress is transmitted from one body to the other.

In this way of regarding the matter, when work is done by one aspect of the stress, say at the surface of the displaced body, equal work is done against the other aspect of the stress. We say that work is done on the body, which either gains motion or is displaced, in both cases relatively to the other bodies which act upon it, as in the case of a body moving under the action of gravity, or of a body displaced along a horizontal table against friction. In all cases the work done has an equivalent arising from a redistribution of what we call the *energy* of the system, that is, the capacity of the parts of the system for the performance of work. [We here regard as " the system " the mutually acting and reacting portions of matter between which the stress considered exists.]

If the motion of a part of the system with regard to the other parts has been increased the body can be made to do work on other parts in having the increase of motion annulled, or any existing relative motion can be annulled in the same way. Again, by the intervention of the action of a body external to the system, a part A of the system may be placed in a new position with respect to the other parts (as when a body is lifted against gravity); then, in virtue of the stresses between the parts work may be done on A in restoring the system to its former configuration, and A may, in consequence, be set in relative motion.

In the former case A is said to have gained *kinetic energy*, or energy of motion; in the latter case A is said to have gained *potential energy*, or energy of configuration. It is possible that in both cases the change is one of kinetic energy, in the first case of a visible molar part of the system, in the latter of invisible parts which constitute the connecting medium.

When work is done between different parts of the same system, a loss of kinetic energy in the system is accompanied by an equal gain ot potential energy, and *vice versa*, so that the total energy of the system remains unchanged in amount. This is the principle of *Conservation of Energy*. Work spent against friction forms no exception.

Energy is measured by the same units as work, and its dimensional formula is the same as that of work, that is

$$[E]=[ML^2T^{-2}].$$

23. Examples of kinematical and dynamical units. We give here as illustrations some examples of the application of dimensional formulae to the solution of problems regarding units. The statements of these problems are taken from Everett's *Units and Physical Constants.*

Ex. 1. If the unit of time be the second, the unit density 162 lb per cubic foot, and the unit of force the force of gravity on an ounce at a place where the change of velocity produced by gravity in one second is 32 feet per second, what is the unit of length ?

Here the change-ratio by which we must multiply the density of a body in the system of units proposed, to find its density in terms of the pound as unit of mass, and the foot as unit of length, is 162. We have therefore, omitting brackets in the dimensional formula,

$$ML^{-3} = 162.$$

Also it is plain that the unit of force is two foot-pound-second units, that is two poundals. Thus, since $T = 1$, we get

$$MLT^{-2} = ML = 2.$$

Thus we get by division

$$L^4 = \tfrac{1}{81}, \text{ or } L = \tfrac{1}{3}.$$

The unit of length is $\frac{1}{3}$ foot, or 4 inches.

Ex. 2. The number of seconds in the unit of time is equal to the number of feet in the unit of length, the unit of force is the gravity of 750 lb ($g = 32$ ft/sec^2), and a cubic foot of the substance of unit density contains 13,500 ounces. Find the unit of time.

We have $$ML^{-3} = \frac{13500}{16}$$

and $$MLT^{-2} = 750 \times 32.$$

Since $L = T$, we get by division

$$T^2 = \frac{750 \times 32 \times 16}{13500} = \frac{16^2}{3^2}.$$

Thus $T = 16/3 = 5\frac{1}{3}$. The unit of time is $5\frac{1}{3}$ seconds.

Ex. 3. When an inch is the unit of length and T seconds the unit of time, the numeric for a certain acceleration is a : when 5 feet and 1 minute are units of length and time respectively, the numeric for the same acceleration is $10a$. Find T.

In the first case $LT^{-2} = T^{-2}/12$ is the change-ratio for reduction to foot-second units, in the second case it is 5/3600. We have therefore

$$\frac{1}{12} T^{-2} a = \frac{5}{3600} 10a$$

or $$T = \sqrt{6}.$$

IV. DERIVED ELECTRICAL UNITS.

There are two systems of electrical units, the electrostatic system, founded on the definition of unit quantity of electricity, and the electromagnetic system founded on the definition of unit magnetic pole. As a rule magnetic units are not included in the electrostatic system, which is convenient only for purely electrical purposes; but there is no difficulty about expressing magnetic units in that system. For distinction we shall use, in the case of those quantities which appear in both systems, small letters for quantities taken in electrostatic measure, and the corresponding capitals of these letters for the same quantities taken in electromagnetic measure.

A. Electrostatic System.

24. Quantity of electricity $[q]$. The following definition is fundamental in this system. *Unit quantity of electricity is that quantity which, concentrated at a point at unit distance from an equal and similar quantity, also concentrated at a point, is repelled with unit force, when the medium across which the electric action is transmitted is a certain standard insulating medium.*

An ideal perfect vacuum may be taken as standard, but as this standard medium is far from being readily realizable, air at temperature 0° C. and at standard atmospheric pressure is taken. We call this simply *air*.

According to Coulomb's law that (the numerical values of) electric attractions and repulsions are directly as the products of the (numerics for the) attracting and repelling point-charges, each of q units, at points A, B and inversely as the square of the (numeric for the) distance L between them, the numeric F for the force of attraction, or repulsion as the case may be, is given by

$$F = \frac{q^2}{\kappa L^2},$$

where κ is a factor which depends on the medium and is called its *electric inductivity*. [The value of κ, it is to be understood, depends on the units adopted. But its dimensional formula may have the form $C[L^l M^m T^t]$; and we can by properly choosing C make κ unity for any medium we please.] Hence we get

$$[q^2] = [F\kappa L^2] = [ML^3T^{-2}\kappa],$$

and therefore $$[q] = [M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\kappa^{\frac{1}{2}}].$$

This leaves the dimensions of κ undetermined, and gives a more general electrostatic system of units than the ordinary one, and reducible at once to the latter. In the absence of special reasons for preferring one dimensional formula for κ to another, we may assign its dimensions according to any convenient hypothesis. One such hypothesis is that which forms the basis of the *ordinary* electrostatic system, namely that κ is, as regards the fundamental units, of zero dimensions, that is has a dimensional formula [1]. But in the ordinary electromagnetic system of units, which has quite a different derivation from the electrostatic, and in which what we call the *magnetic* inductivity (μ)—a quantity exactly analogous to κ—is of zero dimensions, the dimensional formula of κ is $[L^{-2}T^2]$, and the numerical value of κ depends on the choice made of fundamental units.

If we suppose both κ and μ undetermined as regards dimensions,* and investigate in each system the dimensional formula of a particular quantity, say quantity of electricity or electric charge, and suppose that the undetermined dimensions of κ and μ render the dimensions of electric charge (or whatever the quantity may be) really the same in both systems, as for unity of theory they ought to be, we shall find that

$$[\kappa\mu] = [L^{-2}T^2].$$

That is, the dimensions of $1/\kappa\mu$ are those of the square of a speed, which as stated above are exactly the dimensions obtained for $1/\kappa$ when we suppose μ to be of zero dimensions.

We now define the specific inductive capacity of any medium as the ratio of the electric inductivity of the medium in question to that of the standard medium. This is the value obtained as the ratio of the capacity of a condenser with the medium in question as dielectric to the capacity of a condenser the same in every respect except that the dielectric is the standard medium.

25. Electric surface density $[\sigma]$. The density of an electric charge on a surface is measured by the quantity of electricity per unit of area. Hence

$$[\sigma] = [qL^{-2}] = [M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\kappa].$$

26. Electric force and intensity of electric field $[f]$. The electric force (or the intensity of the electric field) at any point, is the force which a unit of positive electricity would experience if placed at that point. Hence, if the numeric for a point-charge at the point P be q, and that for the electric force at that point be f, the numeric F for the dynamical force on the charge is qf. Hence we have

$$[f] = [Fq^{-1}] = [M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\kappa^{-\frac{1}{2}}.$$

27. Electric potential $[v]$. The difference of electric potential between two points is measured by the work which would be done if a unit of positive electricity were placed at the point of higher potential and made to pass by electric force to the point of lower potential. Hence, in the transference of q units of electricity through a difference of potential expressed numerically by v, an amount of work is done for which the numeric W is equal to qv. (It will be observed that it is supposed that the difference of potential is supposed unaffected by the transference.) We have therefore $v = Wq^{-1}$, and

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* This mode of proceeding was advocated, and its advantages pointed out, by the late Sir Arthur Rücker, in a paper on "The Suppressed Dimensions of Physical Quantities," *Phil. Mag.* Feb. 1889.

$$[v] = [Wq^{-1}] = [M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} \kappa^{-\frac{1}{2}}].$$

28. Capacity of a conductor $[c]$. The capacity of an insulated conductor is the quantity of electricity required to charge the conductor to unit potential, all other conductors in the field being supposed at zero potential. Hence, if the numerics for the capacity, charge, and potential of a given conductor be c, q, v we have $c = qv^{-1}$, and therefore

$$[c] = [qv^{-1}] = [L\kappa].$$

The unit of capacity has therefore the dimensions of the unit of length, provided $[\kappa] = 1$, and the capacity of a conductor might then be said to be so many centimetres.

The electrostatic capacity of a conducting sphere insulated and alone in its own field is in electrostatic units numerically equal to the radius if the electric inductivity of the medium occupying the field is unity. A conducting sphere of 1 cm radius in air (for which we take the electric inductivity as 1) has therefore 1 c.g.s. unit of capacity.

29. Specific inductive capacity $[K]$. From what has been stated above in connection with the definition of unit quantity of electricity, it is clear that

$$[K] = 1.$$

30. Electric current $[\gamma]$. An electric current in a conducting wire is measured by the quantity of electricity which passes a given cross-section per unit of time. If q units pass in T units of time, and the numeric for the current be γ, we have $\gamma = q/T$, so that

$$[\gamma] = [qT^{-1}] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2} \kappa^{\frac{1}{2}}].$$

31. Resistance $[r]$. By Ohm's law the resistance of a conductor is expressed by the ratio of the numeric v for the difference of potential between its extremities to the numeric γ for the current flowing through it. It is understood that the current does not vary with the time, and that the conductor is not in motion across the lines of force of a magnetic field, so that there is no inductive action on the wire. We have then $r = v/\gamma$, and

$$[r] = [v\gamma^{-1}] = [L^{-1} T \kappa^{-1}].$$

32. Conductance $[1/r]$. The conductance of a wire of resistance r is $1/r$. Hence

$$[1/r] = [v^{-1}\gamma] = [LT^{-1}\kappa].$$

In the ordinary electrostatic system, in which κ is taken as of zero dimensions, we have

$$[r] = [L^{-1}T],$$

$$[1/r] = [LT^{-1}].$$

A conductance in this system of units may thus be expressed as a speed, *e.g.* so many centimetres per second: a resistance, on the other hand, may be expressed as a slowness, *e.g.* as so many seconds per centimetre.

We may illustrate this result as follows. Let the plates of a condenser be supposed plane and separated by a dielectric of uniform thickness. Further, let the dimensions of the plates be so great that we may take the capacity as inversely proportional to the distance between the plates, that is to the thickness of the dielectric. Or to avoid edge-effects altogether the plates may be taken as the outer surface of an inner sphere and the inner surface of an outer sphere, placed concentrically: in this case the inner plate may be charged by means of a wire let in through a paraffin plug filling a hole cut in the outer plate; the outer plate is not insulated. If A be the area of the charged plate and d the distance between the plates, the capacity c is $CA\kappa/d$, where C is a constant.

Now let the plate which is charged and the uninsulated plate be connected by a long thin wire, so that discharge takes place slowly, and suppose that the plates are receding from one another at such a rate that as the charge q diminishes the difference of potential between them is kept constant in consequence of diminution of the capacity. In the case of the spherical condenser the inner sphere may be supposed to shrink while remaining concentric with the other.

We should have then, since $vc=q$, the charge,

$$\gamma=\frac{v}{r}=-\dot{q}=-v\dot{c}.$$

Thus $1/r=-\dot{c}$, or the conductivity of the wire is measured by the time-rate of diminution of the capacity of the condenser.

If we make the radius of the external surface of the spherical condenser infinite, we have the case of a spherical conductor charged and in its own field. Then $c=\kappa R$, if R be the radius, and $1/r=-\kappa\dot{R}$. If κ be unity the conductivity of the wire is equal to the rate at which each point of the surface is approaching the centre.

A shrinking sphere can be realized by blowing a soap bubble with a tube and then allowing the air to escape slowly. A soap bubble can be charged, and the effect of the charge in increasing the radius observed. Loss of charge by conduction would then result in diminution of radius which would tend to prevent lowering of the potential of the bubble.

B. Electromagnetic System.

33. Magnetic pole. The electromagnetic system of units is based on the unit magnetic pole. This is defined *mutatis mutandis* in precisely the same way as the unit quantity of electricity, on which the electrostatic system is founded; and therefore the purely magnetic quantities here mentioned, which bear the same relations to the unit quantity of magnetism that the corresponding electric quantities bear to the chosen unit quantity of electricity, have, with the substitution of the magnetic analogue to κ, in the electromagnetic system the same dimensional formulae as those just found for the latter quantities in the electrostatic system.

This mode of defining magnetic quantities is distinctly artificial. It depends on the conception of magnetic doublets as elements of a magnet, where each doublet is regarded as a system of two equal and opposite quantities of magnetic matter concentrated at close points on a line which is called the axis of the doublet. The distance ξ between these points is in the limit infinitely small, but the quantities of magnetism are taken so great that the product of either into ξ gives a finite quantity, $m\xi$, which is called the moment of the doublet. A magnetic element, suitable for purposes of calculation, can be constructed; but it must not be taken for granted that this is the real nature of the elements of which an actual magnet is composed.

It is a remarkable fact, which has been made the basis of electromagnetism, that a magnetic element can be imitated exactly by an electric current γ flowing round a small plane circuit, of area A, say, at the position O of the doublet, and set with its plane at right angles to the doublet-axis. The circuit has two aspects, according to the side from which it is viewed, and these, which correspond to the positive and negative magnetisms of the doublet, are inseparable in the idea of the circuit. This element, taken as giving at a point P, for which OP is equal to r and makes an angle θ with the positive direction of the axis, a magnetic potential $\gamma A \cos\theta/r^2$ [see Chap. II.], enables the magnetic field-intensity due to any distribution of magnetism to be found for any point of the field. Here γA is the moment of the element; and so we get at P components of field-intensity, $2\gamma A \cos\theta/r^3$ along OP, and $\gamma A \sin\theta/r^3$ transverse to OP. The resultant field-intensity is

$$(\gamma A/r^3)(4-3\sin^2\theta)^{\frac{1}{2}},$$

and makes the angle $\tan^{-1}(\frac{1}{2}\tan\theta)$ with the outward direction of OP. The differential equation of a "line of magnetic force" due to the element is $dr/2\cos\theta = r\,d\theta/\sin\theta$, which is at once integrable and gives as the polar equation of the curve $\sin^2\theta = Cr$, where C is a constant.

Such an elementary circuit seems to be the natural fundamental thing in magnetism, and its adoption in definitions would avoid some difficulties in the consideration of "magnetic force and magnetic induction" in magnets of steel and other magnetizable substances.

[Although the two kinds of magnetism are inseparable even in an elementary magnet it is possible to realize a single magnetic pole to a high degree of approximation. One end of a long thin uniformly magnetized bar, such that (if the bar is not straight) the other end is distant from the first, is as far as points near it in the field are concerned, a single magnetic pole (see p. 39).]

It is clear that the dimensional formula $[m]$ of quantity of magnetism is given by

$$[m]=[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}].$$

In the ordinary electromagnetic system of units μ is defined (see p. 26 below) so as to be a mere numeric. We shall not make this assumption, but allow μ to appear in the formulae, and its dimensions may be afterwards assigned. By simple deletion of μ from the dimensional formulae they become those for the ordinary electromagnetic system in which $[\mu]=1$.

34. Magnetic moment $[\mathfrak{m}]$. The numeric for the magnetic moment of a uniformly magnetized bar-magnet, is the product of the numerics for the strength of either pole and the length of the magnet. Hence we have

$$[\mathfrak{m}]=[M^{\frac{1}{2}}L^{\frac{5}{2}}T^{-1}\mu^{\frac{1}{2}}].$$

Moment of a doublet or of an elementary current circuit $[\gamma A]$. The dimensional formula is the same as that for magnetic moment.

35. Intensity of magnetization $[\nu]$. The intensity of magnetization of any portion of a magnet is measured by the magnetic moment of that portion per unit of volume. Hence if ν denote the numeric for the intensity of magnetization of a uniformly magnetized magnet, the numerics for the volume and magnetic moment of which are AL^3 and $\mathfrak{m}$, we have $\nu=\mathfrak{m}/AL^3$, and so

$$[\nu]=[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}].$$

It is clear that the intensity of magnetization of a uniformly and longitudinally magnetized bar is equal to the surface density of the magnetic distribution over the ends of the bar, and therefore intensity of magnetization has the same dimensional formulae as surface density of magnetic distribution.

Electric current $[\gamma]$. Also, since $\gamma A\mu$ has the dimensions of magnetic moment,

$$[\gamma A]=[M^{\frac{1}{2}}L^{\frac{5}{2}}T^{-1}\mu^{-\frac{1}{2}}];$$

and we get

$$[\gamma]=[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}].$$

Hence, in the ordinary electromagnetic system, in which $[\mu]=1$, electric current has the dimensions of intensity of magnetization multiplied by a length. If we consider a uniformly magnetized bar of uniform cross-sectional area A, current has then the dimensions of the magnetic moment of the bar taken per unit of A.

36. Magnetic permeability $[\varpi]$. The magnetic inductive capacity, or the *magnetic inductivity*, is the analogue in magnetism of the electric inductivity of a dielectric. We define the specific magnetic inductivity, or the magnetic permeability, of a medium as the ratio of the magnetic inductivity of that medium to the magnetic inductivity of a chosen medium, say air at 0° C., and at standard atmospheric pressure, or, as we shall sometimes suppose, a high vacuum. So far as experiment goes there is little influence of temperature or pressure on the magnetic inductivity of a gaseous medium, and but little variation from one such medium to another. The results of experiments on high vacua seem to show that the magnetic inductivity in a nearly vacuous space is sensibly the same as in air under ordinary temperature and pressure.

If we consider a highly magnetizable material such as iron, and adopt the Amperean hypothesis that it is a congeries of small molecular circuits carrying currents, then the magnetization of the iron, however effected, consists in an alignment of the axes of these circuits, both as to direction and as to aspect. We may suppose then that the congeries of circuits is permeated by ether, and imagine a point in the ether. At such a point there will be a magnetic field-intensity due not merely to magnets at a distance but to the molecular magnets imbedded in the ether. If we call this field intensity in ether H, and that due to magnetic distributions at a distance, estimated in the manner explained below, H', we shall have H/H' equal to the magnetic permeability of the substance.

In each case the point is considered as situated in the ether which pervades the ordinary matter. For H, the field-intensity due to the molecular magnets, is taken account of, and though the value of H may vary very considerably from point to point among the molecular magnets we may assume that in a statistical sense it is definite, and approximately that within a narrow crevasse (or disk-space), with parallel walls at right angles to the direction of magnetization, produced by removing the molecular magnets from that space. For H a tunnel, or space of small breadth and depth, but of length comparatively great and in the direction of magnetization, is supposed cleared of the molecular magnets, and the field-intensity at the centre of that space is taken as H' [see also p. 57].

According to this mode of considering the matter, the permeability ϖ is a mere numeric, and we have $[\varpi]=1$.

If we use the hypothesis of magnetic matter, we have for H the value of H' together with the field intensity due to the unbalanced surface distributions of magnetism produced by the formation of the crevasse, or $H=H'+4\pi\sigma/\mu_0$, or $H=H'+4\pi I/\mu_0$, if μ_0 denote the magnetic inductivity of the medium in the crevasse (ether, we may suppose), and I the intensity of magnetization at the crevasse, since I, as noticed above, must then measure the intensity of magnetization. It is usual to introduce a quantity κ such that $I/\mu_0=\kappa H'$, which gives $H=(1+4\pi\kappa)H'$, and so $\varpi=1+4\pi\kappa$. The multiplier κ is called the magnetic susceptibility.

37. Intensity of magnetic field $[H]$. A quantity m of magnetism placed at a point in a magnetic field, at which the intensity is H, experiences a force mH. Hence we have

$$[mH]=[M^{\frac{1}{2}}L^{\frac{3}{2}}\mu^{\frac{1}{2}}H], \quad \text{and therefore} \quad [H]=[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}].$$

It has been agreed to call the c.g.s. unit of H one gauss.

Magnetic induction $[B]$. The magnetic induction is the product of the magnetic field-intensity by μ. Hence we get

$$[B]=[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}].$$

38. Magnetic susceptibility $[\kappa]$. We have $[\kappa]=[I][H^{-1}][\mu^{-1}]$, and therefore

$$[\kappa]=1.$$

This also follows from the fact that κ occurs in the factor $1+4\pi\kappa$, which is the value of ϖ.

39. Quantity of electricity $[Q]$. The numeric Q for the quantity of electricity conveyed in T seconds by a current, the numeric for the strength of which is γ, is γT. Hence $[Q]=[\gamma T]$, or

$$[Q]=[M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}].$$

40. Electric potential, or **electromotive force** $[V]$. As above (p. 17), but using in this case V for the numerical value of a difference of potential, we get $Work=VQ$. Thus we have

$$[V]=[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}].$$

41. Electrostatic capacity $[C]$. If Q and V be the numerics for the charge and potential of a condenser, we have for the numeric of the capacity $C=Q/V$. Thus we get

$$[C]=[L^{-1}T^{2}\mu^{-1}].$$

42. Resistance $[R]$. If V be the numeric for the difference of potential between the terminals of a conductor (as specified above, p. 17), and γ be that of the current flowing, the numeric for the resistance of the conductor is $R=V/\gamma$. Hence

$$[R]=[LT^{-1}\mu].$$

If $[\mu]=1$, the dimensional formula for resistance is the same as that for velocity, and therefore a resistance can be expressed in ordinary electromagnetic units as a speed, and accordingly in c.g.s. units as so many centimetres per second.

Consider an ideal electromagnetic engine consisting of two massive parallel rails, of copper say, laid parallel to one another so that their plane is at right angles to the lines of force of a uniform magnetic field of intensity H. Let the rails be insulated from one another and be at a distance l apart. Now let a sliding conductor be placed across the rails at right angles, and be moved along them with a steady speed v. After a short time a constant difference of potential will be established between the rails, the numerical value of which in electromagnetic units is Hlv.

If the rails be now joined by a wire of resistance R, and the rails and sliding bar have resistance negligible in comparison with R, the current γ in the wire will be given by the equation

$$Hlv-\gamma\frac{dL}{dt}=R\gamma$$

provided the current remains constant. The quantity L is the *self-inductance* of the circuit. If the rails and slider have sensible resistance, R must of course be the total resistance in the circuit. Since $\gamma dL/dt$ must have the dimensions of Hlv we can find another speed v_1, which would be that of the slider for the same R and the instantaneous value of γ if the circuit were deprived of self-inductance, so that $Hlv_1 = R\gamma$.

Now let the wire be partly contained in the coil of a tangent galvanometer, and the field-intensity which gives the return couple on the needle of the galvanometer, when deflected by the magnetic action of the current, be also H. This can be arranged by running the rails in a magnetic east and west vertical plane, so that the horizontal component of the earth's field is that cut by the sliding bar.

If the turns of wire (n in number) in the coil of the galvanometer form a ring of small cross-section, we may take each as of radius r, and as having all the same effect on the needle. If θ be the deflection of the needle the current is $(Hr/2\pi n)\tan\theta = (Hr^2/L)\tan\theta$, where L is the whole length of wire in the coil. Hence we have

$$R = \frac{Llv_1}{r^2 \tan\theta}.$$

Now we may suppose that the radius r of the coil $= \sqrt{Ll}$, and that v_1 is such that $\theta = 45°$. We then get $R = v_1$.

This illustration of resistance as a speed was given by Sir W. Thomson (Lord Kelvin) [*B.A. Rep. on Electr. Standards*, App. B. § 30]. It is rather remarkable that in the original descriptions of this illustration given in the B.A. Report [*Collected Reports*, pp. 68, 69 and 112, 113] and in accounts of it given by other writers (including that in the first edition of the present work) no account is taken of the self-inductance of the circuit. [See also below, p. 31.]

43. Self-inductance [L_1]. From the above discussion it will be seen that $L_1\gamma$ has the same dimensions as $Hlvt$. But $\gamma = Hlv_1/R$, and therefore L_1 has the dimensional formula of Rt. Thus

$$[L_1] = [L\mu].$$

In the ordinary electromagnetic system of units L_1 has thus the dimensions of a length. In that system, as we shall see, the practical unit of resistance is the *ohm* (10^9 cm/sec) or, as it is sometimes described, *one earth-quadrant per second*. The corresponding unit of L_1 is therefore *one quadrant* in the same system. This is now usually called a *henry*, in honour of Joseph Henry of Washington, who independently discovered the main facts of the induction of currents. (See *Scientific Writings of Joseph Henry*, Washington, 1886.)

The self-inductance, L_1, of a circuit, it will have been seen, is measured by the ratio of the numeric for the magnetic induction embraced by the circuit, and produced by the current flowing round the circuit, to the numeric for that current. Its value may vary, as in the case just considered, with the time; and it may in certain cases be a function of the current-strength. Definitions differing from that here stated have been adopted by various writers for greater convenience of discussion of the cases just referred to; but none of these alters the dimensional formula.

44. Mutual inductance $[M_1]$. The mutual inductance M_1 of two circuits is measured by the ratio of the numeric for the magnetic induction, embraced by either of the circuits and produced by the current in the other, to the numeric for that current. Clearly its dimensional formula is the same as that of self-inductance.

It does not matter (unless there is material within either or both of the circuits the magnetization of which is a function of the current, and except in certain other cases) which circuit carries the current; the mutual inductance is unaffected by this circumstance [See the discussion of the induction coil in Appendix I.]

Vector Potential $[A]$. The components a, b, c of magnetic induction at a point are the components of the curl of the vector-potential. Thus, if F, G, H be the components of A the vector-potential, we have

$$a = \partial H/\partial y - \partial G/\partial z, \text{ etc.}$$

We see therefore, that the dimensional formula of vector-potential is that of magnetic induction multiplied by a length, that is

$$[A] = [M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1} \mu^{\frac{1}{2}}]. \qquad \text{[See Chap. II.]}$$

45. Relation between electrostatic and electromagnetic units. We have now investigated the dimensional formulae of the absolute units of all the principal electric and magnetic quantities, in the electrostatic system or in the electromagnetic system, according as each quantity is measured in practice. Each may, however, be expressed either in electrostatic units or in electromagnetic units. The bridge from one system to the other is the fact that a current in a circuit (defined in electrostatic units by the ratio q/t) gives a magnetic field-intensity (the H' of 36 above) which is independent of the nature of the medium occupying the field, so that if γ be reckoned in electrostatic units we have $[\gamma]\,[L^{-1}] = [H]$, that is in electrostatic units

$$[H] = [q/t]\,[L^{-1}] = [M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-2} \kappa^{\frac{1}{2}}].$$

Or the theorem that the work done in carrying a unit magnetic pole round a current of strength γ is $4\pi\gamma$, whatever the medium, may be taken as the connecting link, with the same result. We give the following tables of dimensional formulae for all the quantities and in both systems. In Tables II. and III. κ and μ have been introduced into the formulae as stated in 24 above, with their dimensions undetermined. The ordinary electrostatic and electromagnetic systems are obtained by supposing κ or μ to be unity as the case may be.

46. Systems reconciled by "suppressed dimensions" of μ and κ. Tables of dimensions. As indicated above, one advantage of thus exhibiting the dimensions is that it enables electrostatic and electromagnetic quantities to be regarded as of the same absolute dimensions, since κ and μ, not being fixed as to dimensions, can, unless restricted by definition, have dimensions assigned to them which fulfil this definition. For example, as suggested by the late Professor G. F. FitzGerald, each may be taken as having the dimensional formula $[TL^{-1}]$. Another advantage is that problems in which passage from one set of units to another is involved, are solved with greater ease from first principles. [See Professor Sir Arthur Rücker's paper, *loc. cit.* 24 above.]

Fundamental Units.

Quantity.	Dimensional formula.
Length	$[L]$
Mass	$[M]$
Time	$[T]$

Derived Units.

(i) *Dynamical.*

Speed	$[LT^{-1}]$
Acceleration	$[LT^{-2}]$
Force	$[MLT^{-2}]$
Work Energy	$[ML^2T^{-2}]$

(ii) *Electrical Units.*

	A, in terms of L, M, T, κ.	*B*, in terms of L, M, T, μ.
Quantity of Electricity	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\kappa^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}]$
Surface Density of Electricity Electric Displacement	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\kappa^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}\mu^{-\frac{1}{2}}]$
Electric Force or Intensity of Electric Field	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\kappa^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}]$
Electric Potential Electromotive Force	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\kappa^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\mu^{\frac{1}{2}}]$
Electrostatic Capacity	$[L\kappa]$	$[L^{-1}T^2\mu^{-1}]$
Electric Inductivity	$[\kappa]$	$[L^{-2}T^2\mu^{-1}]$
Current Strength	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\kappa^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$
Resistance	$[L^{-1}T\kappa^{-1}]$	$[LT^{-1}\mu]$
Specific Resistance	$[T\kappa^{-1}]$	$[L^2T^{-1}\mu]$
Vector-Potential	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}\kappa^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}]$

(iii) *Magnetic.*

	A, in terms of L, M, T, κ.	*B*, in terms of L, M, T, μ.
Quantity of Magnetism or Magnetic Pole - - Electrokinetic Momentum of Circuit - - - -	$[M^{\frac{1}{2}}L^{\frac{1}{2}}\kappa^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}]$
Surface Density of Magnetism Intensity of Magnetization -	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}\kappa^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}]$
Magnetic Moment - - -	$[M^{\frac{1}{2}}L^{\frac{3}{2}}\kappa^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{5}{2}}T^{-1}\mu^{\frac{1}{2}}]$
Magnetic Potential - -	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}\kappa^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$
Magnetic Inductivity - -	$[L^{-2}T^{2}\kappa^{-1}]$	$[\mu]$
Magnetic Force or Magnetic Field-Intensity - -	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}\kappa^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$
Magnetic Induction - -	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}\kappa^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}]$
Self-Inductance - - - Mutual Inductance - -	$[L^{-1}T^{2}\kappa^{-1}]$	$[L\mu]$

It will be noticed that magnetic induction, which is magnetic field-intensity multiplied by magnetic inductivity, has the same dimensional formulae in the two systems as surface density of magnetism and intensity of magnetization.

47. Examples of dimensional formulae. We now take some examples of the use of these formulae. The second and third are solutions of problems stated and solved in Sir Arthur Rücker's paper (*loc. cit.* 24 above).

1. The earth's horizontal magnetic force at Greenwich was given in the *Nautical Almanac* for 1883 as 3·92 in foot-grain-second units. Required its value in c.g.s. units.

Let H be the value sought. Then

$$H = 3{\cdot}92[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}] = 3{\cdot}92\left[\frac{1}{15{\cdot}43235} \times \frac{1}{30{\cdot}47945}\right]^{\frac{1}{2}} = 0{\cdot}18705\,;$$

since 1 gramme $=$ 15·43235 grains, and 1 centimetre $= 1/30{\cdot}47945$ foot.

2. The unit change of electricity is defined to be such that two unit point-charges at a distance of 1 metre, in a medium of electric inductivity twice that of air, repel each other with a force which would give 1 centigramme an acceleration of 1 metre per second per second: find the number of ordinary electrostatic units to which this charge is equivalent. Also find the number of electromagnetic units in the electrostatic unit thus defined.

If q be the number of electrostatic units required we have $\frac{1}{2}q^2/100^2 = 1$, and therefore $q = 100\sqrt{2}$.

If N be the number of electromagnetic units sought, we get by the dimensional formulae, since the units of length and mass for this value of q are now the centimetre and the gramme,

$$N[\mu^{-\frac{1}{2}}]/[\kappa^{\frac{1}{2}}] = 100\sqrt{2}.$$

But $1/\sqrt{\mu\kappa} = 3 \times 10^{10}$. Hence

$$N = \tfrac{1}{3}\sqrt{2} \times 10^{-8}.$$

3. Find the number of c.g.s. electrostatic units of magnetic induction in 10 electromagnetic units of a system in which the units of length, mass, and time are the metre, centigramme, and second respectively, and in which the specific inductive capacity and index of refraction of the standard medium in comparison with air are 2 and 1·5 respectively.

Let N be the number required, and let κ and μ refer to air, while κ', μ' refer to the standard medium. By the dimensional formulae we have

$$N[(\text{grm})^{\frac{1}{2}}\text{cm}^{-\frac{3}{2}}\kappa^{-\frac{1}{2}}] = 10\left[\left(\frac{1}{100}\text{grm}\right)^{\frac{1}{2}}(100\,\text{cm})^{-\frac{1}{2}}\mu'^{\frac{1}{2}}\right]$$

or

$$N = \frac{1}{10}\kappa^{\frac{1}{2}}\mu'^{\frac{1}{2}} = \frac{1}{10\sqrt{2}}[2\kappa\mu']^{\frac{1}{2}}.$$

But

$$\sqrt{2\kappa\mu'} = \sqrt{\kappa'\mu'} = \frac{1\cdot5}{3 \times 10^{10}} = \frac{1}{2}10^{-10}.$$

Hence

$$N = \frac{1}{2\sqrt{2}}10^{-11}.$$

V. UNITS ADOPTED IN PRACTICE.

48. Coulomb, ohm, volt, ampere. In practical work the resistances and electromotive forces occurring to be measured are usually so great that if the absolute electromagnetic units of the c.g.s. system were used, the resulting numerics would be inconveniently large; while, on the other hand, capacities are generally so small that their numerics in c.g.s. units would be small fractions. Accordingly, certain multiples of the c.g.s. units of resistance and electromotive force, and a submultiple of that of capacity have been chosen for use in practice. The derivation of the first two (the *ohm* and the *volt*) together with the practical units of current and quantity (the *ampere* and the *coulomb*) may be illustrated by means of the rails and slider magneto-machine referred to above.

We have seen that if the rails are insulated and only connected by the sliding bar, the difference of potential between them is Hlv. In the c.g.s. system this will be unity if H be one c.g.s. unit of magnetic field-intensity, l be one centimetre, and v be one centimetre per second; that is, the difference of potential between the rails would be one c.g.s. unit.

This difference of potential is too small for use as a practical unit, and instead of it, the difference of potential which would be produced if, everything else remaining the same, the speed of the slider were 10^8 centimetres per second, is taken as the practical unit of difference of potential, or electromotive force, and is called one *volt*. It is a little less than the difference between the two terminals of a Daniell's cell on open circuit.

If the rails be connected by a wire of resistance very great in comparison with that of the rest of the circuit, the constant current produced will, for a given value of v_1 (the speed used to replace v in consequence of the variation of self-inductance of the circuit), vary inversely as the resistance of the wire. Let us suppose that when the slider 1 cm long was moving with a virtual speed v_1 of 1 cm per second, the current in the wire was 1 c.g.s. unit; the resistance of the wire was then 1 c.g.s. unit of resistance.

This resistance, however, is too small to be practically useful, and a resistance 10^9 times as great, that is the resistance of a wire, to maintain 1 c.g.s. unit of current in which it would be necessary that the slider should have a virtual speed v_1 of 10^9 cm (approximately the length of a quadrant of the earth from the equator to the north pole) per second, is taken as the practical unit of resistance, and called one *ohm*.

49. Specification of international ohm, volt, and ampere. An account of experiments which have been made for the realization of the ohm is given in a later chapter, and it will be seen from the results that the ohm to an accuracy of "one-tenth part of one per cent." (*Order in Council Relating to Electrical Standards*, Jan. 10, 1910) is equal to "the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14·4521 grammes in mass of a constant cross-sectional area, and of a length of 106·300 centimetres." The specification here quoted from the *Order in Council* is that of what is called the International Ohm, which, with the International Ampere and the International Volt, was recommended for adoption by the International Congress of Electricians held at London in 1908.

It is Ohm's Law that if the difference of potential V between two points in a homogeneous wire, which is at rest in a magnetic field, be altered in any ratio, the current in the wire will be altered in the same ratio. It is supposed, however, that there is no sensible heating of the wire by the current. If V be 1 volt (10^8 c.g.s. units of potential) and R be 1 ohm (10^9 c.g.s. units of resistance) the current will be $\frac{1}{10}$ of 1 c.g.s. unit of current. A current of this strength has been adopted as the practical unit of current, and called one *ampere*.

The ampere is defined for practical purposes of measurement and reproduction by means of electrolysis. Thus, in the *Order in Council* already quoted, the International Ampere is defined as "the unvarying electric current which, when passed through a solution of nitrate of silver in water, deposits silver at the rate of 0·00111800 of a gramme per second."

The *Order in Council* also defines the International Volt as the difference of potential "which, when steadily applied to a conductor whose resistance is one International Ohm, will produce a current of one International Ampere."

The amount of electricity conveyed in one second by a current of one ampere is called *one coulomb*. This unit, although not so frequently required as the others, is very useful, as, for instance, for expressing the quantities of electricity which a secondary cell is capable of yielding in various circumstances. For example, in comparing different cells with one another, their capacities, or the total quantities of electricity they are capable of yielding when fully charged, are very conveniently reckoned in coulombs per sq cm of the area across which the electrolytic action takes place.

50. Realization of unit of e.m.f. We have now to consider the energetics of a circuit moving in a magnetic field; it will be sufficient for the present purpose, and it will fix the ideas, to take the case of the simple magneto-machine to which reference has already been made above (42, 43). The slider of length l between the rails (supposed horizontal) is moving with speed v across the lines of force of a magnetic field of uniform intensity H. Since H is uniform the lines are uniformly directed; we suppose that they are at right angles to the plane of the rails. If the self-inductance of the circuit be L, the whole resistance R, and the current γ, the equation of currents is

$$Hlv - \frac{d}{dt}(L\gamma) = R\gamma. \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We shall suppose that the current is kept constant, so that the equation becomes

$$Hlv - \gamma\frac{dL}{dt} = R\gamma. \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (2) we derive, since γ is constant, the relation

$$Hl\frac{dv}{dt} = \gamma\frac{d^2L}{dt^2}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

so that unless L increases at a uniform rate (which we shall find is not the case) the speed v of the sliding bar cannot be constant.

The backward drag of the field on the sliding bar is $Hl\gamma$, and a force equal to this in the forward direction must be applied to the bar by an external agent. The agent therefore works at rate $Hl\gamma v$. If for simplicity we suppose that the sliding bar is not resisted by friction, this must be equal to the sum of the rates at which work is spent (1) in heat in the circuit, (2) in increasing the electrokinetic energy of the current, (3) in increasing the kinetic energy of the moving bar. At the instant considered the electrokinetic energy is $\frac{1}{2}L\gamma^2$, and the kinetic energy of the bar is $\frac{1}{2}mv^2$, if m be the mass. Thus, on the expressed condition that the current is constant, we have the activity equation

$$Hl\gamma v = R\gamma^2 + \tfrac{1}{2}\gamma^2\frac{dL}{dt} + mv\frac{dv}{dt}. \qquad \text{.......................(4)}$$

But by (2) we obtain

$$Hl\gamma v = R\gamma^2 + \gamma^2\frac{dL}{dt}, \qquad \text{.......................(5)}$$

which asserts that over and above the rate of dissipation $R\gamma^2$, electrical work is done at rate $\gamma^2 dL/dt$. Equations (4) and (5) give, however,

$$mv\frac{dv}{dt} = \tfrac{1}{2}\gamma^2\frac{dL}{dt}, \qquad \text{.......................(6)}$$

so that half the rate of electrical working $\gamma^2 dL/dt$, goes to increase the kinetic energy of the slider. The other half is employed in increasing the electrokinetic energy.

Now it is known, and will be proved in Chap. XIII., that for two parallel wires of circular section, radius ρ, distance l between their axes, and length x, carrying one a direct the other an equal return current, L is given by the equation

$$L = 4x\log\frac{l}{\rho} + x - 4l + \textit{terms depending on the cross connections.}$$

Hence
$$\frac{dL}{dt} = 4\dot{x}\left(\log\frac{l}{\rho} + \frac{1}{4}\right) = 4v\log\left(\frac{l}{\rho} + \frac{1}{4}\right).$$

Thus, by (6),
$$m\dot{v} = 2\gamma^2\left(\log\frac{l}{\rho} + \frac{1}{4}\right),$$

or if
$$2\gamma^2\log(l/\rho + \tfrac{1}{4}) = ma,$$
$$\dot{v} = a.$$

Thus, if v_0 be the speed at time t_0,

$$v - v_0 = a(t - t_0).$$

Also, since $\dot{v} = v\,dv/dx$, we get

$$\tfrac{1}{2}(v^2 - v_0^2) = a(x - x_0),$$

where $x - x_0$ is the distance travelled by the slider in time $t - t_0$. But

$$x - x_0 = v_0(t - t_0) + \tfrac{1}{2}a(t - t_0)^2,$$

and therefore, in terms of the time, we obtain for the kinetic energy of the slider,

$$\tfrac{1}{2}m(v^2 - v_0^2) = mav_0(t - t_0) + \tfrac{1}{2}ma^2(t - t_0)^2.$$

We obtain also by the value of L, and that of a found above,

$$L - L_0 = 4(x - x_0)\left(\log\frac{l}{\rho} + \frac{1}{4}\right) = \frac{2m}{\gamma^2}av_0(t - t_0) + \frac{m}{\gamma^2}a^2(t - t_0)^2.$$

51. Disk magneto for realization of unit of e.m.f. The rails and slider illustration which we have discussed is distinctly inferior to the arrangement, sometimes substituted, of a metal disk rotating at right angles to the lines of force of an impressed magnetic field, and touched at its centre and circumference, or at the circumference and at an inner concentric circle, by the terminals of the external part of the circuit. If the field be maintained constant and the disk rotate at a uniform rate, and there be no variation of contacts of the wire with the disk, a constant current will be maintained. This of course is the arrangement of a disk magneto-machine, and of the Lorenz apparatus for the determination of the ohm, except that in the latter case the electromotive force in the disk circuit is balanced.

In this arrangement there is no variation of self-induction, inasmuch as the configuration remains unchanged as the rotation proceeds. The electromotive force for total integral magnetic induction through the disk, of amount I, between the circles of contact, and angular speed ω is $I\omega/2\pi$; or if the magnetic field-intensity be uniform and of numeric H, the inductivity be taken as 1, and the outer and inner circles of contact have radii a, a', the electromotive force is $\frac{1}{2}(a^2-a'^2)H\omega$. The current is $\frac{1}{2}(a^2-a'^2)H\omega/R$ in the latter case, or $I\omega/2\pi R$ in the former.

If we consider an external part of the circuit, that is a part not including the slider (or the disk in the arrangement just described)—the seat of the electromotive force—the whole rate at which work is done in that part is $V\gamma$, if V be the difference of potential between the terminals of that part. This rate of working may of course consist of $E\gamma + R'\gamma^2$, where E is the back electromotive force of a motor, or other arrangement, due to the performance of work at rate $E\gamma$, otherwise than in producing heat, and R' is the resistance included between the terminals, so that $R'\gamma^2$ is the rate at which work is spent in heat in that part of the circuit.

52. Absolute units of electrical energy : B.T.U. and watt. One of the advantages of the system of units described here is that the value of the rate at which work is done in a circuit is stated without the introduction of any coefficient such as would have been necessary if the electrical units had been arbitrarily chosen. When the quantities are measured in c.g.s. units the value of $E\gamma$ is given in ergs per second. Results thus expressed may be reduced to horse-power by dividing by $7{\cdot}46 \times 10^9$; or, if E is measured in volts and γ in amperes, $E\gamma$ may be reduced to horse-power by dividing by 746. Thus, if on the terminals of an arc-lamp a difference of potential of 80 volts be maintained and the current be 15 amperes, the rate at which energy is spent on the lamp is 1200/746, or 1·61, horse-power, nearly.

Electrical energy is usually sold in Board of Trade Units (B.T.U.). A B.T.U. is the energy supplied in 1 hour by an e.m.f. of 1000 volts and a current of 1 ampere.

If the rate at which work is done in maintaining a current of one ampere through a resistance of one ohm, when the work is all spent in producing heat in the conductor, is taken as the practical unit of activity, and E is reckoned in volts and γ in amperes, rate of working is simply $E\gamma$, and calculations of electrical work are much simplified. This choice of a unit of activity was proposed by Sir William Siemens (*B.A. Address*, 1882), with the suggestion that the unit should be called a *watt*. Thus the rate of expenditure of energy on the arc-lamp in the example taken above is 1200 watts. A watt is equivalent to 10^7 ergs per second or very nearly $\frac{1}{746}$ horse-power.

53. Kilowatts and joule. The Electrical Congress held at Paris in 1889 adopted the watt as the practical unit of rate of working for electrical purposes, and the term *kilowatt*, proposed by the late Sir W. H. Preece, to designate an activity of 1000 watts, or 10^{10} ergs per second. To a considerable extent the kilowatt is now used instead of the horse-power. An activity given in kilowatts can be reduced to horse-power by dividing by 0·746, or roughly by multiplying by 4 and dividing by 3.

Sir William Siemens also proposed to call the work done in one second when the rate of working is one watt, one *joule*. A joule is therefore equivalent to 10^7 ergs, and the work done in one second in the above example is 1200 joules.

The Electrical Congress of 1889 also adopted the joule as the practical unit of work. The International Congress of Electricians, held at London in 1908, recommended the adoption of an *international watt*, defined as the energy expended per second by an unvarying electric current of one international ampere under a difference of potential of one volt.

54. Unit of capacity. The practical unit of electrostatic capacity is called the *farad*, and is defined as the capacity of a condenser which, when charged by an electromotive force of one volt applied to its terminals, has a charge of one coulomb. If C be the capacity of such a condenser in c.g.s. electromagnetic units of capacity, we have $C = 10^{-1}/10^8 = 10^{-9}$; or one farad is equivalent to 10^{-9} c.g.s.

In some cases, when the quantities to be expressed are very large, units one million times the chosen practical units are employed. These are denoted by the names of the corresponding practical units with *mega* (great) prefixed. On the other hand, for the expression of very small quantities, units one millionth of the practical units are sometimes used, and are denoted by the names of the corresponding practical units with *micro* (small) prefixed.

Such units are however rarely employed, with the exception of the *megohm*, used for expressing the high resistances of insulating substances, and the *microfarad*, which is really the most convenient unit for the expression of capacities. A megohm, in ordinary electromagnetic units, may be expressed as 10^{15} cm per second; one c.g.s. unit of capacity is equivalent to 10^{15} microfarads.

VI. PRACTICAL UNITS AS AN ABSOLUTE SYSTEM.

55. Practical units as an independent system. The practical units which have been adopted may be considered as belonging to an absolute system based on a unit of length equivalent to 10^9 cm (one earth-quadrant), a unit of mass $1/10^8$ of a milligramme, or 10^{-11} gramme, and the second as unit of time. The verification of this in the different cases will furnish some further examples of dimensional formulae.

First let us find what the numerics for resistances and electromotive forces, expressed in terms of c.g.s. units, become when these new units of length and mass are substituted. Let R be the numeric for c.g.s. units and R' the numeric for the new units, of one and the same resistance. Then

$$R' = R[LT^{-1}] = R\frac{1}{10^9}.$$

Calling the unit of resistance in the new system one ohm, we see that

$$1 \text{ ohm} = 10^9 \text{ c.g.s. units of resistance.}$$

Again, let V be the numeric for an electromotive force in terms of the c.g.s. electromagnetic unit, V' the corresponding numeric for the new system. We have

$$V' = V[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}] = V[10^{-\frac{11}{2}} \times 10^{-\frac{3}{2}9}],$$

that is

$$V' = V \times 10^{-8}.$$

Calling the new unit one volt, we see that

$$1 \text{ volt} = 10^8 \text{ c.g.s. units of e.m.f.}$$

The following table gives the numerics for the various practical units in terms of the corresponding c.g.s. units:

Quantity.	Practical Unit.	Equivalent in c.g.s. Units.
Resistance	Ohm	10^9
Electromotive Force	Volt	10^8
Current	Ampere	10^{-1}
Quantity of Electricity	Coulomb	10^{-1}
Electrostatic capacity	Farad	10^{-9}
	Microfarad	10^{-15}

56. Ratio of units. We have seen above that if N, N' be the numerics or two quantities, the dimensional formula of N'/N is $[N']/[N]$, and this of course applies to the expressions of the same quantity in two different systems of units. Thus, if q, Q be the numerics for a quantity of electricity in electrostatic and electromagnetic units respectively (founded of course on the same fundamental units), we have

$$[q] = [M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}] \quad \text{and} \quad [Q] = [M^{\frac{1}{2}}L^{\frac{1}{2}}].$$

The dimensional formula $[q]/[Q]$ is thus $[LT^{-1}]$. Thus q/Q has the dimensions of speed, and as q/Q is the inverse of the ratio of the units employed in the two cases, q/Q expresses a certain definite speed, the numeric for which depends on the fundamental units of length and time employed. In other words, the number of electrostatic units of electricity equivalent to one electromagnetic unit is the numeric for this speed.

The same speed is derivable from the ratios of the numerics for any one of the other electrical or magnetic quantities in the two systems of units. For example, if e, E be the numerics for one and the same electromotive force in electrostatic and electromagnetic units respectively, we have

$$[e/E]=[M^{\frac{1}{2}}L^{\frac{1}{2}}]/[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}]=[L^{-1}T].$$

The ratio e/E has thus the dimensional formula of the reciprocal of a speed, and as this is the reciprocal of the ratio of one electrostatic unit to one electromagnetic unit, we see that the number of electromagnetic units of electromotive force equivalent to one electrostatic unit is a certain definite speed. This speed is identical with the former. For if q, Q be the numerics for one and the same quantity of electricity in the two systems, and e, E are the corresponding numerics for an electromotive force, the work eq must be equal to the work EQ. We get therefore $E/e=q/Q$, that is the two speeds are the same.

By taking the more general dimensional formulae given in the table of 46, we find that when q, Q refer to the ordinary systems,

$$[q/Q]=[\kappa^{-\frac{1}{2}}\mu^{-\frac{1}{2}}].$$

Hence the product $\kappa^{-\frac{1}{2}}\mu^{-\frac{1}{2}}$ has the dimensions of a speed. It is in fact the speed q/Q above referred to.

Denoting this speed by v, we get for the various quantities the following relations, in which the numerator of the ratio on the left of each equation denotes the numeric of the quantity in electrostatic units, the denominator the numeric of the same quantity in electromagnetic units:

A given Quantity of Electricity - - - $q/Q = v$
,, Current - - - - - - - $\gamma/\Gamma = v$
,, Electromotive Force - - - - $e/E = 1/v$
,, Electrostatic Capacity - - - - $c/C = v^2$
,, Resistance - - - - - $r/R = 1/v^2$

Therefore, if q and Q, e and E, or the numerics for any other electrical quantity be determined in the two systems of units, the value of v can be at once obtained. Experiments of this kind have been made by many investigators, and an account of the different methods employed and the results obtained is given in a later chapter. It has been found that $v=3\times 10^{10}$, in cm per second, very approximately, or very nearly the speed of light in air as deduced from experiments made by the methods of Foucault and Fizeau. According to Maxwell's Electromagnetic Theory of Light [*Elec. and Mag.* vol. ii. chap. xx.] this relation should hold, and thus the theory is so far confirmed. It is very remarkable that $(\kappa\mu)^{-\frac{1}{2}}$ should be the speed of light in the ether, and the full significance of the result cannot yet be said to be fully appreciated. In this is no doubt the physical meaning of κ and of μ.

In the present chapter we have considered only the scalar magnitudes of electric and magnetic quantities. For a discussion of dimensions from a vector point of view the reader may refer to a paper by Dr. W. Williams, *Phil. Mag.* Sept. 1892.

CHAPTER II.

Section I.

MAGNETS. MAGNETIC POTENTIAL. POTENTIAL ENERGY OF A MAGNET.

1. Magnetism. Unit of magnetism. We shall suppose the reader to be acquainted with the elementary facts of magnetic phenomena and theory, and shall therefore not devote space to the description of the ordinary phenomena of attraction and repulsion between permanently or inductively magnetized bodies. We recall merely such an outline of theory as may suffice to render intelligible the various methods of magnetic measurement as these occur in the course of our discussion, and define clearly the quantities which are determined by these methods.

It can be shown that magnetic phenomena are capable of being accounted for by supposing the magnetized body or system to be the seat of a distribution of imaginary or fictitious magnetic matter. This matter is of two kinds, each of which repels matter of its own kind, and attracts matter of the other kind. If two portions of this matter be supposed concentrated at points in a uniform medium, the force between them is directly as the product of the quantities, and inversely as the square of the distance between them. To be quite definite we may take, as the medium specified, air at standard atmospheric pressure, and at temperature 0° C. The alteration, however, of the magnetic properties of air, produced by any ordinary change of pressure or temperature is imperceptible. Air is thus the medium with which others are compared, and for which the inductivity μ_0, defined below, is usually taken as unity. Both kinds of matter are always present in the distribution in equal amounts, but the distributions may be different in the two cases. It is to be carefully observed, however, that so far as our knowledge goes, no such *matter* exists. The hypothesis of its existence serves merely to fix the ideas, and afford to them a convenient, but only provisional, mode of expressing the polarity of a magnetized particle.

Between this expression and the, as yet, imperfectly understood physical nature of magnetism, we are able to say that there exists a certain correspondence. It is very important to notice that the presence of equal and opposite amounts of magnetism is essential to the constitution of the particle. The two "polarities," as we describe them, of the particle are just as inseparable and as exactly complementary as are the two aspects which a wheel in rotation presents, according as it is viewed from one side or the other [see p. 20]. If, as seems probable, the magnetization of a body is due to the rotation of particles, these two aspects are the polarities.

Whatever may be the view ultimately established as to the essential nature of magnetism, it is important that the hypothesis of imaginary matter should not be accepted as more than it is—a way of speaking—or be allowed to stand in the way of research as to the physical constitution of magnetized bodies.

We shall, following the ordinary convention, call the magnetism of the same kind as that of the extremity of a magnet which points north positive, and the opposite kind negative. The positive direction of magnetic force will then be that in which a positive magnetic pole is urged to move in the field.

Unit quantity of this magnetic matter (or magnetism as we shall call it) is defined as that quantity which concentrated at a point, at unit distance from an equal quantity of the same kind, also concentrated at a point, is repelled with unit force, when the medium in which both quantities are placed is air. This definition of unit quantity of magnetism, or *unit magnetic pole* as it is sometimes called, is that on which the electromagnetic system of units is founded, and corresponds exactly to the definition of unit quantity of electricity which forms the basis of the electrostatic system. If m, m' be the quantities at the points, r the distance between the points, and μ the magnetic inductivity of the medium, the force F of repulsion (if m, m' be of the same kind) between them is given by

$$F = \frac{mm'}{\mu r^2}.$$

When the distance between the points is 1 centimetre, and the quantities are such that the force between them is 1 dyne, each quantity is 1 c.g.s. unit of magnetism, or, as it is sometimes put, is unit magnetic *pole* in the c.g.s. system of units. In the case of electric force an electric inductivity κ replaces μ for air. It is usual to omit κ and μ for this medium, and then we have, but only apparently, the anomaly of different dimensions for the same physical quantity, according as it is measured in one or the other system of units. The suppressed constants κ and μ are to be understood. [See above, Chapter I. *passim.*]

Thus, if m denote a quantity of magnetism, which, placed at a point distant L units from an equal quantity of the same kind, is repelled with a force of F units, we have $m^2 = FL^2\mu$, and therefore the dimensional formula $[m]$ of quantity of magnetism is $[F^{\frac{1}{2}} L \mu^{\frac{1}{2}}]$, or $[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} \mu^{\frac{1}{2}}]$.

This is a dimensional formula exactly similar to that of the quantity of electricity in the electrostatic system. [See p. 16 above.]

The poles referred to in this definition are of course purely ideal, for, as we have seen, we cannot isolate a quantity of either kind of magnetism from the opposite kind ; but we can by proper arrangements obtain an approximate realization of the definition. Suppose we have two long, very thin, straight steel bars, which are uniformly and longitudinally magnetized ; they may be taken as having poles at their extremities ; in fact, the distribution of magnetism in them is such that the magnetic effect of either bar, at all points external to its own substance, would be perfectly represented by a certain quantity of one kind of magnetism placed at one extremity of the bar, and an equal quantity of the opposite kind of magnetism placed at the other extremity. We may imagine, then, these two bars placed with their lengths in one line, and like poles turned towards one another, and at unit distance apart. If the lengths of the bars be very great compared with this unit distance, say 100 or 1000 times as great, the attraction of the farther pole of one magnet on the unlike pole of the other will be only 1/10,000 or 1/1,000,000 of the repulsion between the near poles of the two magnets, and so, the farther poles will have no effect on the others comparable in practice with the repulsive action of the latter on one another. But there will be an inductive action between the two near poles which will tend to diminish their mutual repulsive force, and this we cannot in practice get rid of. The magnitude of this inductive effect is, however, less for hard steel than for soft steel, and we may therefore imagine the steel of the magnets such that the action of one on the other does not appreciably affect the distribution of magnetism in either. If, then, two equal like poles repel one another with unit force, each, according to the definition, has unit strength.

2. Magnetic field. Magnetic field intensity. Equilibrium of a magnet in a magnetic field. The whole space surrounding a distribution of magnetism is called the magnetic field of the distribution, and the intensity of the field at any point is measured by the force which unit quantity of magnetism, or unit pole, would experience if placed at the point. A magnetic field intensity (or, as it is often called, a *magnetic force*) is therefore a directed quantity. If its value at a point P in the field be $\mathbf{H}$, the force F on a quantity m of magnetism placed at P will be $m\mathbf{H}$. Hence the dimensional formula $[\mathbf{H}]$ of $\mathbf{H}$ is $[F/m]$ or $[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$. [See also p. 22.]

If $\mathbf{H}$ be the same in magnitude (and therefore also in direction) at each point in the field, the field is said to be uniform. Since there is as much magnetism of one kind in a magnetic distribution as of the other kind, a magnetized body, placed in a uniform field, will, if not in equilibrium, experience only a couple, and will, if not prevented by applied forces, turn round until a certain determinate direction in the magnet is parallel to the direction of the magnetic force in the field. This direction in the body is called the magnetic axis.

For example, if a magnet be suspended so as to be free from the action of all except the magnetic force due to the earth, it is found to experience no sensible force of translation as a whole, but takes up a position of directional equilibrium; that is, there is a direction round which if the magnet be turned through any angle it remains in equilibrium in the new position. This direction is that of the magnetic axis of the magnet.

The magnet is also in equilibrium (stable or unstable) if turned through 180° round an axis at right angles to the magnetic axis. Any angular displacement of the magnet not compounded of the two which have just been specified will leave it under the influence of a couple the moment of which depends on (1) the magnet itself, (2) the angle which the new direction of the magnetic axis makes with its direction of stable equilibrium, (3) the intensity of the magnetic field.

In general, for a magnet placed in a uniform magnetic field of intensity **H** so that its axis makes an angle θ with its position of stable equilibrium, that is with the direction of the force, the moment of the couple is $\mathbf{MH}\sin\theta$, where **M** is a quantity depending on the magnet, and called its *magnetic moment*. This couple is a directed quantity, being in fact the vector product of **M** and **H**, and could be represented graphically by a line drawn perpendicular to the axis of the magnet and to **H**, towards the side of the plane of the couple from which the turning action of the couple appears to be counter-clockwise directed.

3. Potential energy of a magnet. Couple on a magnet in a magnetic field. We denote the scalar (numerical) values of **M** and **H** by M and H. If we assume that the magnet has zero potential energy when its axis is at right angles to the lines of force, its potential energy E in the given position is plainly given by the equation

$$E = \int_{\pi/2}^{\theta} MH\sin\theta\,d\theta = -MH\cos\theta. \quad \dots\dots\dots\dots\dots\dots(1)$$

We shall see in II. 13 below that this is the value of the work done in bringing any magnet into a uniform field, and placing it with its axis inclined at an angle θ to its position of stable equilibrium. For certain simple cases such as symmetrical bar-magnets, etc., it is clear that this is the physical meaning of the potential energy defined with reference to the position of zero potential energy above chosen.

If the (scalar) components of the magnetic force **H** referred to three rectangular axes, one, say that of x, drawn in the true north direction, another, that of y, drawn east, and the third, that of z, drawn downwards, be α, β, γ respectively, and the direction cosines of the magnetic axis referred to the same axes be l, m, n, the equation for E becomes

$$E = -M(l\alpha + m\beta + n\gamma). \quad \dots\dots\dots\dots\dots\dots\dots\dots(2)$$

For the moment K of the couple tending to bring the magnetic axis into coincidence with the direction of the resultant force, we have the value

$$K = M\{(m\gamma - n\beta)^2 + (n\alpha - l\gamma)^2 + (l\beta - m\alpha)^2\}^{\frac{1}{2}}. \quad \text{.........} (3)$$

The component N of this couple round the axis of z is given by

$$N = M(l\beta - m\alpha). \quad \text{..............................} (4)$$

If the angle which the total magnetic force makes with a horizontal plane, or the *dip*, be ζ, and the angle between a north and south vertical plane, and a vertical plane through the direction of the magnetic force, or the azimuth of the latter plane, be ϕ, and the angles for the magnetic axis be η and ψ (ψ taken from Ox towards Oy), we have plainly [Fig. 1]

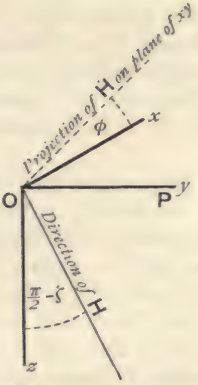

FIG. 1.

$$\alpha = H\cos\zeta\cos\phi, \quad \beta = -H\cos\zeta\sin\phi, \quad \gamma = H\sin\zeta,$$

$$l = \cos\eta\cos\psi, \quad m = \cos\eta\sin\psi, \quad n = \sin\eta.$$

The preceding equations become

$$E = -MH\{\cos\zeta\cos\eta\cos(\phi+\psi) + \sin\zeta\sin\eta\}, \quad \text{..........} (5)$$

$$N = -MH\cos\zeta\cos\eta\sin(\phi+\psi). \quad \text{.................} (6)$$

4. Uniform magnetization. Uniform magnetization has been referred to in p. 39 above, and we shall now consider it a little more fully. A uniformly magnetized magnetic filament is an infinitely thin bar (not necessarily straight nor of uniform cross-section), so magnetized that its action at any external point can be represented by a certain quantity of one kind of magnetism concentrated at one extremity of the bar, and an equal quantity of the opposite magnetism concentrated at the other extremity. Such a filament, if divided across, would be converted into two uniformly magnetized filaments, and each of these in turn, if divided, into two such filaments, and so on. In short, each small element of the filament is to be supposed magnetized in the same way as the whole bar, and to be indivisible, so that, when the elements are united, the action of the polarity of any end of an internal element is annulled by the equal and opposite action of the adjacent end of the next element. Thus the equal and opposite polarities of the ends of the complete filament are left unbalanced.

We may suppose, to make this clearer, that each small element of the filament has equal and opposite distributions of magnetic matter over its two ends, so that the total quantity on two end faces in contact is zero. Of course, as we have seen above, this is only a way of figuring the distribution to the mind; what we really have is no doubt something essentially different from an actual distribution of matter.

Any uniformly magnetized bar may be supposed made up of uniformly magnetized filaments put together with their ends in the surface of the bar. We have in this case a surface distribution of magnetism only.

A non-uniformly magnetized bar may be regarded as one in which the polarities of the elements in contact do not counteract one another; in this case we have, besides the end distributions (which are generally opposite but not necessarily equal), a diffused distribution of magnetism throughout the substance of the bar. Here also the distribution is to be regarded as due to uniformly magnetized filaments, the terminals of which give the surface and body distributions.

5. Magnetic potential. Lines of magnetic force. Equipotential lines and surfaces. This subject is more easily understood when considered mathematically. We shall investigate first the potential and force due to an infinitely short and uniformly magnetized filament, and then consider the general case of a magnet made up of such elements. The magnetic filament is its own magnetic axis, and its magnetic action may be supposed due to equal and opposite quantities of magnetism placed at its two extremities. For brevity we shall call this elementary magnet in what follows a *magnetic doublet*. Its magnetic moment we define as the product of either of these quantities of magnetism into the distance between the extremities, and for our present purpose we shall suppose this product finite. Denoting by δx the length of the filament, which we take in the plane of the paper and parallel to the axis of x, with its centre at the origin of coordinates, we have for the coordinates of its extremities $-\frac{1}{2}\delta x$, $\frac{1}{2}\delta x$. The potential at a point in the plane of the paper the coordinates of which are ξ, η, due to unit quantity of positive magnetism at the origin, is $(\xi^2+\eta^2)^{-\frac{1}{2}}$. Hence if m be the moment of the short magnet, and the positive magnetism correspond to the point $\frac{1}{2}\delta x$, the potential V of the two equivalent point distributions is given by

$$V=\frac{m}{\delta x}\left\{\frac{1}{\{(\xi-\frac{1}{2}\delta x)^2+\eta^2\}^{\frac{1}{2}}}-\frac{1}{\{(\xi+\frac{1}{2}\delta x)^2+\eta^2\}^{\frac{1}{2}}}\right\}=\frac{m\xi}{(\xi^2+\eta^2)^{\frac{3}{2}}}. \quad \ldots(7)$$

This may be written in either of two other equivalent forms, viz.:

$$V = -m\frac{\partial}{\partial\xi}\frac{1}{(\xi^2+\eta^2)^{\frac{1}{2}}} = \frac{m\cos\theta}{r^2}, \quad \text{.....................}(8)$$

where θ is the angle between the axis of the magnet and the line drawn from the centre to the point (ξ, η) and r is the length of that line.

The components X, Y, of magnetic force at the point ξ, η, are given by differentiation of (7), and are

$$\left.\begin{aligned} X &= -\frac{\partial V}{\partial\xi} = -\frac{m(\eta^2-2\xi^2)}{(\xi^2+\eta^2)^{\frac{5}{2}}}, \\ Y &= -\frac{\partial V}{\partial\eta} = \frac{3m\xi\eta}{(\xi^2+\eta^2)^{\frac{5}{2}}}. \end{aligned}\right\} \quad \text{.....................}(9)$$

It is easy to verify that these values of X, Y satisfy the differential equation

$$\frac{\partial X}{\partial\xi} + \frac{\partial Y}{\partial\eta} + \frac{Y}{\eta} = 0, \quad \text{.........................}(10)$$

which is the well-known form which the equation

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0$$

takes in the case of a force system symmetrical round an axis. It is to be noted that in (10) the coordinate η is the distance of the point considered from the axis of symmetry, taken as axis of x, and that therefore Y in (10) above represents $(Y^2+Z^2)^{\frac{1}{2}}$, where Y, Z are taken as the component forces along two other axes of η and ζ at right angles to one another and to that of x.

To find the equation of the lines of force we have for any one line $X/d\xi = Y/d\eta$. Hence by (9) the differential equation in its simplest form is

$$3\xi\eta \, . \, d\xi + (\eta^2 - 2\xi^2)\, d\eta = 0. \quad \text{.......................}(11)$$

This equation may be integrated either by the ordinary method of separation of the variables, or by restoring the omitted common factor $1/r^5$, and remembering that by (10), η is an integrating factor of the equation thus modified. The integral is

$$\frac{\eta^2}{(\xi^2+\eta^2)^{\frac{3}{2}}} = \frac{1}{c}, \quad \text{...............................}(12)$$

in which c is a parameter constant for any one line, but variable from one line to another. [See also the discussion on p. 20.]

6. Graphical construction of lines of force and equipotential lines. This equation may obviously be written in the form

$$r = c\sin^2\theta, \quad \text{.................................}(13)$$

which is very convenient for the graphical description of the curves.

For let O, Fig. 2, be the position of the small magnet, OX the direction of its axis, OY an axis in the plane of the paper at right angles to OX. From O as centre and with c as radius describe a semicircle upon the axis of x. Then draw any line OA intersecting the semicircle in A. From A let fall a perpendicular on OY meeting it in B, and from B a perpendicular to OA intersecting it in P. P is a point on the line of force whose parameter is the value of c chosen. The oval curve in Fig. 2 represents a complete line of force, successive points on which were found in this way. It will be seen that the curve, as is also evident from its equation, is symmetrical about its maximum radius vector, OG, which lies along OY, and is equal in length to c. Points on the curve near G cannot be found with accuracy by the method just described,* but this part of the curve can be filled in with sufficient accuracy, by drawing a circular arc from the centre of curvature for the point G. The radius of curvature for any point is easily found from (13) and is

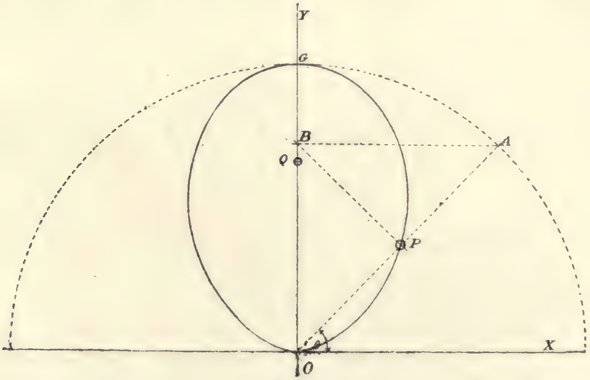

FIG. 2.

$$c \sin \theta \, (\sin^2 \theta + 4 \cos^2 \theta)^{\frac{3}{2}} / 3 \, (\sin^2 \theta + 2 \cos^2 \theta).$$

For the point G this is $c/3$, and the centre is on OG.

Fig. 3 shows lines of force for different values of the parameter. The points F_1, F_2, etc., are the points of maximum radius of curvature for the several curves.

The direction of the magnetic force at any point may easily be obtained in the following manner. If ϕ be the angle between the radius vector and the tangent to the curve at P, we have

$$\tan \phi = r \, d\theta / dr = \tfrac{1}{2} \tan \theta,$$

by (13). Hence the following construction. Draw from the point of trisection of OP nearest O a perpendicular to OP; then if M be the point in which this perpendicular cuts the axis of the magnet, PM is the direction of the line of force at P.

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* This elegant method of describing these curves is due to Mr. John Buchanan, B.Sc. *Nature*, vol. xxi. p. 371.

This construction gives also the magnitude of the force at P, for by (9) we get $X^2+Y^2=m^2(4\xi^2+\eta^2)/(\xi^2+\eta^2)^4$, and this is easily proved to be $m^2 . PM^2/(OM . OP^3)^2$. Hence the magnitude of the force is

$$m . PM/(OM . OP^3).$$

Of course a family of lines of force exists on each side of the line OX, and a similar double family in every plane which contains OX. If we suppose the diagram of Fig. 3 to make a complete turn about OX, each line of force will sweep out a surface at every point of which there will be no component magnetic field intensity normal to the surface. Thus no work would be done against magnetic forces in carrying a unit of magnetism along an element of the normal to the surface at any point, and if such an element were continued into a curve cutting the successive surfaces at right angles, no work would be done in carrying a unit of magnetism along the curve. Such a curve would be what is called an *equipotential line.* Such lines cut the family of surfaces at right angles: we shall see presently that there exists also a family of equipotential surfaces, which are intersected at right angles by the lines of force. Clearly there exists a reciprocal relation which suggests interesting mathematical consequences; but we do not pursue the matter further at present.

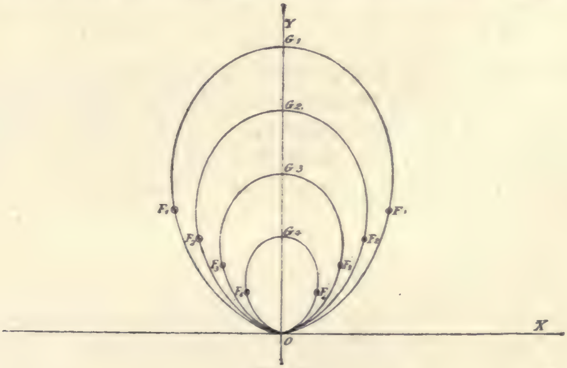

FIG. 3.

The equipotential curves in the plane of the paper are obtained by putting $V=$const. in (7) or (8). It is easy to verify by (9) and (12) that these curves cut the lines of force, as they ought, at right angles. They may be constructed graphically in the following manner. Draw with $\frac{1}{2}\sqrt{m/V}$ as radius, from a centre on the axis of x, a circle (Fig. 4) passing through the position, O, of the centre of the magnet. Then draw any line from O to meet this circle in A. The length of this line is $\sqrt{m/V}.\cos\theta$ if θ be the angle which OA makes with Ox. Lay off a distance OB along the axis of x equal to OA, and on the other segment of the diameter describe a semicircle and draw to it a tangent from O. The length of this tangent is the length of the radius vector r, which laid off from O along OA will give P a point on the curve. Or the construction may sometimes be more conveniently performed as follows: Lay off the length $OB=OA$ as in Fig. 5, and describe a circle on the line made up of OB and the diameter of the former circle. The length of the tangent OC gives the distance OP. The curves, like the lines of force, are symmetrical about the axes of x and y and all pass through the origin. Both sets of curves are shown in Fig. 6.

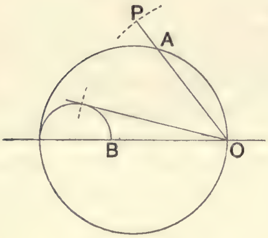

FIG. 4.

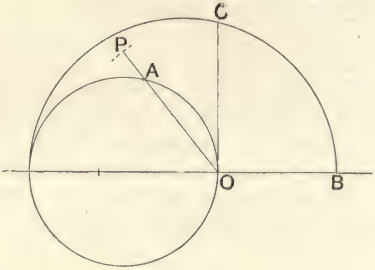

FIG. 5.

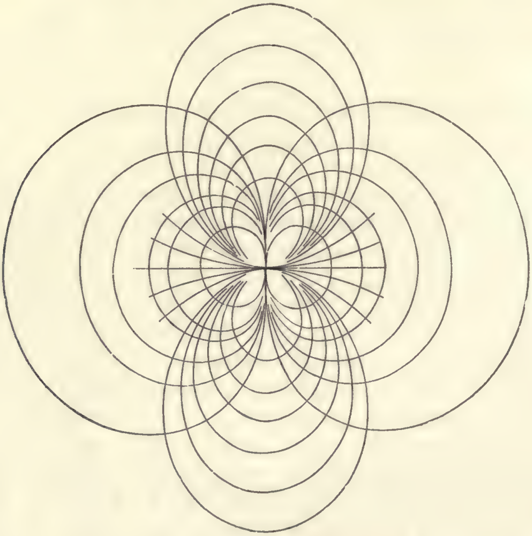

FIG. 6.

7. Analogies with fluid motion. The lines of force and equipotential surfaces due to a small magnet coincide for all external points with those of a uniformly magnetized sphere, a case approximately realized when a ball of iron is placed in a uniform magnetic field, and also with those of a conducting or dielectric sphere placed without charge in a uniform field of electric force. They further correspond exactly to the lines of flow and equipotential surfaces within a large mass of a frictionless incompressible fluid, kept flowing continuously in steady motion through an infinitely short, straight, narrow tube. We have discussed them with some fulness, on account of their theoretical importance. We shall consider them again later in connection with inductive magnetism, and with the investigations of Hertz on the radiation of electric and magnetic energy, if we have space for this latter subject.

8. Potential of a magnetic filament. Let us now consider a magnetic filament regarded as made up of an infinite number of infinitely short magnets placed end to end. Let x, y, z be the coordinates of the centre of one of these elementary magnets, ds its length, dm its moment, λ, μ, ν the cosines of the angles which the axis of the element measured in the direction along the filament from the negative extremity to the positive, makes with the axes: the potential dV at a point (ξ, η, ζ) external to the filament, produced by the element, is by (8) given by

$$dV = \frac{dm}{r^3}\{\lambda(\xi - x) + \mu(\eta - y) + \nu(\zeta - z)\}, \quad \text{............(14)}$$

since $\{\lambda(\xi - x) + \mu(\eta - y) + \nu(\zeta - z)\}/r$ is now the value of $\cos\theta$. But if I denote the magnetic moment of the element per unit of length, taken positive when the direction of the axis, as specified above, is from the negative end of the element to the positive, or, which is the same, if I denote the quantity of magnetic matter on the positive end of the axis, we have

$$dV = \frac{I}{r^3}\{(\xi - x)dx + (\eta - y)dy + (\zeta - z)dz\}, \quad \text{...........(15)}$$

where dx, dy, dz, are the projections of ds on the axes. But since $r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2$ we can write this equation in the form

$$dV = I\frac{d}{dr}\frac{1}{r}dr.$$

Hence integrating by parts we get

$$V = \frac{I_2}{r_2} - \frac{I_1}{r_1} - \int \frac{1}{r}\frac{dI}{ds}ds, \quad \text{......................(16)}$$

where I_2, I_1, r_2, r_1 are the values of I and r at the positive and negative ends respectively. If I be uniform along the filament the last term vanishes, and (16) becomes

$$V = I\left(\frac{1}{r_2} - \frac{1}{r_1}\right), \quad \text{........................(17)}$$

or the potential is that due to the two end distributions alone, as stated above, p. 42.

Since the potential of a quantity of magnetism $dI/ds \, . \, ds$ at the distance r is $dI/ds \, . \, ds/r$ the interpretation of the third term in (16) is that $-dI/ds$, if not zero, is the linear density of magnetism diffused throughout the filament. Hence in the general case the total potential is that due to the end distributions together with that produced by the diffused distribution.

Let any path be drawn in a magnetic field starting at a point A and terminating at another point B. If H be the magnetic field-intensity at any point P of the path, the component of field-intensity along any element ds of the path at P is $H \cos \theta$, if θ be the angle which ds makes with the element ds. But if V be the magnetic potential, we have

$$H \cos \theta = -\frac{dV}{ds}.$$

Also if we take the line integral of $H \cos \theta$ from A to B we get in the field of magnetic matter

$$\int_A^B H \cos \theta \, . \, ds = -(V_B - V_A).$$

This integral is very important. As we shall see in the case of a magnetic field produced by a current of electricity, the integral

$$\int H \cos \theta \, . \, ds$$

taken round a closed path is not zero if the path be carried round the circuit, but has the value $4\pi\gamma$ if γ be the current which circulates through the path. In fact we shall see that the magnetic potential is in this case multiple valued in the sense that the value of this line integral taken round a closed path is zero or $4\pi n\gamma$ according as it threads not at all round the current, or does so n times. The line integral has been called the *circuitation* of the path, and the unit of difference of potential in the c.g.s. system one *gauss*. The name *gauss* is however now used for the c.g.s. unit of magnetic field intensity, and the use of "*gaussage*" in the sense here indicated is generally given up.

9. Lines of force of a uniformly magnetized bar. The equation of the lines of force due to a uniformly magnetized filament is of interest, and may be easily found in a variety of ways. The most elegant is perhaps the following. It is evident that the system of lines is symmetrical about the straight line joining the ends A, B, of the filament. Describe circles from A, B (Fig. 7), as centres with any radii the sum of which is greater than the distance AB. They will intersect in two points which will be points on two lines of force having the same parameter, but on opposite sides of the axis. The circles may be regarded as the intersection with the plane of the paper of two spheres having A, B, as centres, and intersecting in a circle through which pass all the lines of force which can be drawn in space for the magnet

AB, and which have a certain parameter. PQ are two points on such a circle, and of all such circles AB is the common axis. Now considering the total flux of magnetic force (that is, the surface integral of normal magnetic force) in the same direction through any surface having this circle as boundary, and the two centres on the same side of it, it is clear that it may be taken as that due to the quantity of magnetism $-I$ at A, outwards through the segment PRQ of the sphere described from A, and bounded by the circle of intersection, together with that due to $+I$ at B taken outwards through the corresponding segment PSQ of the other sphere. If the angles PAQ, PBQ be respectively $2\theta_1$, $2\theta_2$, these fluxes are respectively

$$-2\pi I(1-\cos\theta_1) \text{ and } 2\pi I(1-\cos\theta_2).$$

Hence the total flux is

$$2\pi I(\cos\theta_1-\cos\theta_2).$$

Now let two other spheres be described in the same way; then if the flux through a corresponding surface bounded by the circle of intersection is the same as that just found, the two circles of intersection may be supposed joined by a surface generated by the revolution of a line of force round AB as an axis. Hence the equation of a line of force is

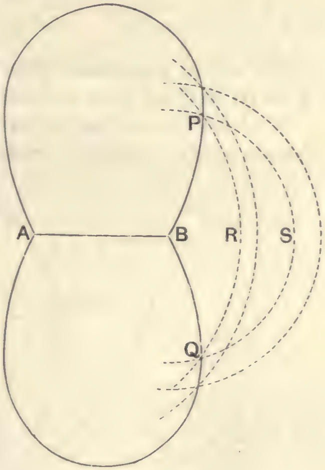

FIG. 7.

$$\cos\theta_1-\cos\theta_2=c, \quad\text{..............................(18)}$$

where c is a parameter varying from one curve to another.*

10. Construction for lines of force of a uniformly magnetized bar. To construct the lines of force in this case we may proceed as follows: Describe a circle on AB (Fig. 8 †) as diameter, and lay off a distance AM such that $AM=c\,.\,AB$. Then draw any line from A to cut the circle in Q, and lay off Aq along AB equal to AQ. From B as centre with radius Mq describe a circle cutting the former circle in R. Hence, since $\cos BAP+\cos ABR=AQ/AB+BR/AB=(Aq+qM)/AB=c$, the point in which AQ and BR intersect is a point on the curve. The curve in the vicinity of A or B must be drawn from a knowledge of its inclination θ to the axis of x. This is given by the equation $\cos\theta=c-1$.

The cut shows curves numbered 1, 2, 3, 4, 5, drawn for the corresponding values of c, $\frac{3}{2}$, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$. When $c=2$, AB (the axis) is the curve, when $c=0$, the productions of the axis to the right of B and the left of A are the curves.

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* See also Chapter III. below.

† This figure is taken by permission from *Constructive Geometry of Plane Curves*, by T. H. Eagles, M.A. (London, Macmillan & Co.). The method of construction here adopted is that given in the same work.

The dotted curves intersecting at right angles the lines of force in Fig. 8 are the equipotential curves, which are given by the equation

$$\frac{1}{r}-\frac{1}{r'}=c, \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (19)$$

where c is a parameter (the potential per unit of magnetic matter at A or B) varying from one curve to another.

That the dotted curves are intersected at right angles by the lines of force is easily verified by considering that if $f(r, r')=0$ be the equation of a curve, and lengths df/dr, df/dr' be laid off along r and r', the resultant of these lines is in the direction of the normal. We have from (19) $df/dr=-1/r^2$, $df/dr'=1/r'^2$. Hence laying off $1/r^2$ from a point on the curve along r towards A, and $1/r'^2$ from the same point along r' in the direction from B, we find that the normal to the equipotential curve (19) is in the direction of the resultant force due to the equal quantities of opposite kinds of magnetic matter at A and B respectively.

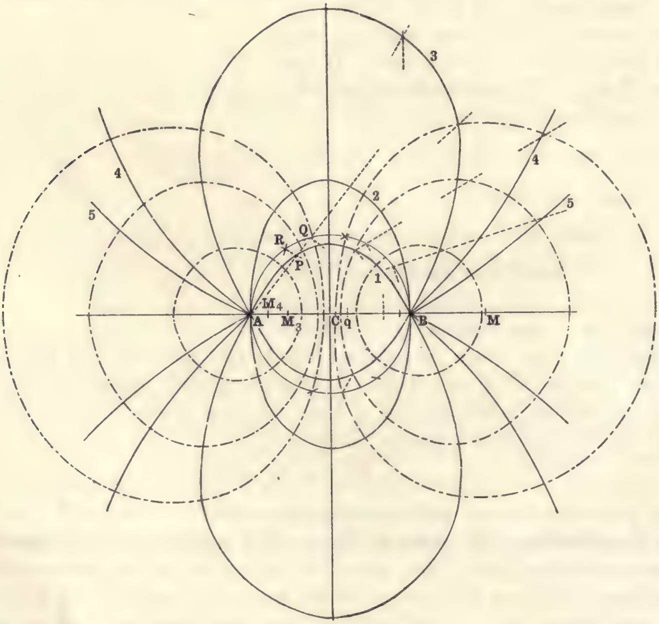

FIG. 8.

11. Resultant field obtained by superimposing the field of a bar-magnet on a uniform field. The resultant field of a magnet placed in an already existing field is important. We consider here only the case in which the axis of the magnet and the field are parallel. Let H be the value of the uniform field-intensity, and M that of the moment of the magnet, supposed uniformly magnetized. Take any plane containing the magnetic axis, and let ξ, η be the coordinates of the point in that plane at which the coordinates X, Y of resultant field-intensity are taken, l the half length of the magnet. We have then

$$\left.\begin{aligned} X &= \frac{M}{2l}\left[\frac{\xi - l}{\{(\xi - l)^2 + \eta^2\}^{\frac{3}{2}}} - \frac{\xi + l}{\{(\xi + l)^2 + \eta^2\}^{\frac{3}{2}}}\right] + H, \\ Y &= \frac{M\eta}{2l}\left[\frac{1}{\{(\xi - l)^2 + \eta^2\}^{\frac{3}{2}}} - \frac{1}{\{(\xi + l)^2 + \eta^2\}^{\frac{3}{2}}}\right]. \end{aligned}\right\} \quad \text{.........(20)}$$

We consider the points, called *neutral points*, at which $X = 0$, and also, (1) $\xi = 0$, (2) $\eta = 0$. In case (1) we get

$$M = H(l^2 + \eta^2)^{\frac{3}{2}},$$

and in case (2)

$$M = -H\frac{(\xi^2 - l^2)^2}{2\xi}, \quad \text{........................(21)}$$

where ξ, taken positive, is the distance of the point at which $X = 0$ from the centre of the magnet.

Thus in case (1) there is a circle of neutral points round the axis of the magnet as geometric axis, and lying in what may be called the equatorial plane through the magnet centre. In the other case, if M and H have the same sign there is no neutral point in the line of the axis; if M and H have opposite signs there are two such neutral points, at distances $\pm\sqrt[3]{2M/(-H)}$ from the centre, if l be very small.

A useful laboratory exercise consists in laying down the lines of force on a horizontal sheet of paper, on which lies a bar-magnet in the magnetic meridian. These lines are obtained by marking the direction of a small compass needle at any point P, then moving the centre of the needle a small distance in that direction from its first position, which is also marked. The direction of the needle and the position of its centre are again marked, the needle is then moved on through an element of distance, and so on. Thus a line of force is mapped out. When this has been done over the sheet of paper, the family of lines has been obtained, and the lines disclose the neutral points.

Reversal of the magnet and repetition of the process of mapping gives the pair of neutral points described above.

12. Potential of a magnetized body in a magnetic field. We shall now find the potential at any point in a magnetic field produced by a body magnetized in any given manner. As we shall see later, we are led by magnetic phenomena to suppose a magnetized body made up of an infinitely large number of infinitely small magnetized molecules, each of which may be considered a magnetic doublet, as defined above. We shall also suppose that the magnetic axes of these molecules have in each small element of the body a common direction, of which the cosines for a given element are λ, μ, ν, and which varies from point to point continuously in the body. This is called the direction of magnetization at the element.

We consider an element of the body, in shape a rectangular parallelepiped with its edges parallel to the axes, large enough to contain a very great number of molecules, but not so large that the direction of magnetization varies in it to a sensible extent. Let n be the number of molecules in the element, m their average magnetic moment, then the magnetic moment of the element is nm, and it may be regarded as a small magnet of this moment, with its centre at the point x, y, z, and its axis in the direction λ, μ, ν. For nm we shall write $I\,dx\,dy\,dz$, where I is the magnetic moment of the element per unit of volume, or, as it is usually called, the intensity of magnetization at the element, and $dx\,dy\,dz$ is the volume of the element. I is thus the scalar value of a directed quantity $\mathbf{I}$.

By (14) above we have for the magnetic potential produced by the element at the point (ξ, η, ζ) the expression

$$I\,dx\,dy\,dz\{(\xi-x)\lambda+(\eta-y)\mu+(\zeta-z)\nu\}/r^3,$$

where $r^2=(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2$. Writing $I\lambda$, $I\mu$, $I\nu=A$, B, C, so that A, B, C are what are called the components of magnetization at the point x, y, z, and integrating throughout the body we get for the total potential V at (ξ, η, ζ) the equation

$$V=\iiint \frac{1}{r^3}\{A(\xi-x)+B(\eta-y)+C(\zeta-z)\}\,dx\,dy\,dz. \quad \ldots\ldots(22)$$

This may be written in the form :

$$V=\iint \frac{A}{r}\,dy\,dz+\iint \frac{B}{r}\,dz\,dx+\iint \frac{C}{r}\,dx\,dy$$
$$-\iiint \frac{1}{r}\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right)dx\,dy\,dz, \quad \ldots\ldots(22')$$

in which the first three integrals are confined to the surface and are reckoned in the following manner. Taking the first of the three, conceive a prism of cross-section $dy\,dz$, and length parallel to the axis of x, drawn in the body. The area $dy\,dz$ is the projection at right angles to the axis of x of the element dS of the surface intercepted at either end by the prism ; and the element of the integral corresponding to the negative or left-hand end of the prism is to be taken negative, the element for the other end positive. Now if l_1, l_2 be the x direction cosines of the normals drawn outwards from the surface elements at these ends respectively, dS_1, dS_2 the corresponding areas, we have $dy\,dz=l_2dS_2=-l_1dS_1$; so that the elements of the integral are $A_2l_2dS_2/r_2+A_1l_1dS_1/r_1$. Hence we may write the first integral in the form $\int Al/r\,.\,dS$, in which the integration is to be extended over the whole surface. The other surface integrals may be similarly transformed, and we get

$$V=\int \frac{1}{r}(Al+Bm+Cn)\,dS-\iiint \frac{1}{r}\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}\right)dx\,dy\,dz, \quad \ldots(23)$$

in which l, m, n are the direction cosines of the normal to an element dS of the surface of the body.

Clearly we may interpret the quantity $Al+Bm+Cn$ as a surface density σ of magnetic distribution, equal at each surface element to the normal component of intensity of magnetization.

The expression $-(\partial A/\partial x+\partial B/\partial y+\partial C/\partial z)$ is interpretable in the same way as the volume density ρ of a distribution of magnetism throughout the substance of the body.

From these expressions we get by direct integration, as we clearly ought, the total magnetism of the body equal to zero.

It is almost needless to say that these results are consequences of our suppositions as to the structure of the magnetized body, and that the interpretations just stated are to be regarded merely as convenient modes of expressing the outcome of the analysis. If, however, as seems certain, the magnetized body be made up of polarized molecules of some kind, the surface and body distributions found, will correspond to unbalanced surface and body polarities respectively.

If the potential at (ξ, η, ζ) due to the body is expressed by the surface integral alone, then

$$\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}=0. \quad \ldots\ldots\ldots\ldots(24)$$

A distribution of magnetism fulfilling this condition is said to be *solenoidal*.

13. The mutual potential energy of a magnet and a magnetic field. We have now to consider the potential energy of a magnet situated in a magnetic field. By this we mean the work which has been done against magnetic forces, in bringing the magnet into the given field and placing it in the given position. The potential energy of unit quantity of negative magnetism at a point P, at which the potential is V, is of course simply $-V$; and hence that of a unit of positive magnetism at a point at distance ds from this point is $V+dV/ds\,.\,ds$. The potential energy of a magnetic doublet with its extremities at these points is therefore $m\,dV/ds$, where m is the moment of the doublet. If the direction cosines of the axis of the doublet be λ, μ, ν, we have of course

$$m\frac{dV}{ds}=m\left(\lambda\frac{\partial V}{\partial x}+\mu\frac{\partial V}{\partial y}+\nu\frac{\partial V}{\partial z}\right). \quad \ldots\ldots\ldots\ldots(25)$$

Now, as above (p. 52), we may regard this small magnet as a magnetic molecule of a body of finite size, and take a parallelepiped of the body, large enough to contain a great number of such molecules, but not so large that the direction of magnetization, that is the common direction of the axes of the molecules, varies to a sensible extent. The potential energy of the element will be proportional to the number of such molecules contained in the element. Hence by the expression above, the potential energy dE of an element of volume $dx\,dy\,dz$, is given by

$$dE = I\left(\lambda\frac{\partial V}{\partial x} + \mu\frac{\partial V}{\partial y} + \nu\frac{\partial V}{\partial z}\right)dx\,dy\,dz,$$

where I denotes the intensity of magnetization as defined above. Writing as before A, B, $C = \lambda I$, μI, νI, and integrating throughout the magnet we get

$$E = \iiint\left(A\frac{\partial V}{\partial x} + B\frac{\partial V}{\partial y} + C\frac{\partial V}{\partial z}\right)dx\,dy\,dz. \quad \text{............}(25')$$

Integrated by parts this becomes

$$E = \iint V(A\,dy\,dz + B\,dz\,dx + C\,dx\,dy) - \iiint V\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right)dx\,dy\,dz. \quad \text{.........}(26)$$

The triple integration is taken throughout the space occupied by the magnet; the double integrations give, when l, m, n, are put for the direction cosines of the normal to an element dS of the surface, an integration over the whole surface of the magnet, so that

$$E = \int V(lA + mB + nC)\,dS - \iiint V\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right)dx\,dy\,dz. \quad \text{...}(27)$$

By the interpretations, stated on p. 53 above, of the quantities in brackets, this may be written

$$E = \int V\sigma\,dS + \iiint V\rho\,dx\,dy\,dz, \quad \text{..................}(28)$$

which is the energy equation. The field in the present case is independent of the distribution brought into it: if the distribution and the field grew up together, so that the distribution came into its own field, the energy would have half the value here given.

It it to be noted that V is the potential due to the magnetic system producing the field, and that therefore $-\partial V/\partial x$, etc., are the components α, β, γ, parallel to the axes, of the magnetic field intensity due to this system. Hence we may write

$$E = -\iiint(A\alpha + B\beta + C\gamma)\,dx\,dy\,dz. \quad \text{..................}(29)$$

In the case of a uniform field intensity for every part of the magnet this becomes

$$E = -(\alpha M_1 + \beta M_2 + \gamma M_3), \quad \text{......................}(30)$$

where M_1, M_2, M_3 denote the integrals $\int A\,dx\,dy\,dz$, etc. Now we can find three quantities p, q, r fulfilling the equation $p^2+q^2+r^2=1$, and such that

$$M_1 = pM, \quad M_2 = qM, \quad M_3 = rM,$$

so that we have

$$E = -M(p\alpha + q\beta + r\gamma). \quad \text{........................}(31)$$

M is what has been defined above (p. 40) as the value of the magnetic moment of the magnet, and p, q, r are the direction cosines of its axis. If H be the value of the resultant magnetic intensity of the field, its direction cosines are α/H, β/H, γ/H, and (31) may be written

$$E = -MH\left(p\frac{\alpha}{H} + q\frac{\beta}{H} + r\frac{\gamma}{H}\right) = -MH\cos\theta, \quad \ldots\ldots\ldots\ldots(32)$$

the equation (1) already obtained in p. 40.

14. Potential energy of a magnet in the field of a single magnetic pole. It is instructive and useful to consider as a particular case the potential energy of a magnet in the field due to a single magnetic pole, as this gives the potential of the magnet at the point at which the pole is situated, and supplies conditions by which the centre and axes of the magnet may be determined. In this case if PP' be the distance of the point $P(\xi, \eta, \zeta)$, at which the pole is situated, from the point $P'(x, y, z)$ of the magnet, we have for the potential V at P', due to the unit pole at P, the value $1/PP'$, that is

$$V = \frac{1}{PP'} = (r^2 - 2\mu rr' + r'^2)^{-\frac{1}{2}} = \frac{1}{r}\left(Z_0 + Z_1\frac{r'}{r} + Z_2\frac{r'^2}{r^2} + \text{etc.}\right), \quad \ldots(33)$$

where r, $r'(r' < r)$ are the distances OP, OP' from the origin of coordinates to the points P, P', and Z_0, Z_1, Z_2, etc., are zonal surface harmonics * of the orders specified by their suffixes, and having their pole at P. Here μ is the cosine of the angle POP', and

$$Z_0 = 1, \quad Z_1 = \mu, \quad Z_2 = \tfrac{1}{2}(3\mu^2 - 1), \quad Z_3 = \tfrac{1}{2}(5\mu^2 - 3\mu), \text{ etc.}$$

Wherever $r' > r$ we must, of course, use (33) as altered by writing r' for r and r for r'.

Substituting these values of Z_0, Z_1, etc., in (33), then putting for μ its value $(\xi x + \eta y + \zeta z)/rr'$, and differentiating, we evaluate $\partial V/\partial x$, $\partial V/\partial y$, $\partial V/\partial z$. Using these in (25), and putting

$$P_1 = \iiint Ax\,dx\,dy\,dz, \quad P_2 = \iiint By\,dx\,dy\,dz, \quad P_3 = \iiint Cz\,dx\,dy\,dz,$$

$$Q_1 = \iiint (Bz + Cy)\,dx\,dy\,dz, \quad Q_2 = \iiint (Cx + Az)\,dx\,dy\,dz,$$

$$Q_3 = \iiint (Ay + Bx)\,dx\,dy\,dz,$$

we get (with r used in two senses, that is, with p, q as a direction cosine, and in $1/r^3$, $1/r^5$, etc., as a distance)

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* For the theory of Spherical Harmonics which we shall frequently have to employ in what follows, the student may consult Thomson and Tait's *Nat. Phil.* vol. i. part i., or Ferrers's *Spherical Harmonics*. A clear and brief account of the subject is given in Minchin's *Statics*, vol. ii. 3rd edition. A short explanation, covering the theorems used in this work, is given in an Appendix to the present volume.

$$E = M(p\xi + q\eta + r\zeta)\frac{1}{r^3} + \{\xi^2(2P_1 - P_2 - P_3) + \eta^2(2P_2 - P_3 - P_1)$$
$$+ \zeta^2(2P_3 - P_1 - P_2) + 3(Q_1\eta\zeta + Q_2\zeta\xi + Q_3\xi\eta)\}\frac{1}{r^5} + \text{etc.} \quad \ldots\ldots(34)$$

The quantities P_1, P_2, etc., are functions of the coordinates x, y, z, and of A, B, C; hence we may change the origin to another point (x', y', z'), and take the direction of the axis of the magnet as that of x, so that $p = 1$, $q = 0$, $r = 0$. This makes

$$\iiint A\,dx\,dy\,dz = M, \quad \iiint B\,dx\,dy\,dz = 0, \quad \iiint C\,dx\,dy\,dz = 0.$$

We have then new values P', etc., given by the equations

$$P_1' = P_1 - Mx', \quad P_2' = P_2, \quad P_3' = P_3,$$
$$Q_1' = Q_1, \quad Q_2' = Q_2 - Mz', \quad Q_3' = Q_3 - My'.$$

Hence, if the new origin be taken so that

$$x' = \frac{2P_1 - P_2 - P_3}{2M}, \quad y' = \frac{Q_3}{M}, \quad z' = \frac{Q_2}{M},$$

we get $$P_1' = \tfrac{1}{2}(P_2 + P_3), \quad Q_2' = 0, \quad Q_3' = 0,$$

and (34) takes the simplified form (accents omitted)

$$E = M\frac{\xi}{r^3} + \frac{3}{2}\frac{(P_2 - P_3)(\eta^2 - \zeta^2) + 2Q_1\eta\zeta}{r^5} + \text{etc.}, \quad \ldots\ldots\ldots(35)$$

in which ξ, η, ζ have of course the proper values for the new origin.

The origin thus found is called the centre of the magnet, and the definition enables us to specify the position of the magnetic axis, as well as its direction. The magnetic axis is sometimes called the principal axis of the magnet.

If we turn the axes of y and z round that of x, through the angle $\frac{1}{2}\tan^{-1}\{Q_1/(P_2 - P_3)\}$, (35) takes the form

$$E = M\frac{\xi}{r^3} + \frac{3}{2}\frac{R(\eta^2 - \zeta^2)}{r^5} + \text{etc.}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(36)$$

where R is the quantity which replaces $P_2 - P_3$. These directions of the axes of y and z are called the secondary axes of the magnet.

In the case of symmetry round the axis of x, the second term of the expression on the right of (36) is zero, since then whatever magnetization at right angles to the axis there be throughout the body, it must be such that the coefficient R vanishes identically. To a close approximation therefore for a unit pole placed at a point (ξ, η, ζ), the distance r of which from the origin is considerably greater than that of any part of the magnet from the same point, the mutual potential energy is $M\xi/r^3$.

Since the potential energy is mutual, the equations (34), etc., found for E, give the potential energy of the unit pole in the field due to the magnet, that is the potential due to the magnet at the point (ξ, η, ζ).

Section II.

MAGNETIC INDUCTION. VECTOR POTENTIAL. MAGNETIC ENERGY.

15. Magnetic induction and magnetic force. When a substance capable of being magnetized is placed in a magnetic field, it becomes magnetic, and a definite relation in general exists between the magnetization produced at each part and the field-intensity. The determination of this relation has been, especially in late years, the subject of much careful investigation, and if space allows we shall give an account later of methods of measurement employed. With this in view, we deal here with the theory of certain given cases of magnetization. In general, in the substances with which we have to deal in practice, the magnetization is in the direction of the magnetic force, and we shall first consider this case.

We have already (p. 22) referred to the force in the interior of a magnet. Here, as in all other cases, the magnetic force is that which would be exerted on a unit magnetic pole if placed at the point, and, since we could make no experiment as to the internal state of the body, except within a cavity hollowed out within it, we imagine a small portion of the magnetized body excavated so as to give a space in which the force might be measured. The formation of this cavity leaves unbalanced the magnetism on the extremities of the molecules which abut against its surface. We shall suppose it formed without disturbing the magnetization of the rest of the body, and since we cannot divide a magnetic molecule the signs of the surface distributions will be perfectly definite. Thus for a crevasse* cut at right angles to the direction of magnetization there is positive magnetism on the face next the negative end of the magnet, and negative magnetism on the opposite face. We shall suppose the crevasse filled with the standard medium, of inductivity μ_0. On a surface the normal to which, drawn into the cavity, is inclined at an angle ϵ to the direction of the intensity of magnetization **I** [scalar magnitude I], taken as positive when drawn in the magnet from the negative pole to the positive, the density of distribution is $I \cos \epsilon$, and is positive therefore when ϵ is acute, and negative when ϵ is obtuse.

The force within the cavity depends upon the shape and dimensions of the cavity, and upon the position of the pole within it. In the first place we shall consider a cylindrical cavity of finite length and diameter, cut with its axis in any given direction, in a uniformly magnetized body (Fig. 9). If the intensity of magnetization of the body be I, and θ be the angle which the axis of the cylinder makes with the direction AB of

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* A narrow cavity with parallel plane faces, every dimension of which is great in comparison with the width of the cavity.

I, we have for the density of the distribution on the curved surface of the cavity the value $I \sin\theta$ at points in a plane through the axis parallel to the direction of magnetization. In another plane through this axis making an angle ϕ with the former plane the density is $I\sin\theta\cos\phi$. Now the force which this distribution exerts at right angles to the axis on a unit pole placed at the centre of the axis, is, if $2l$ be the length of the cylinder, and $2r$ its diameter,

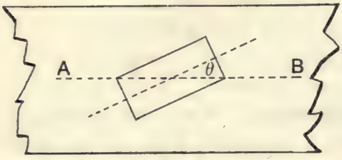

FIG. 9.

$$\frac{1}{\mu_0} 2r^2 I\sin\theta \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2\phi\, d\phi \int_{-l}^{+l} \frac{dx}{(r^2+x^2)^{\frac{3}{2}}} = \frac{2\pi}{\mu_0} I\sin\theta \frac{l}{(l^2+r^2)^{\frac{1}{2}}}.$$

The ends of the cylinder give a resultant force along the axis of amount $4\pi I\cos\theta\{1-l/(r^2+l^2)^{\frac{1}{2}}\}/\mu_0$; and the total force within the cylinder at the centre of the axis is the resultant of these two components. Hence if l be great in comparison with r, the force is at right angles to the axis and of amount $2\pi I\sin\theta/\mu_0$. Hence if $\theta=\pi/2$, the force is $2\pi I\mu_0$. If l be small in comparison with r, the force is $4\pi I\cos\theta/\mu_0$.

If $\theta=0$, so that the axis of the cylinder is parallel to I, the force becomes $4\pi I\{1-l/(r^2+l^2)^{\frac{1}{2}}\}/\mu_0$, and is therefore $4\pi I/\mu_0$ or zero, according as l is small or great in comparison with r. Also the force is $4\pi I/\mu_0$ in any narrow crevasse bounded by planes at right angles to I, and is plainly zero in any elongated narrow cavity with its length parallel to I.

In the important case of a spherical hollow the surface distribution follows the law of variation from point to point of a material distribution formed by placing two spheres of equal uniform densities $+\rho$ and $-\rho$ in coincidence, and displacing the positive sphere in the direction of I through a small distance δx. We may suppose ρ very great, and δx very small, so that $\rho\,\delta x=I$, and take in this case $\mu_0=1$. The potential due to the inner nucleus of the positive sphere at a point distant r from the centre is $\frac{4}{3}\pi I r^2/\delta x$. The potential due to the shell beyond r is $2\pi I(R^2-r^2)/\delta x$. Hence the whole potential is $2\pi I(R^2-\frac{1}{3}r^2)/\delta x$. The potential at the same point due to the negative sphere is plainly

$$-2\pi I(R^2-\tfrac{1}{3}r^2)/\delta x+\tfrac{4}{3}\pi I r\, dr/dx.$$

Hence the total potential is $\frac{4}{3}\pi Ix$. The force within the spherical hollow produced by the surface magnetization is therefore in the direction of magnetization, and equal to $\frac{4}{3}\pi I$.

In the case of a non-uniformly but continuously magnetized body these cavities have only to be taken small enough to enable the average value of I over each to be used in the values of the force.

The cases most important for our present purpose are (1) the comparatively long narrow cylinder, (2) the short comparatively wide cylinder, both with axes parallel to I. In each case the force within the hollow, due to the surface distribution upon it, must be increased by the resultant force at the point due to the distribution producing the magnetic field and to the rest of the magnetic distribution of the magnet. If we call this force H, the force within the cavity is in case (1) simply H, in case (2) it is $H + 4\pi I/\mu_0$, if μ_0 denote the magnetic inductivity of air. H is thus the magnetic force within the magnet, apart from any action of unbalanced polarity produced by cutting a hollow in the substance. The quantity $\mu_0(H + 4\pi I/\mu_0)$ is called the *magnetic induction* within the magnet. We shall denote it by B.

In the case in which the magnetization is induced by the magnetizing force H, and has the same direction, if we put $I/\mu_0 = \kappa H$, we get

$$B = \mu_0(1 + 4\pi\kappa)H. \qquad \text{.............................(37)}$$

16. Magnetic susceptibility. Magnetic permeability. We denote the multiplier $\mu_0(1 + 4\pi\kappa)$ by μ, and call it the *magnetic inductive capacity*. The factor κ is called the *magnetic susceptibility*. In general, as we shall see below, it is a function of H.

It is clear that as here defined κ is a mere number. The quantity μ is also a mere number when defined by the equation $1 + 4\pi\kappa$. Now μ has a definite value for every medium, and it is possible that that property of the medium (say some form of motion), which makes the magnetic inductive capacity vary from medium to medium, may give to it certain dimensions at present unknown. We may use μ as the absolute magnetic inductive capacity depending on this property, that is the magnetic inductive capacity with reference to an absolutely unmagnetizable medium as standard, and regard its dimensions as unknown; but we shall in the account of magnetic measurements which follows, in general employ it to denote $1 + 4\pi\kappa$. According to the relations used above, $1 + 4\pi\kappa$, is the value of μ/μ_0. We call this the *permeability* of the magnetized substance, and denote it by ϖ. Thus we have

$$\varpi = 1 + 4\pi\kappa. \qquad \text{.................................(38)}$$

Since **B**, **H**, and **I** are vectors we may replace each by three components along the axes. We have then instead of (37) the equation

$$\left.\begin{aligned} a &= \alpha + 4\pi A, \\ b &= \beta + 4\pi B, \\ c &= \gamma + 4\pi C, \end{aligned}\right\} \qquad \text{..........................(39)}$$

where $a, b, c, \alpha, \beta, \gamma, A, B, C$, are the components of **B**, **H**, and **I** for the point considered, and μ_0 is taken as unity.

It is easy to prove that the magnetic induction fulfils the solenoidal condition. We have from (39), with $\mu_0 = 1$,

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = \frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} + 4\pi\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right). \qquad \text{.......(40)}$$

Now remembering that the quantity within the brackets may be regarded as a volume density $(-\rho)$ of magnetism, and that by the definition of **H** we must have by the characteristic equation[*] of electric and magnetic potential

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}+4\pi(-\rho)=0,$$

and therefore

$$\frac{da}{dx}+\frac{db}{dy}+\frac{dc}{dz}=0. \quad \text{..........................(41)}$$

In the space surrounding the magnetized body, **B** coincides with **H** in all respects. The transition in value from one side of the surface to the other, takes place differently in the two cases. The normal component of **B** varies continuously from one side of the surface to the other, the tangential component discontinuously; and the reverse is the case with the value of **H**. To prove this we have only to notice that by (39) if θ be the angle between the normal to the surface drawn outwards, and the common direction of **B**, **H**, and **I**, we have for the normal component of **B** in the interior

$$B\cos\theta=(H+4\pi I)\cos\theta, \quad \text{.......................(42)}$$

and that if H' be the magnetic force just outside the surface at the same place, and θ' its inclination to the normal, the characteristic equation of the potential gives, since $I\cos\theta$ is a surface density of magnetism,

$$\cdot\, H'\cos\theta'=H\cos\theta+4\pi I\cos\theta. \quad \text{..................(43)}$$

Since B and H coincide outside the magnet the quantity on the left is the normal component of the magnetic induction. The expression on the right, therefore, shows the normal continuity of B, and at the same time the normal discontinuity of H.

The tangential component of B is $(H+4\pi I)\sin\theta$ inside the surface, and $H\sin\theta$ outside the surface. The latter is the value of the tangential component of the magnetic force on both sides.

17. Vector potential. Since the magnetic induction fulfils the solenoidal condition, it follows that the surface integral of magnetic induction taken over any closed surface whatever, whether wholly within or wholly without, or partly within and partly without the magnetized body, is zero. This is clear from the following equation,

$$\iint(la+mb+nc)\,dS=-\iiint\left(\frac{da}{dx}+\frac{db}{dy}+\frac{dc}{dz}\right)dx\,dy\,dz, \quad \text{.......(44)}$$

in which the quantity on the right is zero identically.

The truth of equation (44) (apart from the special value of the right-hand side in the present case) may be seen from the following considerations. The expression $(da/dx+db/dy+dc/dz)\,dx\,dy\,dz$ represents the sum, for a small rectangular parallelepiped of the substance having its edges parallel to the axes, of the products of the average value of the component of induction, at each surface of the element into the area of the face. The integral on the right of (44) simply expresses the aggregate value of these sums for such elements making up the portion of the body considered. Now clearly if we imagine the body divided into small elements, then each face of these will be common to two elements, except those faces which abut on the surface. For every common face the products of induction into area for the two elements are equal and opposite, and cancel one another. We are left then with the aggregate of the products for the faces at the surface, and it is clear by projection that the sum of the products of induction and area for these faces is

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[*] *Poisson's Theorem*, see Appendix, *Notes.*

$$\iint (la + mb + nc)\, dS.$$

Hence the theorem.

We may of course imagine a magnetic field divided up into unit tubes of induction, that is, tubular surfaces bounded by lines of induction, and such that the magnetic induction over the cross section of each is everywhere unity. The magnetic induction over any surface is then measured by the number of unit tubes (or, as it is frequently put, by the number of "lines") of induction which pass through it.

It is clear from the result that the magnetic induction over any closed surface is zero, that the surface integral of magnetic induction over an unclosed surface depends only on the bounding curve. For consider the surface closed by a cap fitted to the boundary and not enclosing any part of the magnetic distribution, and let the integration be extended to the whole surface. The total integral is then zero, and therefore the integral taken over the cap is equal and opposite to that over the original surface. This holds if the cap close the surface, whatever be its form and position otherwise; hence the integral taken over the surface depends only on the form and position of the boundary.

It follows that we can express the surface integral of magnetic induction over an unclosed surface by the integral of a certain quantity taken round the bounding curve. This quantity must be directed, since its sign must change with that of the magnetic induction. The sign of the integral will therefore depend on the direction of integration round the curve. Thus let F, G, H be functions of the coordinates of a point (x, y, z) on the curve, dx, dy, dz, the projections on the axes of an element ds of the curve; we have

$$\iint (la + mb + nc)\, dS = \int (F\, dx + G\, dy + H\, dz). \quad \text{............}(45)$$

18. Values of components of magnetic induction in terms of vector potential. F, G, H have been called by Clerk Maxwell the components of the vector potential of magnetic induction. We shall now find the values of a, b, c in terms of these quantities. We assume throughout the discussion that $\mu_0 = 1$.

It is evidently possible to draw on the surface a series of curves cutting at right angles, so as to divide the surface into a series of rectangular areas (so small that each may be taken as plane) with incomplete rectangles round the bounding curve. The area of these incomplete elements is evidently vanishingly small in comparison with the sum of the areas of the complete elements, and therefore the induction over that portion of the area may be neglected. Now we can find the line integral of the vector potential round any element traced on the surface by calculating its average component along each side of the element, multiplying by the length of the side, and adding the results. Thus let *du*, *dv* be two adjacent sides of an elementary rectangle, and *U*, *V* be the mean values of the components of vector potential along *du* and *dv* respectively; then for the integral round the element we have

$$U\,du + V\,dv + \frac{dV}{du}du\,dv - \left(U\,du + \frac{dU}{dv}dv\,du\right) - V\,dv$$

$$= \left(\frac{dV}{du} - \frac{dU}{dv}\right)du\,dv. \quad \text{..................(46)}$$

Now writing *dS* for the area *dudv* of the element, and equating the magnetic induction over the element to the value just found, we get

$$\left(\frac{dV}{du} - \frac{dU}{dv}\right)dS = (la + mb + nc)\,dS, \quad \text{..................(47)}$$

if *l*, *m*, *n* be the direction cosines of the normal to *dS*. Taking the line integral as above, and in the same direction, round all the elements of area into which the surface is divided, and adding the results together, we have plainly only the integral round the bounding curve, since each side which is common to two elements of surface contributes two equal and opposite elements to the sum, and it is easy to see that for each triangle left round the edge the line integral along the two rectangular sides can, in the limit, be replaced by the integral along the third side formed by the boundary, so that a complete series of elementary integrals, having the same direction round the boundary, is obtained. Hence integrating round the curve, and over the surface, we have finally

$$\int A\cos\phi\,ds = \iint (la + mb + nc)\,dS, \quad \text{..................(48)}$$

where *A* is the numerical vector potential **A**, and ϕ the angle between its direction and the element *ds* of the curve. Substituting the components of **A** parallel to the axes, we have

$$\int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds = \iint (la + mb + nc)\,dS. \quad \text{..........(49)}$$

If now our circuit be a small rectangle of sides $d\eta$, $d\zeta$ at right angles to the axis of x, we get at once from (47)

$$a=\frac{dH}{d\eta}-\frac{dG}{d\zeta},$$

and in the same way

$$\left.\begin{aligned} b&=\frac{dF}{d\zeta}-\frac{dH}{d\xi},\\ c&=\frac{dG}{d\xi}-\frac{dF}{d\eta}.\end{aligned}\right\} \qquad \text{.............................(50)}$$

It is clear, as we have seen, that F, G, H are directed quantities, and their signs must be reversed by reversing the signs of a, b, c. In what follows we shall take the positive direction of integration round any circuit as the direction in which a person must be imagined to go round the circuit so as to have the area always on his left, and the positive direction of the magnetic induction as across the element from the person's feet to his head.*

19. Specification of vector potential. The vector potential **A** (scalar value A) may be specified as follows. Consider an element, volume δv, of the magnetized substance, at which the intensity of magnetization is **I**. The magnetic moment of the element is $\mathbf{I}\delta v$. Then (as will be seen below) the vector potential produced by this element at a point distant r from it is numerically $I\,\delta v.\sin\theta/r^2$, where θ is the angle between the positive direction of magnetization, and the radius vector r. The direction of the vector potential is at right angles to the plane passing through the directions of **I** and r; and by the convention stated above appears to an eye looking in the negative direction of **I** to be drawn in the counter-clockwise direction.

To verify this specification let λ, μ, ν be the direction cosines of **I**, x, y, z, the coordinates of the magnetic element, ξ, η, ζ, those of the point considered; then we have

$$I\,\delta v\frac{\sin\theta}{r^2}=\frac{I\,\delta v}{r^3}[\{\mu(\zeta-z)-\nu(\eta-y)\}^2+\text{etc.}]^{\frac{1}{2}}, \quad \text{...........(51)}$$

from which the values of dF, dG, dH can be inferred by inspection.

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*The quantities on the right of (50) are called the components of the *curl* of vector potential. As components of curl occur in other connections we interpolate the following explanation of the origin of such components. They occur always as part of the result of the linear vector operation $i\partial/\partial x+j\partial/\partial y+k\partial/\partial z$ (where i, j, k are unit vectors along the axes) performed on the vector $iX+jY+kZ$. The complete result is the sum of a scalar part $-(\partial X/\partial x+\partial Y/\partial y+\partial Z/\partial z)$ and a vector part

$$i\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)+j\left(\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\right)+k\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right).$$

The latter part is the curl, written curl **F**, if **F** be the vector quantity of which X, Y, Z, are the components. We shall see later that, to a factor 4π, the electric current in three dimensions is the curl of the magnetic force; here we see that, to a factor μ, the magnetic force is the curl of the vector potential. Thus we have here

$$4\pi\mu(\text{electric current})=\text{curl}^2(\text{vector potential}).$$

Writing in (51) u for $1/r$, and for $I\lambda$, $I\mu$, $I\nu$, their values A, B, C, and integrating throughout the whole magnetized body, we get for a finite magnet

$$\left.\begin{aligned} F&=\iiint\left(B\frac{du}{dz}-C\frac{du}{dy}\right)dx\,dy\,dz,\\ G&=\iiint\left(C\frac{du}{dx}-A\frac{du}{dz}\right)dx\,dy\,dz,\\ H&=\iiint\left(A\frac{du}{dy}-B\frac{du}{dx}\right)dx\,dy\,dz.\end{aligned}\right\}\qquad\text{..................(52)}$$

From the equation $a=dH/d\eta-dG/d\zeta$ we get by (52), remembering that $du/d\xi=-du/dx$, etc.,

$$a=-\frac{d}{d\xi}\iiint\left(A\frac{du}{dx}+B\frac{du}{dy}+C\frac{du}{dz}\right)dx\,dy\,dz-\iiint A\nabla^2u\,dx\,dy\,dz.\quad\text{......(53)}$$

The first term of this expression is (for $\mu_0=1$) simply the force α at the point (ξ, η, ζ), since the first integral is the potential at the point (ξ, η, ζ). The second term of the expression is zero unless the point (ξ, η, ζ) fall within the limits of integration. In the latter case it is $-4\pi A'$ if A' be the value of A at the point (ξ, η, ζ), for evidently we may regard u as the potential at (x, y, z), due to a pole of strength A' at (ξ, η, ζ), and we know by Poisson's theorem that then the integral has the value stated. Hence in general we have by (53) $a=\alpha+4\pi A'$, where A' is the component of magnetization, and α the magnetic force, where a is taken. Similarly we could find from (53) $b=\beta+4\pi B'$, $c=\gamma+4\pi C'$, where β, γ, B', C' are the corresponding components of force and magnetization. Thus the general expressions (52) for the components of the vector potential are completely verified.

20. Energy of two magnetic distributions in presence of one another. Returning now to the determination of the energy of a magnet in a magnetic field, we have proved (p. 54 above) that

$$E=-\iiint(A\alpha+B\beta+C\gamma)\,dx\,dy\,dz.\quad\text{........ ......(54)}$$

From the manner in which this expression has been found it is plain that it measures the increase of potential energy which takes place when the magnet is caused to take up the given position against the action of magnetic forces, that is, it is equal to the work which must be done by external forces in bringing the magnet into the field. We shall now apply this result to the determination of the whole work done in this way [see below] in building up any two distributions (A) and (B) of magnetism. Plainly this may be regarded as consisting of three parts, E_1, the work done if (A) be supposed given in an infinite number of small parts at an infinite distance from one another, which are then put together to form the distribution, that is the work done in bringing these elements into the field simultaneously created by their aggregation to form the magnet :* E_2, the work done in similarly building up the other distribution ; and E_3, the work done in carrying one magnetic distribution into the field of the other. Calling the components of force due to the distribution (A) α_1, β_1, γ_1, those due to the distribution (B) α_2, β_2, γ_2, and denoting by A_1, B_1, C_1, A_2, B_2, C_2, the corresponding magnetization components, we have

$$E_1 = -\tfrac{1}{2}\iiint (A_1\alpha_1 + B_1\beta_1 + C_1\gamma_1)\,dx\,dy\,dz, \quad \ldots\ldots\ldots\ldots(55)$$

$$E_2 = -\tfrac{1}{2}\iiint (A_2\alpha_2 + B_2\beta_2 + C_2\gamma_3)\,dx\,dy\,dz. \quad \ldots\ldots\ldots\ldots(56)$$

Also we have $$E_3 = -\iiint (A_2\alpha_1 + B_2\beta_1 + C_2\gamma_1)\,dx\,dy\,dz, \quad \ldots\ldots\ldots\ldots(57)$$

in which the integration is extended throughout the volume of the magnet B. We have of course also by Green's theorem, or by the principle that the energy of (A) in the field of (B) must be equal to the energy of (B) in the field of (A),

$$E_3 = -\iiint (A_1\alpha_2 + B_1\beta_2 + C_1\gamma_2)\,dx\,dy\,dz. \quad \ldots\ldots\ldots\ldots(58)$$

The coefficient $\frac{1}{2}$ in the two first expressions arises from the fact that with the annulment of the distribution its field disappears. Hence the total energy may be written

$$E = E_1 + E_2 + E_3 = -\tfrac{1}{2}\iiint \{(A_1 + A_2)(\alpha_1 + \alpha_2) + (B_1 + B_2)(\beta_1 + \beta_2) \\ + (C_1 + C_2)(\gamma_1 + \gamma_2)\}\,dx\,dy\,dz, \quad \ldots\ldots(59)$$

or $$E = -\tfrac{1}{2}\iiint (A\alpha + B\beta + C\gamma)\,dx\,dy\,dz,$$

if A, B, C, α, β, γ be put for $A_1 + A_2$, etc., $\alpha_1 + \alpha_2$, etc.

The integral may evidently be taken throughout all space, since at any point not within either of the distributions of magnetism, each of the quantities A, B, C is identically zero.

We may put this expression into another form, thus : substituting for A, B, C, their values $(a - \alpha)/4\pi$, $(b - \beta)/4\pi$, $(c - \gamma)/4\pi$, we find

$$E = \frac{1}{8\pi}\iiint_{-\infty}^{+\infty} (a^2 + \beta^2 + \gamma^2)\,dx\,dy\,dz - \frac{1}{8\pi}\iiint_{-\infty}^{+\infty} (a\alpha + b\beta + c\gamma)\,dx\,dy\,dz. \quad (60)$$

Now remembering that $\alpha = -dV/dx$, $\beta = -dV/dy$, $\gamma = -dV/dz$, and integrating the second integral by parts, we see that it vanishes, since

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* According to Lord Kelvin, *Electrostatics and Magnetism*, 2nd edition, p. 441, if the magnet be broken up into an infinite number of infinitely thin filaments (each very long in comparison with its thickness) taken along the lines of magnetization, and these be then separated to infinite distance from one another, the work done has the value given in the text. Proportionality of magnetization to field intensity is assumed above. The subject of magnetic energy requires further discussion.

Va, Vb, Vc, are each zero at an infinite distance, and a, b, c, fulfil the solenoidal condition (41) above. Hence we have

$$E=\frac{1}{8\pi}\iiint_{-\infty}^{+\infty} H^2\,dx\,dy\,dz, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(61)$$

where H denotes the magnitude of the resultant magnetic force at the point x, y, z.

If H_1, H_2 denote the resultant forces produced at the point x, y, z, by the distributions (A) and (B) respectively, and θ the angle between H_1 and H_2, we have, by elementary trigonometry,

$$H^2=H_1{}^2+H_2{}^2+2H_1H_2\cos\theta.$$

$$\left.\begin{aligned} E_1&=\frac{1}{8\pi}\iiint_{-\infty}^{+\infty} H_1{}^2\,dx\,dy\,dz,\\ E_2&=\frac{1}{8\pi}\iiint_{-\infty}^{+\infty} H_2{}^2\,dx\,dy\,dz,\\ E_3&=\frac{1}{4\pi}\iiint_{-\infty}^{+\infty} H_1H_2\cos\theta\,dx\,dy\,dz,\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(62)$$

which can be verified by (55), (56), (57), (58), it being remembered that by the definition of E_1, E_2 we must take

$$a_1=\alpha_1+4\pi A_1,\ \text{etc.,}\quad a_2=\alpha_2+4\pi A_2,\ \text{etc.}$$

These expressions are obviously capable of generalization for any number of magnetic distributions, or a single distribution regarded as composed of any number of parts. They may be taken as expressing the fact that the energy may be regarded as residing in the medium in which the magnetized bodies are placed, and, of course, in these bodies themselves.

The energy stored in the field is not, however, to be taken as the whole work done in magnetizing the bodies and the medium: the amount of stored energy in the case of iron is a matter of uncertainty. [See further remarks on this subject in the discussion of hysteresis.]

We shall see later that magnetic force exists at every point in the space surrounding a conductor carrying an electric current, that in fact the molecular magnets composing any magnetized body are most probably produced by electric currents flowing in molecular circuits, which are devoid of resistance, so that the current continues to flow without diminution of strength from generation of heat. We shall then find that if a, b, c, be the components of magnetic induction **B**, and α, β, γ, those of magnetic intensity **H** (scalar values B and H), at any point in the field, the total magnetic energy E is given by the equation

$$\left.\begin{aligned} E &= \frac{1}{8\pi}\iiint_{-\infty}^{+\infty}(a\alpha + b\beta + c\gamma)\,dx\,dy\,dz, \\ \text{or}\qquad E &= \frac{1}{8\pi}\iiint_{-\infty}^{+\infty} BH\cos\theta\,dx\,dy\,dz. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(63)$$

Assuming this, we see that by drawing successive equipotential surfaces so that the difference of potential between each pair of consecutive surfaces is unity, and supposing these cut by unit tubes, we can divide the whole field up into cells, each of which may be regarded as containing $1/8\pi$ of a unit of magnetic energy.

Section III.

APPLICATIONS OF GENERAL THEORY. MAGNETIC SHELLS. LAMELLAR DISTRIBUTION. UNIFORMLY MAGNETIZED ELLIPSOID.

21. Magnetic shells. A most important form of magnetic distribution for consideration is that in which we have a thin sheet of matter magnetized normally to its surface. Such a sheet is called a *magnetic shell.* Its importance arises from the fact proved by Ampère that every linear circuit carrying a current is equivalent in magnetic action to a magnetic shell of a certain uniform intensity of magnetization, and having its bounding edge coincident with the circuit. A magnetic shell, it may be here stated, may be altered in position, elsewhere than at its boundary, in any way whatever, without affecting its magnetic action at any given point, provided only the shell be not so changed in position as to cause the point to pass through it, and that its magnetic moment per unit of area be uniform, and kept constant throughout the changes of position. The chief properties of magnetic shells are investigated in what immediately follows, and the results will be directly available when we come to consider the magnetic action of electric currents.

If $d\nu$ be the thickness of the sheet at any element dS, the volume of the element is $d\nu\,.\,dS$. If $\mathbf{I}$ then be the intensity of magnetization at the element, the magnetic moment of this portion is $\mathbf{I}\,d\nu\,.\,dS$. The product $\mathbf{I}\,d\nu$ is called the *strength* of the shell, and is usually denoted by Φ. This may vary from point to point of the shell.

The sheet here considered is supposed to fulfil certain conditions not usually stated. It must be impossible to pass from a point P to another

P' separated from P by the thickness of the sheet, without passing through the sheet or following a path round its edge. In other words the sheet must have two faces, which are distinct in the sense here indicated.*

We shall consider first a simple shell, that is one for which Φ has the same value at every point. By (14) above, if we consider any element dS of the shell, and θ be the angle between the direction of magnetization of the shell, taken positive when drawn from the negative to the positive side, and a line drawn from the element to a point P at distance r, the potential at P due to the element is $\Phi\, dS \cos\theta/r^2$. But $dS\cos\theta$ is the projection of the element at right angles to r, and therefore $dS\cos\theta/r^2$ is the area $d\omega$, traced out on the surface of a sphere of unit radius, having its centre at P, by a line passing through P, and carried round the boundary of the element, that is, it is the solid angle subtended at P by the element. It follows therefore that the potential V at P produced by the whole shell is given by the equation

$$V = \Phi\omega, \quad \text{..................................(64)}$$

where ω is the total solid angle subtended by the shell at P.

This is also, of course, the potential energy of the shell in the field due to unit magnetic pole placed at P.

It is evident that the value of V depends only on the strength of the shell and its boundary, and hence we have the remarkable result, that any two shells of equal strength, which have the same boundary, produce equal potentials at the point P, provided P does not lie between them.

If the shell be closed its potential at any external point is zero, since the solid angle is then zero. Such a shell therefore produces no magnetic effect at any external point. At every internal point in such a shell however the potential is $-4\pi\Phi$ (if the positive side be outwards, or $+4\pi\Phi$ if the positive side be inwards) since the solid angle is then 4π. There is therefore no magnetic force at any internal point.

In the reckoning of solid angles in this connection we shall adhere to the following convention. Let P, P' be adjacent points on opposite sides of a shell S (Fig. 10), of which P is on the positive side. Then supposing the solid angle subtended at P by the shell to be ω, that subtended by the shell at P' is to be taken as $\omega - 4\pi$; for, plainly, if the generating lines of the cone which meet at P' were turned round the edge of the shell from meeting at P' to meeting at P the solid angle would change in the process by 4π, and we must take it as being increased by that amount.

---

* It is possible to construct a surface which, in this sense, has only *one* face. Take a ribbon of paper, give it a half-turn of twist, or an odd number of half-turns of twist, and then gum the two ends together. The result will be a surface which may be said to have only one face and one edge. It will in this case be possible to pass from a point P to a point P', situated as in Fig. 10, by a path lying wholly in the surface. The edge may be taken to indicate the position of a closed circuit carrying a current: the surface, which it is here considered as bounding, cannot be taken as that of a magnetic shell, equivalent in magnetic action to the current. The construction indicated in Fig. 10 is that which must be used for the shell. Take any point P, not on the surface or its edge, and draw lines from it to successive points of the edge. These lines will indicate the solid angle subtended at P by the circuit, and the conical surface which they give will have two faces, and may be taken as representing the equivalent shell. An attempt to represent the action of the current by magnetization of the unifacial surface would result in exactly no magnetization at all, as the reader may verify.

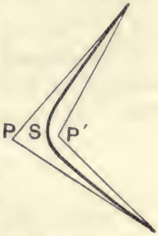

FIG. 10.

Or, the difference between the solid angles may be seen thus: consider the two simple shells A, B, of which a section by the plane of the paper is shown in Fig. 11, which have a common boundary b, b, and form a closed simple shell, the positive face of which is the outside. Let P, P' be infinitely near points, the former on the outside, the latter on the inside of A. Let the potential due to A at P be V_1, and at P', V_2. The potentials at P and P' produced by B will be the same, V', say. But we have $V_1 = \Phi\omega$, $V_2 + V' = -4\pi\Phi$, and $V_1 + V' = 0$. Thus we get $V_2 = \Phi(\omega - 4\pi)$ as already stated.

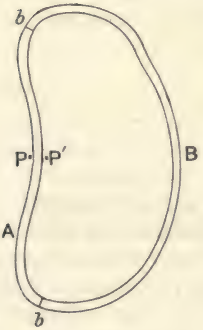

FIG. 11.

Hence the potential of the shell V varies, as the point at which it is measured changes in position from P to P' round the edge of the shell, from the value $\Phi\omega$ to the value $\Phi(\omega - 4\pi)$. If the point pass from a position infinitely near the negative side through the shell to an adjacent position on the positive side, the potential increases by the amount $4\pi\Phi$.

22. Lamellar magnets. In some cases of magnetization, as for example the induced magnetization of soft iron in certain circumstances, the body may be regarded as made up of simple magnetic shells, either closed or having their edges in the surface of the body; in such cases the magnetization is said to be lamellar. If we take ϕ to denote for such a body the sum of the strengths of the shells encountered by a point made to pass within the magnet from any given position to any other position (x, y, z), we easily see that

$$A = \frac{d\phi}{dx}, \quad B = \frac{d\phi}{dy}, \quad C = \frac{d\phi}{dz}. \quad \text{......................(65)}$$

ϕ is called the *potential of magnetization*, since the quantities A, B, C, are derived from it by differentiation. When they can be so derived they are said to fulfil the lamellar condition. Now we have seen, (22) above, that the potential V at any point (ξ, η, ζ) due to a finite magnet is given by the equation

$$V = \iiint \left(A\frac{du}{dx} + B\frac{du}{dy} + C\frac{du}{dz} \right) dx\, dy\, dz$$

if u be written for the reciprocal of the distance r from (x, y, z) to (ξ, η, ζ). Hence for a lamellar magnet this becomes

$$V = \iiint\left(\frac{d\phi}{dx}\frac{du}{dx} + \frac{d\phi}{dy}\frac{du}{dy} + \frac{d\phi}{dz}\frac{du}{dz}\right)dx\,dy\,dz. \quad \text{............(66)}$$

Integrating this expression by parts, and putting l, m, n, for the direction cosines of the normal drawn outwards to an element dS of the surface, we get

$$V = \iint\phi\left(l\frac{du}{dx} + m\frac{du}{dy} + n\frac{du}{dz}\right)dS - \iiint\phi\nabla^2 u\,dx\,dy\,dz, \quad \text{.........(67)}$$

in which the first integral is taken over the surface of the magnet, the second through its substance. Each element of the surface integral may be written in the form $\phi\cos\theta\,dS/r^2$, where θ is the angle between the normal and the direction of r. Each element of the second integral is zero unless the point (ξ, η, ζ) fall within the limits of integration. In the latter case the integral has the value $-4\pi\phi'$ if ϕ' be the value of ϕ at (ξ, η, ζ). Hence in general we have, for a lamellar magnet,

$$V = \iint\frac{1}{r^2}\phi\cos\theta\,dS + 4\pi\phi'. \quad \text{....................(68)}$$

The value of V given in (68) is continuous at the surface of the magnet. For plainly we may regard the surface integral as the potential at P of a magnetic shell coinciding with the surface, and of strength ϕ, varying from point to point. The potentials of this shell at two adjacent points, one just outside, the other just inside, differ only by the potential due to the portion of the shell immediately between the points. Thus denoting the surface integral by Ω, if Ω_e, Ω_i denote the values of the surface integral at the external and internal points respectively, we have

$$\Omega_e = \Omega_i + 4\pi\phi', \quad \text{..............................(69)}$$

and as the term $4\pi\phi'$ of (68) disappears in the passage from the inside to the outside of the surface, the potential is unchanged by the passage.

But the value of V whether at an internal or an external point at first sight seems indefinite, since the value of ϕ depends upon the zero of reckoning chosen for it. This is, however, not the case, for if any arbitrary value of ϕ be taken for a point in the surface, its value is thereby fixed for any other point, and it is clear that by choosing any other value for that point we should simply increase the strength of the shell by the same amount at every point, that is, would superimpose a simple closed shell of strength c_1, say, on the former. The value of ϕ at every internal point would also be increased by the amount c_1. Hence, for the potential V at an internal point we should have

$$V = \iint\frac{1}{r^2}\phi\cos\theta\,dS - 4\pi c_1 + 4\pi(\phi' + c_1),$$

that is, its value would remain unaltered. At an external point the additional potential would be that of a simple closed shell of constant strength, which is zero.

The external and internal action of the lamellar magnet thus depends only on the variation of strength from point to point, and not on its actual value. For an external point therefore it depends only on the variation of ϕ from point to point along the surface. But by the values of A, B, C in (65) it is clear that the rate of variation of ϕ in any direction along the surface is the tangential component of magnetization in that direction. Hence the external action of the shell is given if the tangential component of magnetization is given for every point on the surface.

Since in a lamellar distribution of magnetism we have

$$V=\Omega+4\pi\phi$$

and A, B, $C=d\phi/dx$, etc., α, β, $\gamma=-dV/dx$, etc., we have

$$a, b, c=-\frac{d\Omega}{dx}, \quad -\frac{d\Omega}{dy}, \quad -\frac{d\Omega}{dz} \quad \text{.....................(70)}$$

respectively. Ω is called the potential of magnetic induction.

It is plain that in a lamellar distribution the direction of magnetization is everywhere at right angles to the surfaces $\phi=c_1$, that is, the surfaces of equal potential of magnetization.

The potential energy of a simple magnetic shell in a magnetic field is given by equation (29) above, modified so as to suit the case of the shell. If dS be an element of area, l, m, n the direction cosines of the normal to the shell drawn from its negative to its positive side, Φ the (uniform) strength of the shell, and $\delta\nu$ its thickness, we have $A\,\delta\nu=l\Phi$, etc., and therefore

$$E=-\Phi\iint(l\alpha+m\beta+n\gamma)\,dS, \quad \text{.....................(71)}$$

that is, it is the product of the strength of the shell into the surface integral of magnetic induction over the surface. Hence, by (45) above, the energy of the shell in the field may be expressed by a line integral taken round its boundary.

23. One magnetic shell in the field of another. We have an interesting and extremely important case when the field is produced by another simple shell. In this case the mutual energy of the shells is expressible as a double line integral taken round their boundaries. Calling the energy in this case E_{ss}, we have at once by (45) and (71),

$$E_{ss}=-\Phi\int\left(F\frac{dx}{ds}+G\frac{dy}{ds}+H\frac{dz}{ds}\right)ds, \quad \text{..................(72)}$$

where ds is an element of the boundary of the shell, and F, G, H are given by (see equations (52), p. 64)

$$\left.\begin{aligned} F &= \Phi' \iint \left(m' \frac{du}{dz'} - n' \frac{du}{dy'} \right) dS', \\ G &= \Phi' \iint \left(n' \frac{du}{dx'} - l' \frac{du}{dz'} \right) dS', \\ H &= \Phi' \iint \left(l' \frac{du}{dy'} - m' \frac{du}{dx'} \right) dS', \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(73)$$

in which the accented letters and the integrations refer to the shell producing the field, and u is the reciprocal of the distance between a point (x, y, z) in one shell and a point (x', y', z') in the other. Now by writing in the first of (73) $m'dS' = dz'dx'$, $n'dS' = dy'dx'$, it is easy to see that F is equal to the line integral of $u dx'/ds' . ds'$ taken round the boundary of the shell. The same thing may be proved by equations (49) and (50) of 18, by putting there $F = u$, $G = 0 = H$, and using accented variables. Similarly G and H in (73) may be dealt with. Hence we find for E_{ss} the equation

$$\begin{aligned} E_{ss} &= -\Phi \Phi' \iint u \left(\frac{dx}{ds} \frac{dx'}{ds'} + \frac{dy}{ds} \frac{dy'}{ds'} + \frac{dz}{ds} \frac{dz'}{ds'} \right) ds\, ds' \\ &= -\Phi \Phi' \iint \frac{1}{r} \cos\theta \, ds\, ds', \quad \ldots\ldots\ldots\ldots(74) \end{aligned}$$

where θ is the angle between ds and ds'.

For a lamellar distribution of magnetism we have, by (54) and (65),

$$E = \iiint \left(\frac{d\phi}{dx} \frac{dV}{dx} + \frac{d\phi}{dy} \frac{dV}{dy} + \frac{d\phi}{dz} \frac{dV}{dz} \right) dx\, dy\, dz,$$

which integrated by parts becomes, since $\nabla^2 V = 0$,

$$E = \iint \phi \frac{dV}{d\nu} dS$$

where $dV/d\nu$ is the rate of variation of V along a normal to the shell drawn from the negative to the positive side.

Hitherto we have dealt only with simple shells, or with lamellar distributions built up of simple shells either closed or having their edges in the surface of the magnet. A complex shell is a thin plate of substance normally magnetized, but varying in strength from point to point. It may be conceived as made up of overlapping simple shells. A magnet made up of complex shells fulfils the condition that the direction of magnetization at every point is normal to a family of surfaces; but the intensity is not derivable from a potential of magnetization. But complex shells, as such, play no part in magnetic measurements, and further discussion of this subject is therefore omitted.

24. Potential of a uniformly magnetized body. We shall now consider the potential and force at external and internal points in one or two important cases of magnetization. We shall deal first with the magnetization of a body of uniform susceptibility when placed in a uniform field, a case which is important for magnetic measurements. The magnetization of the body will also be uniform, and we shall suppose it known in amount. We shall deal with its relation to the magnetizing force later when we consider determinations of susceptibility.

Any case of uniform magnetization may be regarded as produced by supposing two uniform volume distributions of magnetism, equal in density but opposite in sign, to be made coincident with the body and the negative distribution to be then displaced [15 above] a small distance in the direction opposite to that of magnetization. The (finite) product of the volume density ρ, supposed infinitely great, into this displacement, supposed infinitely small, is the intensity **I** of magnetization. Now if ρU be the potential at the point P produced by the positive distribution, the potential at the same point produced by the negative distribution displaced relatively through a distance $-\delta s$, will be equal in amount and opposite in sign to that which the positive distribution would produce at P, if P were displaced through an equal and opposite distance $+\delta s$, that is, $-\rho(U+dU/ds\,.\,\delta s)$. Hence if V denote the potential at P due to the magnet, we have

$$V=-\frac{dU}{ds}\,.\,\rho\,.\,\delta s=-I\frac{dU}{ds}. \qquad \text{......................(75)}$$

If λ, μ, ν be the direction cosines of **I** (scalar value I) we have A, B, $C=\lambda I$, μI, νI, as before, and therefore (75) may be written

$$V=-\left(A\frac{dU}{dx}+B\frac{dU}{dy}+C\frac{dU}{dz}\right)=AX+BY+CZ \qquad \text{.........(76)}$$

if X, Y, Z, be the components of force at P due to the positive distribution. From this expression the components of magnetic force can be obtained by differentiation.

We might also obtain equation (76) by remembering that any element of volume, δv, distant r from P, has the magnetic moment $I\,\delta v$, and produces therefore a magnetic potential at P, of amount $I\,\delta v\cos\theta/r^2$, where θ is the angle between I and r. But this is the component force at P (on unit mass) in the direction of **I** due to a material element of volume δv and density I. Hence the whole magnetic potential at P is numerically equal to the resultant force at P due to a uniform distribution of matter, coinciding with the body, and of density I; which is what (76) expresses.

25. Case of a uniformly magnetized ellipsoid. We shall now consider the case of a uniformly magnetized ellipsoid. Let the axes be a, b, c, and the intensity of its magnetization **I**; it is required to find the magnetic potential of the ellipsoid at an external point $P(\xi, \eta, \zeta)$. By the last paragraph this problem will be solved if we find the axial components of force at P due to a uniform ellipsoid of any density ρ.

We know that the force at the surface of a thin elliptic homœoid* is at right angles to the surface, and equal to $4\pi\tau\rho$, where τ is the thickness of the homœoid at the point. Now, by Maclaurin's theorem of attractions (extended to confocal homœoids), the attraction of an elliptic homœoid at a point $P(\xi, \eta, \zeta)$ is equal to the attraction at P due to a confocal homœoid the external surface of which passes through P. The equation of the surface of the given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad\ldots\ldots\ldots\ldots(77)$$

and the equation of a similar and similarly situated surface within and concentric with it is

$$\frac{x^2}{\theta^2a^2}+\frac{y^2}{\theta^2b^2}+\frac{z^2}{\theta^2c^2}=1, \quad\ldots\ldots\ldots\ldots(78)$$

where $1>\theta>0$. If $\delta\theta$ be small the equation of the inner surface of a thin homœoid for which (78) is the equation of the outer surface is got from (78) by multiplying θ^2 by $1-2\delta\theta/\theta$. Hence the thickness of the homœoid at any point is $p\,\delta\theta/\theta$, where p is the perpendicular let fall from the centre on the tangent plane at the point.

Again the equation of an ellipsoid confocal with the outer surface, and passing through P, is

$$\frac{\xi^2}{a'^2}+\frac{\eta^2}{b'^2}+\frac{\zeta^2}{c'^2}=1, \quad\ldots\ldots\ldots\ldots(79)$$

where $a'^2=\theta^2(a^2+\phi^2)$, $b'^2=\theta^2(b^2+\phi^2)$, $c'^2=\theta^2(c^2+\phi^2)$, so that ϕ is a function of θ given by (79). Let p' be the perpendicular let fall from the centre of this ellipsoid to a tangent plane touching at the point (ξ, η, ζ), then the thickness of a thin elliptic homœoid having (ξ, η, ζ) on its outer surface is $\frac{1}{2}p'\nu$, where ν is a constant small quantity. Now the mass of this homœoid is to be the same as that of thickness $p\,\delta\theta/\theta$, so that we are to have, if dS', dS be elements of the areas of the surfaces,

$$\tfrac{1}{2}\nu\iint p'dS'=\frac{\delta\theta}{\theta}\iint p\,dS$$

or

$$\nu=2\frac{abc}{a'b'c'}\theta^2\delta\theta.$$

Hence the attraction is $2\pi\rho p'\nu$ or $4\pi\rho p'\theta^2\delta\theta\,.\,abc/a'b'c'$, and its direction is along p'. But the direction cosines of p' are $p'\xi/a'^2$, $p'\eta/b'^2$, $p'\zeta/c'^2$, and therefore the x-component of attraction is

$$4\pi\rho\xi p'^2\theta^2\delta\theta\,abc/a'^3b'c'.$$

Hence we get for the x-component of attraction due to the whole ellipsoid

$$X=4\pi\rho abc\xi\int_0^1\frac{p'^2\theta^2d\theta}{a'^3b'c'}. \quad\ldots\ldots\ldots\ldots(80)$$

---

* A shell bounded by two similar and similarly situated concentric ellipsoidal surfaces.

We may simplify this equation by substituting for $p'^2\theta^2 d\theta$ its value $-\theta^5\phi\, d\phi$ obtained by differentiating (79). Equation (80) becomes then

$$X = 2\pi\rho abc\xi\int_{\phi_1^2}^{\infty}\frac{d(\phi^2)}{\{(a^2+\phi^2)^3(b^2+\phi^2)(c^2+\phi^2)\}^{\frac{1}{2}}}, \quad \text{.........(81)}$$

where $\phi_1{}^2$ is the positive root of the cubic for ϕ^2 given by (79) when $\theta^2 = 1$. The corresponding expressions for Y and Z may at once be written down by symmetry. It may be noticed that if (ξ, η, ζ) be on the surface of the given ellipsoid the limits of ϕ^2 are 0 and ∞.

Writing now $X = L\xi$, $Y = M\eta$, $Z = N\zeta$, we have by (76)

$$V = AL\xi + BM\eta + CN\zeta.$$

Hence the components of magnetic force α, β, γ at the point (ξ, η, ζ) are

$$\alpha = -A\left(L + \xi\frac{\partial L}{\partial\xi}\right) - \left(B\eta\frac{\partial M}{\partial\xi} + C\zeta\frac{\partial N}{\partial\xi}\right), \text{ etc.} \quad \text{.........(82)}$$

At an internal point the force due to the elliptic homœoid external to the point is zero, and we have only to calculate the force due to a uniform ellipsoid similar and similarly situated to the given ellipsoid. Let the semi-axes be θa, θb, θc respectively, and substitute in (96), observing that the limits of integration are now 0 and ∞.

Then

$$X = 2\pi\rho\theta^3 abc\xi\int_0^{\infty}\frac{d(\phi^2)}{\{(\theta^2a^2+\phi^2)^3(\theta^2b^2+\phi^2)(\theta^2c^2+\phi^2)\}^{\frac{1}{2}}},$$

or writing χ^2 for ϕ^2/θ^2, we get

$$X = 2\pi\rho abc\xi\int_0^{\infty}\frac{d(\chi^2)}{\{(a^2+\chi^2)^3(b^2+\chi^2)(c^2+\chi^2)\}^{\frac{1}{2}}}, \quad \text{................(83)}$$

and similarly for Y and Z. It is to be observed that the integrals are now independent of (ξ, η, ζ). Hence we have, writing L, M, N for the multipliers of ξ, η, ζ, in these expressions,

$$V = AL\xi + BM\eta + CN\zeta \quad \text{........................(84)}$$

and

$$\alpha = -AL, \quad \beta = -BM, \quad \gamma = -CN, \quad \text{..................(85)}$$

that is, the magnetic force is uniform in value within the ellipsoid, and is the same for the same intensity of magnetization within similar ellipsoids. The direction of the force is not however that of magnetization unless the latter coincide with the direction of one of the axes of the surface; then the force acts in the opposite direction.

The integral can be easily evaluated in finite terms when the ellipsoid is one of revolution. Thus to find L, we write $(a^2+\chi^2)^{\frac{1}{2}} = 1/v$, and the integral then reduces to a form at once integrable. Similarly M and N may be dealt with. The results are

(1) For a prolate ellipsoid of eccentricity e ($b=c=a\sqrt{1-e^2}$),

$$\left.\begin{aligned} L &= 4\pi\rho\frac{1-e^2}{e^2}\left(\frac{1}{2e}\log\frac{1+e}{1-e}-1\right), \\ M=N &= 2\pi\rho\frac{1}{e^2}\left(1-\frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right). \end{aligned}\right\} \quad \text{..............(86)}$$

(2) For an oblate ellipsoid of eccentricity e ($b=c=a/\sqrt{1-e^2}$),

$$\left.\begin{aligned} L &= \frac{4\pi\rho}{e^2}\left(1-\frac{\sqrt{1-e^2}}{e}\sin^{-1}e\right), \\ M=N &= 2\pi\rho\frac{\sqrt{1-e^2}}{e^2}\left(\frac{1}{e}\sin^{-1}e-\sqrt{1-e^2}\right). \end{aligned}\right\} \quad \text{...........(87)}$$

From these results (writing 1 for ρ) we can easily find formulae for special cases. Thus, if the ellipsoid be infinitely long, (1) gives $L=0$, $M=N=2\pi$.

This shows that the magnetic force within an infinitely long uniformly magnetized cylinder is zero if the magnetization is parallel to the axis, and is perpendicular to the axis and equal to $-2\pi I$ if the cylinder is magnetized transversely.

Again let the ellipsoid be spherical, that is, let $e=0$, and let the direction of magnetization be parallel to the axis of x. Then the force is

$$-IL=-\tfrac{4}{3}\pi I, \quad \text{..............................(88)}$$

since $4\pi/3$ is the value of the vanishing fraction which L is in this case.

These two results might have been inferred from the investigation on p. 58 above, of the force within a spherical or cylindrical hollow cut within a magnet.

Lastly, let the ellipsoid be very oblate, a disk in fact; then $M=N=\pi^2a/c$, and the force at right angles to it is

$$-IL=-4\pi I. \quad \text{..............................(89)}$$

These forces are all in the opposite direction to that of the magnetization, and therefore act as demagnetizing forces. We shall consider them fully when we deal with induced magnetization and measurements connected therewith.

Section IV.

INDUCED MAGNETIZATION.

26. Induced magnetization in a uniform field. Weber's theory. We now pass to the consideration of induced magnetization, and shall consider here only the problem of the magnetization produced in a homogeneous body when placed in a uniform magnetic field. The essential nature of the magnetization of the body is not known to us; but probably Weber's theory is substantially true, viz. that the body consists of particles already magnetized, but so arranged (not simply mixed up) in the unmagnetized mass as to give no external magnetic effect. The magnetization of each of these small particles may consist in rotatory motion of the ether; and if this be true the direction of rotation is what corresponds to the notion of polarization.

According to Weber's theory, when a body, unmagnetized in mass, is submitted to the action of a magnetic field, the molecular magnets undergo an alignment, so that like extremities are turned preponderatingly in the same direction. Each particle experiences a couple tending to turn its axis into coincidence with the direction of the magnetic force, and, unless prevented from turning by frictional or other resistance, it moves towards that position until brought to rest by an equilibrating couple due to mutual action between the molecule and the surrounding particles. Thus the molecular magnets are in general prevented from coming every one into coincidence with the direction of the magnetic force, in which case no further magnetization would be possible, and we know that by increasing the magnetic force we can increase the magnetization, although not in an unlimited degree.

Again, when the magnetizing force is removed the substance does not in general return to its former unmagnetized state, but does so only to a certain extent, retaining under some circumstances a very considerable amount of magnetization.

27. "Coercive force." Hysteresis in changes of magnetization. Demagnetizing forces. This property of resisting magnetizing action, and of retaining residual magnetization, is sometimes called *coercive force.* It has been attributed to something analogous to frictional resistance, which prevents the magnetic particles from moving freely in obedience to the magnetizing force and from returning freely when it is removed. A theory in which the mutual action of the molecular magnets plays the chief part, will (if there is space) be considered later, but it may be stated here that mechanical agitation, such as jarring or tapping an iron wire or bar, in general increases the magnetization while the body is under the influence of magnetic force, and diminishes the magnetization when the magnetizing force has been removed. The mechanical disturbance enables the particles to obey more completely the magnetizing or demagnetizing action, as the case may be, by changing their configuration.

If a piece of iron be subjected to a gradually increasing magnetic force, and then to a gradually decreasing one, the two magnetizations for the same magnetic force are, in consequence of residual magnetism, not identical. This phenomenon we shall see indicates dissipation of energy in the magnetized iron of an amount which, except under certain special conditions, as already remarked, does not seem to bear a fixed relation to the energy stored in the field in consequence of the magnetization. We shall deal with this, and with other phenomena when treating of the experimental work on this subject.

We shall consider first the case of a spherical portion of an æolotropic body placed in a uniform magnetic field, and examine the magnetization which it receives, on the following supposition :

The total magnetization which the magnet receives is the resultant of the magnetizations which the several parts of the magnetizing system would produce if each acted alone.

This implies, first, that if the intensity of the field at each point is altered in any ratio, the magnetization is simply altered in intensity in this ratio at each point without change of direction ; second, that magnetizations in different directions are produced in the substance, and are superposed as if no other magnetization were present. In point of fact these conditions do not hold, and their assumption gives only an approximation to the result in certain cases ; but in many important practical cases it does not give anything approaching the actual result. The magnetic susceptibility is, as we shall see, a function of the magnetizing force ; and the magnetic behaviour of the material is further complicated in a great many ways not here taken into account.* The investigation now to be given however yields results of great theoretical interest which are of much importance in the theory of diamagnetism and magnecrystallic action.

The supposition made above gives for A, B, C expressions of the form

$$\left.\begin{aligned} A &= p\alpha + u\beta + t\gamma, \\ B &= u'\alpha + q\beta + s\gamma, \\ C &= t'\alpha + s'\beta + r\gamma, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (90)$$

where α, β, γ are as usual the components of the resultant magnetic force H, and p, q, r, s, s', t, t', u, u', coefficients. We shall see directly that $s=s'$, $t=t'$, $u=u'$, that is, that a magnetic force of a certain intensity acting in the direction of the axis of y produces the same intensity of magnetization parallel to the axis of x, as is produced parallel to the axis of y, by an equal magnetic force acting parallel to the axis of x, and so for any other two axes.

Taking the sphere as of unit volume the component magnetic moments due to the magnetization are simply A, B, C. Hence the sphere in the field is acted on by a couple round each of the axes, the moment of which is, for the z axis, $\beta A - \alpha B$ ($=\mathsf{N}$, say). Now let the axes be fixed in the sphere, and let it then be turned round the axis of z. If θ be the angle which the direction of H makes with the axis of z, ϕ the angle which the projection of H on the plane of xy makes with the axis of x, we have

$$H \sin\theta \cos\phi = \alpha, \quad H \sin\theta \sin\phi = \beta, \quad H \cos\theta = \gamma.$$

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* For a detailed examination of the theory of inductive magnetization the reader may refer to Duhem's *L'Aimantation par Influence.* Paris, 1888.

Hence the work done in increasing ϕ by the small angle $d\phi$ is

$$\mathsf{N}\,d\phi = -H^2[\{u'\cos^2\phi - u\sin^2\phi + (q-p)\sin\phi\cos\phi\}\sin^2\theta + (s\cos\phi - t\sin\phi)\sin\theta\cos\theta]\,d\phi. \quad\text{.............(91)}$$

The work done in turning the body through a complete revolution is therefore

$$\int_0^{2\pi}\mathsf{N}\,d\phi = \pi H^2\sin^2\theta\,(u-u'). \quad\text{........................(92)}$$

28. Induced magnetization of an aeolotropic body. Now, since the body has come back to its former position its magnetization is by hypothesis the same as before, and no work can, on the whole, have been spent or gained in the revolution, otherwise the body would be either a continual source of energy, that is, a perpetual motion, or a place where energy is continually dissipated. We shall see later that there can be no such dissipation of energy on the supposition of constant magnetic susceptibility. Assuming then that the work is zero, we have $u'=u$, and in the same way we could show that $s=s'$, $t=t'$. The equations (90) above can therefore be written

$$\left.\begin{aligned} A &= p\alpha + u\beta + t\gamma, \\ B &= u\alpha + q\beta + s\gamma, \\ C &= t\alpha + s\beta + r\gamma. \end{aligned}\right\} \quad\text{..........................(93)}$$

The magnetization in any direction, the cosines of which are l, m, n, is

$$lA + mB + nC = pl\alpha + qm\beta + rn\gamma + u(l\beta + m\alpha) + s(m\gamma + n\beta) + t(n\alpha + l\gamma). \quad\text{...........(94)}$$

If l, m, n be the direction cosines of H this equation becomes

$$lA + mB + nC = H\,(pl^2 + qm^2 + rn^2 + 2ulm + 2smn + 2tnl). \quad\text{......(95)}$$

Now, if we consider the quadric surface of which the equation is

$$R^2\,(pl^2 + qm^2 + rn^3 + 2ulm + 2smn + 2tnl) = 1, \quad\text{...............(96)}$$

we see that the quantity within the brackets in (95) is inversely proportional to the square of the radius R of this surface drawn in the direction of l, m, n. Hence for different directions of H the magnetic moment of the sphere in the direction of H is H/R^2, where R is the radius in that direction of the quadric surface of which (96) is the equation.

Further, since by (93) $A = H(pl + um + tn)$, etc., we see that the resultant magnetization is in the direction of the normal drawn to the quadric at the point at which the radius in the direction of H cuts the surface; or, in other words, is at right angles to the diametral plane of all radii in the direction of l, m, n.

It follows from this, that along the principal axes of the quadric surface represented by (96) the magnetization coincides in direction with the magnetizing force. Now the directions of the axes of this quadric are given by the equations

$$\left.\begin{aligned} pl+um+tn&=kl,\\ ul+qm+sn&=km,\\ tl+sm+rn&=kn,\end{aligned}\right\} \qquad\qquad (97)$$

where k is a constant. Eliminating l, m, n, we get

$$\begin{vmatrix} p-k, & u, & t, \\ u, & q-k, & s, \\ t, & s, & r-k, \end{vmatrix} =0, \qquad\qquad (98)$$

which is a cubic from which k can be found. The three roots k_1, k_2, k_3, of this equation successively substituted in (97) enable l, m, n to be calculated for each of the three axes.

Now let H be in the direction of one of the axes (l, m, n) thus found. The magnetization in that direction is $lA+mB+nC$. Substituting the values of A, B, C given by (93) and having regard to (97), we see that the magnetization has the value kH. Hence k_1, k_2, k_3 are the magnetic susceptibilities in the direction of the axes just found. They are called the principal magnetic susceptibilities of the substance. If the axes just found be chosen for reference the coefficients s, t, u of (93) vanish, and we have $p=k_1$, $q=k_2$, $r=k_3$. We have thus also three principal magnetic inductive capacities, viz.

$$\mu_1=1+4\pi k_1, \quad \mu_2=1+4\pi k_2, \quad \mu_3=1+4\pi k_3. \qquad (99)$$

The physical meaning of the principal susceptibilities is apparent from their mode of derivation; that of a principal magnetic inductive capacity μ can at once be inferred. Cut a crevasse at right angles to the axis in question, and suppose the magnetization unaffected in the remainder of the body. Then if H be the scalar value of the magnetizing force in the direction of the axis and B that of the induction across the crevasse, we have $B=\mu H$.

Choosing now the principal axes as axes of reference, and putting α, β, γ for the components of the intensity of the externally produced field, we get by (88) for the components of magnetic force within the sphere the values

$$\alpha-4/3\,.\,\pi A, \quad \beta-4/3\,.\,\pi B, \quad \gamma-4/3\,.\,\pi C.$$

Hence $A=k_1(\alpha-4/3\,.\,\pi A)$, $B=$ etc., and therefore

$$A=\frac{k_1}{1+\frac{4}{3}\pi k_1}\alpha, \quad B=\frac{k_2}{1+\frac{4}{3}\pi k_2}\beta, \quad C=\frac{k_3}{1+\frac{4}{3}\pi k_3}\gamma. \qquad (100)$$

29. Couples on a magnetized aeolotropic body. The sphere is acted on as we have seen by three component couples, of values

$$\gamma B-\beta C, \quad \alpha C-\gamma A, \quad \beta A-\alpha B,$$

round the axes of x, y, z respectively, in the positive direction, viz. from x to y, y to z, z to x. Denoting these by L, M, N, we get by (100)

$$\left.\begin{aligned} \mathsf{L} &= \frac{k_2 - k_3}{(1+\frac{4}{3}\pi k_2)(1+\frac{4}{3}\pi k_3)}\beta\gamma, \\ \mathsf{M} &= \frac{k_3 - k_1}{(1+\frac{4}{3}\pi k_3)(1+\frac{4}{3}\pi k_1)}\gamma\alpha, \\ \mathsf{N} &= \frac{k_1 - k_2}{(1+\frac{4}{3}\pi k_1)(1+\frac{4}{3}\pi k_2)}\alpha\beta. \end{aligned}\right\} \quad \text{.................}(101)$$

The sphere thus tends to turn so as to bring the axis of greatest magnetic susceptibility into coincidence with the direction of the magnetizing force, and is therefore in stable or in unstable equilibrium according as the axis of greatest or the axis of least susceptibility is in this direction.

The total magnetic induction through the sphere across a central section at right angles to **B** is $\pi r^2 B$, where r is the radius of the sphere. Now the components of **B** are $\mu_1\alpha'$, $\mu_2\beta'$, $\mu_3\gamma'$, where α', β', γ' are the magnetic forces within the sphere in the directions of the principal axes. Let for simplicity the sphere be so placed that the field of force is at right angles to the axis of z. Then $B = (\mu_1^2\alpha'^2 + \mu_2^2\beta'^2)^{\frac{1}{2}}$. But

$$\alpha' = \alpha - 4\pi A/3 = \alpha/(1 + 4\pi k_1/3) = 3\alpha/(\mu_1 + 2)\,;$$

and similarly $\beta' = 3\beta/(\mu_2 + 2)$. Hence substituting, and putting ϕ for the angle between the direction of **H** and the axis of x, and S for the surface integral $\pi r^2 B$, we find

$$S = 3\pi r^2 H\left\{\left(\frac{\mu_1}{\mu_1 + 2}\right)^2 \cos^2\phi + \left(\frac{\mu_2}{\mu_2 + 2}\right)^2 \sin^2\phi\right\}^{\frac{1}{2}}. \quad \text{.......}(102)$$

If μ_1 be the greatest magnetic inductive capacity, and μ_2 the least, the greatest number of unit tubes of induction (or, as they are commonly called, "lines of magnetic force") which can pass through the sphere per unit of the impressed magnetizing force is $3\pi r^2\mu_1/(\mu_1 + 2)$, and the least number $3\pi r^2\mu_2/(\mu_2 + 2)$. The sphere therefore tends to set itself so that the magnetic induction through it is a maximum.

We may express the resultant couple in terms of S and the magnetic inductive capacities. Clearly if $\gamma = 0$, its value is N of (101), and this may be written

$$\begin{aligned} \mathsf{N} &= \frac{9}{4\pi}\,\frac{\mu_1 - \mu_2}{(\mu_1 + 2)(\mu_2 + 2)}H^2 \sin\phi\cos\phi \\ &= -\frac{1}{24\pi^2 r}\,\frac{(\mu_1 + 2)(\mu_2 + 2)}{\mu_1 + \mu_2 + \mu_1\mu_2}\,\frac{d(S^2)}{d\phi}, \end{aligned} \quad \text{................}(103)$$

by (102) above. Here πr^3 has been put equal to its value 3/4, so that the formula is suited to a sphere of any radius.

In the particular case of an isotropic sphere $k_1 = k_2 = k_3$, $= k$ say, and the susceptibility and magnetic inductive capacity are the same in all directions. Thus the coefficients of α, β, γ, in (100) are the same, and the common value is $k/(1+4\pi k/3)$. If k be great this is approximately $3/4\pi$. Hence the magnetization intensity of a highly susceptible sphere is always less than, but nearly equal to $3H/4\pi$. It is thus useless to attempt to determine the susceptibility of a highly susceptible substance by experiments on a portion of it of a spherical shape. In the comparison of different specimens, the influence of slight differences of form would completely mask differences of susceptibility.

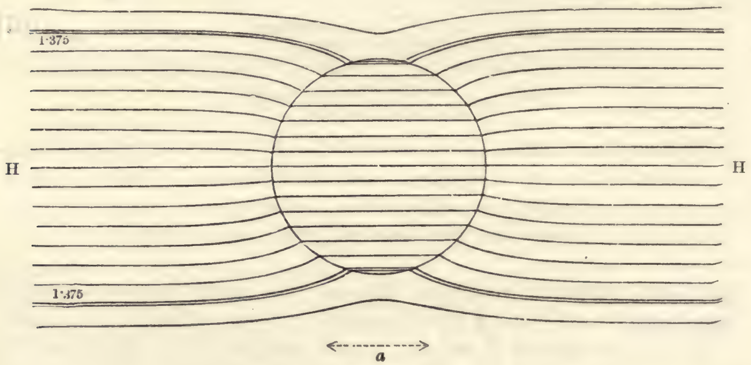

FIG. 12.

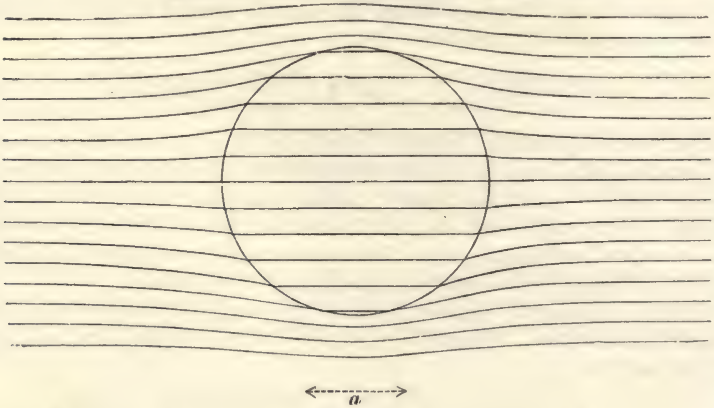

FIG. 13.

The couples calculated above vanish for an isotropic sphere, and the sphere is in equilibrium in all positions. The magnetic induction through a central section at right angles to the field is now $3\mu/(\mu+2) . \pi r^2 H$. This is greater or less than $\pi r^2 H$ according as $\mu >$ or < 1. In the former case the body is said to be paramagnetic, in the latter diamagnetic. Thus the number of tubes of induction through the central section is increased by the presence of the substance if paramagnetic, diminished if it is diamagnetic. The field outside and inside in both cases is shown in Figs. 12 and 13, which are taken from Lord Kelvin's *Electrostatics and Magnetism*, 2nd edition.

30. An æolotropic ellipsoid in a uniform field. We shall here consider, very briefly, the problem of an æolotropic ellipsoid in a uniform field. We shall suppose the ellipsoid cut with its axes of figure coincident with the principal axes of susceptibility. The force within the body has now the values in the direction of the axes

$$\alpha' = \alpha - AL, \quad \beta' = \beta - BM, \quad \gamma' = \gamma - CN, \quad \ldots\ldots\ldots\ldots(104)$$

where L, M, N have the values given in (86) above. The second term in each expression is the component force due to the magnetization. Its effect is to oppose the magnetizing component; that is, AL, BM, CN are components of a demagnetizing force. Therefore if k_1, k_2, k_3 be the three principal susceptibilities the values of A, B, C are given by the equations

$$A = \frac{k_1\alpha}{1 + k_1L}, \quad B = \frac{k_2\beta}{1 + k_2M}, \quad C = \frac{k_3\gamma}{1 + k_3N}. \quad \ldots\ldots\ldots\ldots(105)$$

Hence if the field be at right angles to the axis of z, $\gamma = \gamma' = 0$, and the couple on the ellipsoid is by (120) (if f, g, h be the semi-axes),

$$\mathsf{N} = \tfrac{4}{3}\pi fgh(\beta A - \alpha B) = \tfrac{4}{3}\pi fgh \frac{k_1 - k_2 - k_1k_2(L - M)}{(1 + k_1L)(1 + k_2M)}\alpha\beta. \quad \ldots\ldots(106)$$

From (120) it follows that if k_1, k_2, k_3 be so small that their second powers may be neglected, $A = k_1\alpha$, $B = k_2\beta$, $C = k_3\gamma$; that is, the internal demagnetizing forces AL, BM, CN, are without sensible effect. These forces depend (see p. 75) upon the form of the body; hence in weakly magnetic substances it is of little consequence whether the body be of elongated shape or not. In fact, for such bodies the shape of the specimen experimented on is without influence on the magnetization.

If however the values of k_1, k_2, k_3 be very great the magnetization of the body depends almost entirely on the shape of the body, since then the values of A, B, C depend mainly on L, M, N. Thus in highly magnetic bodies such as iron, the magnetization is principally affected by the shape of the specimen. For example, the magnetization in the direction of the axis (say that of x) of a very elongated ellipsoid is practically independent of L, since by (105) we have $A = k_1\alpha$ simply. On the other hand, for a very short ellipsoid (or disc) we have, since $L = 4\pi$, $A = k_1\alpha/(1 + k_1L) = (\mu_1 - 1)\alpha/4\pi\mu_1$. In the case of a diamagnetic body (see 31 below) k_1 is negative, and hence if the body were shaped so as to give $L = -1/k_1$, a finite magnetizing force would give infinite diamagnetization.

The couple acting on the body has two different limiting values according as the susceptibilities are very small or very great. If the former the couple is

$$\mathsf{N} = \tfrac{4}{3}\pi fgh \frac{(k_1 - k_2)\alpha\beta}{1 + k_1L + k_2M}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(107)$$

and the ellipsoid tends to turn its axis of greatest susceptibility parallel to the direction of the field.

If the susceptibilities be very great the couple is approximately $\frac{4}{3}\pi fgh(M-L)\alpha\beta/LM$, and this is positive or negative according as $M >$ or $< L$. Hence by the values of M and L (see above, p. 76) the ellipsoid will set itself with its longest dimension parallel to the lines of force.

The influences of form and æolotropy may be made to counteract one another, and, under certain circumstances, by properly shaping the body it may be made to remain in neutral equilibrium when movable, as supposed above, about an axis in the magnetic field.

In the case of a homogeneous isotropic ellipsoid the values of A, B, C and of the turning couple are obtained by putting $k_1 = k_2 = k_3$, $= k$ say, in (105), (106) above. It is obvious at once that the magnetization is not parallel to the resultant magnetic force, but makes with it the angle

$$\cos^{-1}(\alpha A + \beta B + \gamma C)/\{(\alpha^2 + \beta^2 + \gamma^2)(A^2 + B^2 + C^2)\}^{\frac{1}{2}},$$

which vanishes for a sphere. Further the ellipsoid will turn its longest axis parallel to the lines of force, and this whether k be positive or negative, provided in the latter case the field be intense enough.

Regarding k as a constant, and taking as hitherto $\mu_0 = 1$, we have, if a, b, c, α, β, γ be the components of magnetic induction and magnetic force at any point of the medium, the equations

$$a = \alpha(1 + 4\pi k), \quad b = \beta(1 + 4\pi k), \quad c = \gamma(1 + 4\pi k), \quad \ldots\ldots(108)$$

and therefore, since the solenoidal condition holds for the magnetic induction (a, b, c), it also holds for the magnetic force (α, β, γ). Hence also it holds for the induced magnetization (A, B, C), that is, this magnetization is solenoidal.

But since $\alpha = -dV/dx$, etc., and $A = k\alpha$, etc., where k is a constant, we see that the magnetization is also lamellar (p. 69).

It is to be very carefully observed that these results follow from the fact that k is a constant, and do not hold in the general case (unless the magnetization be uniform) in which k is a function of the magnetization, and which is the case of actual practice. The *total* magnetization, made up of the pre-existent magnetization, if any, and the induced magnetization, is not solenoidal, unless the former is itself solenoidal.

31. Paramagnetism and diamagnetism. We have considered incidentally above the consequences of a negative value of k, and have stated that in that case the substance is said to be diamagnetic. The phenomena of diamagnetism are to some extent explicable on a theory of negative or differential susceptibility, and other theories have lately been advanced. It is not necessary to discuss any of these here. The substance placed in the field behaves as if its polarity were opposite to that which an ordinarily magnetic, or *paramagnetic*, body would receive in the same circumstances.

Hitherto we have been supposing that the medium in which the magnetized substance is placed is of zero susceptibility, that is, possesses unit magnetic inductive capacity. We shall now show that a paramagnetic body placed in a medium of greater magnetic inductive capacity than its own will behave diamagnetically. The medium, being in the field, will be magnetized; and if μ, μ' be the magnetic inductive capacities of the medium and the substance imbedded in it, respectively, we have, from the continuity of the normal component of magnetic induction at every point of the separating surface, the equation

$$\mu \frac{dV}{dr} + \mu' \frac{dV'}{dr'} = 0, \quad \text{.........................} (109)$$

in which r, r' denote normals drawn from the surface into the respective media. But we have also the characteristic equation of the surface

$$\frac{dV}{dr} + \frac{dV'}{dr'} + 4\pi\sigma = 0, \quad \text{.........................} (110)$$

where σ is the surface density of free magnetism on the separating surface at the point where the normals are drawn. These two equations give

$$\sigma = \frac{\mu' - \mu}{4\pi\mu} \frac{dV'}{dr'} = \frac{\mu - \mu'}{4\pi\mu'} \frac{dV}{dr}. \quad \text{...................} (111)$$

The first multiplier is positive and the second negative if $\mu' > \mu$, that is if the substance be of higher susceptibility than the medium. Hence the surface density of magnetization where the lines of force pass from the medium to the body is positive or negative, and where they pass from the body to the medium negative or positive, according as $\mu' <$ or $> \mu$. In the latter case the body behaves as a paramagnetic, in the former as a diamagnetic substance. If μ be put $= 1$ in the above results, we have the case of a paramagnetic or diamagnetic body in a medium of zero susceptibility. It is the case, however, that this explanation does not account for all the facts of diamagnetism.

These results are in accordance with and explain the behaviour of a solution of a magnetic salt of iron suspended in a tube within another solution of greater or less strength, and the whole placed in a magnetic field. In the former case the suspended salt behaves as a diamagnetic, in the latter as a paramagnetic.

32. Distribution of magnetism in bar-magnets. Question of "centroids" of magnetism. A great deal of research and theoretical discussion has been spent on the question of the distribution of magnetism in straight bar-magnets of rectangular or circular section, and it may be well, before leaving the subject of magnetic theory, to refer to the widely prevalent idea of the existence of poles in ordinary magnets. It has been tacitly supposed by many persons that there are two definite points or poles, one near each end of a regularly* magnetized bar at which the whole of the free magnetism of the bar may be supposed concentrated, the negative at one, the positive at the other, and much time and labour have been spent in determining the positions of these poles. Now certainly, according to the theory given above, there is a certain amount of free positive and an equal amount of free negative magnetism in every magnet, but so far as the action of the magnet at external or internal points is concerned, there are no such definite points, except in the theoretical case of an infinitely thin and uniformly magnetized filament, in which case the poles are at its extremities.

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* That is, without "consequent points," not necessarily uniformly.

In an accurate sense the magnet can be said to have poles or points at which the free magnetism may be supposed concentrated, when the couple it experiences when placed in a uniform field is considered. If its axis is at right angles to the lines of force, the couple experienced is equal to the magnetic moment of the magnet, and this may be regarded as due to two forces, each of which is the resultant of the parallel forces on the elements of free magnetism. The centres of these two systems of parallel forces, or, which is the same thing, the "centres of mass" of the two distributions of free magnetism are the poles in this connection. The idea of pole is not here of any utility, as the poles could not be determined; what we are concerned with is the magnetic moment only, which is the resultant of the couples exerted on the molecular magnets composing the body.

But as a matter of approximation the existence of poles in the sense of points at which, if the free magnetism were concentrated, the action of the magnet at an external point would be the same as it actually is, can be assumed in certain cases, and their position assigned. For example, when we consider the mutual forces between two magnets, each symmetrical about its axis and about a plane at right angles to the axis, and at a distance apart which is great in comparison with any dimension of either, such positions of the poles can be found, and the distance between them used as the virtual length of the magnet.

CHAPTER III.

DEFINITION OF UNIT CURRENT. DETERMINATION OF THE HORIZONTAL COMPONENT OF THE EARTH'S MAGNETIC FIELD-INTENSITY.

1. Measurement of currents. The measurement of a magnetic field-intensity is of importance for many purposes, and especially for the measurement of currents in absolute units. For the fundamental definition of unit current, as *that current which flowing in a thin wire, bent into a circle of radius r, produces at the centre of the circle a magnetic field-intensity of* $2\pi/r$ *units* (or, in other words, the statement that a current of strength γ flowing in such a circle produces at the centre a magnetic field-intensity of $2\pi\gamma/r$ units) presupposes that some method of comparing the intensities of the fields which different currents produce is known.

This definition may be regarded as founded on the law given by Laplace for the action of an element of a linear circuit in producing a magnetic field. This law for an element of a circuit gives by integration a correct result for the magnetic field produced by a complete circuit of any form. To any expression for the field due to an element which fulfils this necessary condition may be added any term which, when the whole circuit is considered, does not alter the integral effect. For we have as a rule to consider only closed circuits, or circuits which are virtually closed, and have no absolute knowledge of the effect of an element; so that the adoption of this or that expression for such an effect is a matter merely of simplicity and convenience. In the theory of the magnetic effects of convection currents there is something of the same ambiguity.

2. Law of Laplace. Laplace's law asserts that if ds be the length of an element of a linear conductor (a thin wire for example) carrying a current γ, r be the length of the line CP from the centre C of the element to any point P at which the magnetic field is to be considered, and θ be the angle which the line CP makes with the direction of flow in the element, the field-intensity at P due to the element is proportional to $\gamma\, ds \sin\theta/r^2$. To define the unit current we take this field-intensity as *equal to* $\gamma\, ds \sin\theta/r^2$. The direction of the intensity is at right angles to the plane containing the element and the point P, and, when the direction of the current is that settled by the usual convention, is towards the left hand of a mannikin supposed placed in the element so that the current flows past him from his feet to his head and with his face turned toward P.

We thus derive the formal definition: *Unit current is that current which, flowing in an element ds of a conductor, produces at distance r from the element a magnetic field-intensity* $ds\sin\theta/r^2$. For the complete circle referred to above θ is $\frac{1}{2}\pi$ for every element if P be at the centre, or be any point on the axis of the circle, and so for the centre we obtain by integration $2\pi r/r^2$ or $2\pi/r$; and the direction is at right angles to the plane of the circle. Other forms of the definition of unit current will arise in different connections, but they will be all consistent with this. [See also V. 34, and the historical note there given.]

The force exerted at P on m units of magnetism there placed, in consequence of the current in the element, is thus $m\gamma\, ds\sin\theta/r^2$. Regarding this as applied to the pole m by the medium, we must suppose that at P there is exerted a reaction on the medium of the same numerical value but in the opposite direction. This may be transferred to the element ds (centre C) of the circuit by applying at C two equal and opposite forces, of amount $m\gamma\, ds\sin\theta/r^2$, parallel to the line of the magnetic intensity at P. These with the reaction at P give a couple of moment $m\gamma\, ds\sin\theta/r$ in the plane containing CP, and at right angles to ds, and a force at C equal and opposite to the force $m\gamma\, ds\sin\theta/r^2$, due to the existence of the magnetism at P.

This leads us to the result that if the element of the circuit is situated in a magnetic field of intensity H, the element is, according to Laplace's law, acted on by an electromagnetic force $\gamma H\, ds\sin\theta$, where θ is the angle between the direction of H and that of γ. For we may suppose the field at C produced by a positive magnetic pole of chosen finite strength m placed at a point Q, situated at a distance $\sqrt{m/H}$ (great in comparison with ds) on a straight line coinciding at C with an element of a line of magnetic force.

3. Electromagnetic force. The direction of the electromagnetic force at C may be specified by reference to the mannikin placed as supposed above in the element of the conductor, with his face turned towards Q. The element of the conductor would be urged towards the right hand of the figure.

By taking $\theta=\frac{1}{2}\pi$, and $H=1$, we get the following alternative definition of unit current. *Unit current is the current flowing in an element of a conductor, which, placed at right angles to the direction of the resultant force in a field of unit intensity at the element, is acted on by an electromagnetic force, the amount of which per unit of length of the element is equal to unity.* The force $\gamma H\, ds\sin\theta$ is of electromagnetic origin, but it is a force in the true dynamical sense and has the dimensional formula $[MLT^{-2}]$.

4. Determination of the earth's horizontal magnetic force. The determination of the horizontal component H of the earth's magnetic field-intensity is necessary for the measurement of currents in absolute units by means of a standard galvanometer. Hence, before proceeding to consider the measurements of currents by galvanometers, we devote the remainder of the present chapter to the study of methods for the determination of H. These methods differ in various respects from those employed at magnetic observatories for the measurement of the magnetic elements and their variation from day to day.

One of the first methods proposed is that due to Poisson (*Connaissance des temps*, 1828); but apparently it has been very little used. We shall indicate it here with some suggestions as to practical details.

5. Poisson's method. A very short thin magnetic needle, suspended by a torsionless fibre of silk or quartz, hangs horizontally with its centre at a point C and its magnetic axis in the magnetic meridian. With its centre at another point P, on a line through C at right angles to the axis of the needle, but with its magnetic axis horizontal in the magnetic meridian at P, is placed a bar-magnet of moment M. This magnet, if short in comparison with the distance $CP\,(=r)$, produces a field of intensity h approximately equal to M/r^3, or, if the magnet be nearly uniformly magnetized and of length $2l$, more nearly $(M/r^3)(1-3l^2/2r^2)$. We suppose that M is the moment of the magnet when so directed that the earth's field enhances the magnetization. The magnetic field h at C, due to the bar-magnet is oppositely directed to the component H of the earth's field.

The needle is made to oscillate (1) in the field of intensity H (that is with the bar-magnet at a great distance), then (2) in the field of intensity $H-h$ produced by placing the bar-magnet at P, as described. The oscillations are kept of very small range on either side of the meridian, and are magnified and rendered easily observed by means of a ray of light thrown by a fixed source on a mirror carried by the needle and reflected to a scale placed at a suitable distance. [The small magnetometer described in 7 below, is a very suitable arrangement for the purpose.] The time required for a considerable number of vibrations, say 50, is obtained by means of a stop-watch, or is recorded if a chronograph is available; and the periods T_1, T_2 are obtained for the two cases (1) and (2) specified above.

If M' is the moment of the needle in the field of intensity H, the moment in the field of intensity $H-h$ will be $M'(1-ah)$, where, if the needle is of hard steel well magnetized, a is a small constant multiplier. If μ' be the moment of inertia of the vibrating mass for these oscillations, we have

$$\frac{4\pi^2}{T_1^2}=\frac{M'H}{\mu'},\quad \frac{4\pi^2}{T_2^2}=\frac{M'(1-ah)(H-h)}{\mu'}. \quad\ldots\ldots\ldots\ldots(1)$$

These equations enable us to eliminate M'/μ', so that we obtain

$$\frac{(1-ah)(H-h)}{H}=\frac{T_1^2}{T_2^2}. \quad\ldots\ldots\ldots\ldots(2)$$

Previous experiments of the sort described in 13 and 21 below enable the value of a to be assigned for a given value of h, and when this has been done the value of $1-ah$ can be inserted in (2). We shall see presently how the correction can be approximately made.

Now let the needle be removed and the bar-magnet suspended horizontally in its place, and made to perform horizontal oscillations under the influence of the field H. If not too massive, the magnet can also be suspended by a fibre which can safely be regarded as devoid of torsion. We get then, without any correction for inductive variation of the moment M of the magnet, provided the terrestrial field at P may, as in practice is always the case, be regarded as the same as that at C, for the period T the equation

$$\frac{4\pi^2}{T^2}=\frac{MH}{\mu}, \tag{3}$$

where μ is the moment of inertia in this case.

To obtain an approximate evaluation of the factor $1-ah$ in (2), let this factor be taken as unity for a first approximation. We obtain the equation

$$H=h\frac{T_2^2}{T_2^2-T_1^2}. \tag{2'}$$

But

$$h=\frac{M}{r^3}\left(1-\frac{3}{2}\frac{l^2}{r^2}\right); \tag{4}$$

and therefore (2′) can be written

$$H=\frac{M}{r^3}\left(1-\frac{3l^2}{2r^2}\right)\frac{T_2^2}{T_2^2-T_1^2}. \tag{5}$$

By (3) we eliminate M and obtain

$$H^2=\frac{4\pi^2\mu}{r^3}\left(1-\frac{3}{2}\frac{l^2}{r^2}\right)\frac{T_2^2}{T^2(T_2^2-T_1^2)}, \tag{6}$$

or we can eliminate H and obtain

$$M^2=4\pi^2\mu\frac{r^3}{1-\frac{3}{2}\frac{l^2}{r^2}}\frac{T_2^2-T_1^2}{T^2T_2^2}. \tag{7}$$

The last equation gives, by substitution in (4),

$$h=\frac{2\pi\sqrt{\mu}}{T_2T}\left\{\frac{1-3l^2/2r^2}{r^3}(T_2^2-T_1^2)\right\}^{\frac{1}{2}}. \tag{8}$$

Hence, if we know a from experiment, we have

$$1-ah=1-\frac{2\pi a\sqrt{\mu}}{TT_2}\left\{\frac{1-3l^2/2r^2}{r^3}(T_2^2-T_1^2)\right\}^{\frac{1}{2}}. \tag{9}$$

Denoting the expression on the right by A, we get by (4) and (2)

$$H=\frac{M}{r^3}\left(1-\frac{3l^2}{2r^2}\right)\frac{AT_2^2}{AT_2^2-T_1^2}. \tag{10}$$

The exact value of M is $4\pi^2\mu/T^2H$, by (3), and therefore

$$H^2 = 4\pi^2\frac{\mu}{r^3}\left(1 - \frac{3}{2}\frac{l^2}{r^2}\right)\frac{AT_2^2}{T^2(AT_2^2 - T_1^2)}. \quad \text{..............(11)}$$

Also we have, by (3) and (10),

$$M^2 = 4\pi^2\mu\frac{r^3}{1 - \frac{3}{2}\frac{l^2}{r^2}}\,\frac{AT_2^2 - T_1^2}{AT^2T_2^2}. \quad \text{..................(12)}$$

Thus (11) and (12) give H and M.

It is most carefully to be observed that the needle is supposed to be so small that it can have no sensible effect on the bar-magnet. This is the case if the length of the needle be only 3 or 4 mm, which it need not exceed. If this condition is not fulfilled the formulae become too complicated for practical use.

The bar-magnet might be placed with its magnetic axis in the horizontal north and south (magnetic) line through C, with its centre at a point P distant r from C. It should be so directed that the field h produced at C is in the same sense as H; the magnet when suspended at C will then be directed as it was at P, and no inductive correction will be required in the oscillation equation. The field h then gives a factor $1 + ah$ in (2) instead of the $1 - ah$ of the former case. We have

$$h = \frac{2M}{r^3}\left(1 + 2\frac{l^2}{r^2}\right). \quad \text{..........................(13)}$$

In this case for a given value of r nearly twice the former value of h is obtained, so that with the magnet in the north and south line through the needle a large value of r may be employed to give the same value of h as before, which is a distinct advantage.

Equation (13) is now to be used instead of (4) to obtain an equation for H from (2). Thus we get

$$H = 2\frac{M}{r^3}\left(1 + 2\frac{l^2}{r^2}\right)\frac{T_2^2}{T_2^2 - T_1^2}. \quad \text{......................(14)}$$

Again, by (3), we eliminate M and obtain

$$H^2 = 8\frac{\pi^2\mu}{r^3}\left(1 + 2\frac{l^2}{r^2}\right)\frac{T_2^2}{T^2(T_2^2 - T_1^2)}, \quad \text{..............(15)}$$

or we eliminate H and find

$$M^2 = 2\frac{\pi^2\mu r^3}{1 + 2\frac{l^2}{r^2}}\,\frac{T_2^2 - T_1^2}{T^2T_2^2}. \quad \text{..........................(16)}$$

This gives

$$h = \frac{\sqrt{2\mu}\,\pi}{TT_2}\left\{\frac{1 + 2l^2/r^2}{r^3}(T_2^2 - T_1^2)\right\}^{\frac{1}{2}}. \quad \text{.............(17)}$$

Denoting now $1 + ah$ by A, we obtain

$$H = 2\frac{M}{r^3}\left(1 + 2\frac{l^2}{r^2}\right)\frac{AT_2^2}{T_1^2 - AT_2^2}; \quad \text{....................(18)}$$

and therefore, by (3),

$$H^2 = 8\pi^2 \frac{\mu}{r^3}\left(1 + 2\frac{l^2}{r^2}\right)\frac{AT_2^2}{T^2(T_1^2 - AT_2^2)}. \quad \text{............}(19)$$

Also, by (3) and (18), we get

$$M^2 = 2\pi^2\mu \frac{r^3}{1 + 2\frac{l^2}{r^2}} \frac{T_1^2 - AT_2^2}{AT^2T_2^2}. \quad \text{....................}(20)$$

Thus (19) and (20) give H and M.

6. Method of Gauss. The most convenient method for the determination of H in a physical laboratory is that suggested by Gauss.* It consists in finding (1) the angle through which the needle of a magnetometer in the earth's field is deflected by a magnet placed in a chosen position at a given distance, (2) the period of vibration of the magnet when suspended horizontally in the earth's field, so as to be free to turn about a vertical axis. The first operation gives an equation involving the ratio of the magnetic moment of the magnet to the horizontal component H of the terrestrial field, the second an equation involving the product of the same two quantities.

7. The magnetometer. A very convenient form of magnetometer is that indicated by M in Fig. 14. Within a small closed chamber is hung, by a single fibre of washed silk, or a fibre of quartz, a small mirror with the needle, preferably a single short thin bar, either cemented to its back or, better, attached to a short rigid strip of aluminium hung from the lower end of the fibre and also carrying the mirror. The silk fibre may with advantage be replaced by a thin fibre of quartz. The suspension thread is very carefully attached to the back of the mirror so that the needle, which should be a very short piece of fine steel wire, tempered glass-hard, hangs horizontally when the front of the mirror is vertical. The closed chamber for the mirror and fibre is easily made by cutting a narrow groove to within a short distance of

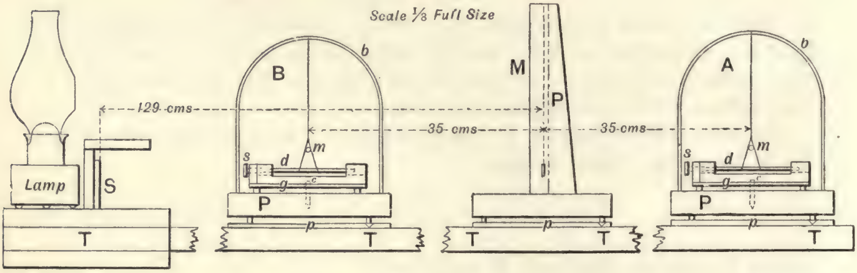

FIG. 14.

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*"Intensitas vis magneticae ad mensuram absolutam revocata."—*Comment. Soc. Reg.*, Götting. 1833.

each end along a piece of mahogany of length an inch or so greater than that proposed for the fibre. The groove is widened at one end to a circular space a little greater in diameter than the mirror. The piece of wood is then fixed, with that end of the groove down, to a horizontal base-piece of wood furnished with three levelling screws. The groove is thus made, to begin with, very nearly perpendicular to the base piece. It is then set up in a vertical position, and the fibre, to which the mirror has previously been attached, is suspended within it by passing the free end through a small hole at the upper end of the groove. It should be attached to an eye at one end of a pin of brass or copper (not shown in the cut), the other end of which is provided over half an inch with a screw thread to receive a nut. The pin is preferably made of square section with the screw thread cut on the edges, slightly rounded in the upper part to receive it. By making the pin pass through a square hole in a washer fixed to the top of the stand, the pin can be raised or lowered, without being rotated, by turning a nut fitting the screw thread and resting on the washer. Thus the height of the mirror can be conveniently adjusted.

The chamber is closed in front by covering the face of the piece of wood with a strip of glass, which may be kept in its place either by cement or by proper fastenings which hold it tightly against the wood. By making the distance between the front and back of the mirror-space small and its diameter little greater than that of the mirror, the instrument can be made nearly *dead-beat*, that is, the needle when deflected through any angle comes to rest almost at once in its true position of equilibrium, that is almost without oscillation.

This magnetometer can be constructed at a trifling cost, and is much more accurate and convenient than magnetometers furnished with long magnets, the indications of which must be reduced by the application of elaborate formulae involving quantities difficult of exact estimation.

The instrument is set up with its glass front in the magnetic meridian, and levelled so that the mirror hangs freely inside its chamber. The foot of one of the levelling screws should rest in a small trihedral hollow cut in the table, or platform, of another in a V-groove, the axis of which is in line with the vertical axis of the hollow, and the third on the plane surface of the table or platform. The hollow and V-groove are best made in brass or copper and carefully inserted in their proper positions in the wood of the platform. When thus set up with six points of support the instrument is perfectly steady, and if disturbed can be replaced in an instant in exactly the same position as before.

8. Mode of measuring deflections. A beam of light passes through a slit, in which a thin vertical cross-wire is fixed, from a lamp placed in front of the magnetometer. It passes through a small convex lens of, say, rather less than half a metre focal length, placed a little in front of the mirror, and is reflected from the mirror to a scale attached to the lampstand and facing the mirror. The lamp and scale are moved nearer to or farther from the mirror until the position at which the image of the cross-wire of the slit is most distant is obtained. The lampstand should also have three levelling screws, for which the arrangement of trihedral hollow, V-groove, and plane, should be adopted. The scale should be straight (the deflections measured are all small) and placed with its length in the magnetic north and south line; and the lamp should be placed so that when the mirror is undeflected the incident and reflected rays of light are in an east and west vertical plane, and the spot of light falls near the middle of the scale. To avoid errors due to variations of length in the scale it should be glued without stretching to the wooden backing which carries it, not simply fastened with drawing-pins. This wooden backing should be a massive piece of carefully dried and seasoned wood, the pores of which have been filled with fine spirit varnish. Scales well graduated on paper to half-millimetres can be bought; but each, after being glued and allowed to dry, should be carefully compared with a standard scale.

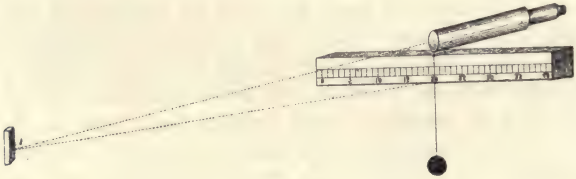

FIG. 15.

Instead of a lamp giving a ray of light, a telescope [Fig. 15] may be mounted above the centre of the scale with its axis pointing to the mirror, and focussed so that the divisions of the scale (illuminated by a lamp suitably placed) are distinctly seen, without parallax, along with a vertical cross-wire in the focal plane of the instrument. The lens in front of the mirror is of course dispensed with in this arrangement.

9. Construction of deflecting magnets. The magnetometer has now been set up and is ready for the measurement of deflections. Four or five magnets, each about 10 cm long and 1 mm thick, and tempered glass-hard, are made from steel wire. This is done as follows. From ten to twenty pieces of steel wire, each perfectly straight and with its ends carefully filed so that they are at right angles to the length, are prepared. They are tied tightly in a bundle within a serving of iron wire, and heated to redness in the heart of a mass of uniformly glowing coals. The bundle, while in the fire, should be kept in a vertical position to avoid any bending of the wires by their weight. When sufficiently heated the bundle is quickly removed from the fire and plunged with its length vertical into cold water. The wires are thus tempered glass-hard without sensible warping. They are then magnetized to saturation individually in a helix which is a good deal longer than the wire, and closely and uniformly wound with copper wire stout enough to stand a strong current of electricity. The piece of steel is placed well within the helix, and the current turned on, and allowed to flow for a few seconds. The magnets thus prepared are laid aside on a rack made by cutting narrow grooves well apart with a saw across a piece of wood, and are made to point east and west, but alternately in opposite directions, so that any inductive action between adjacent magnets may tend to augment the magnetization.

10. Placing of deflecting magnets in position. A horizontal east and west (magnetic) line is now laid down on a convenient platform (made of wood) put together without iron and extending east and west on both sides of the magnetometer. The platform is best held together with tight-fitting wooden pins, as ordinary brass screws very frequently contain a core of iron or steel. The part on which the magnetometer stands should be a little lower than the rest of the platform, so that the centre of the mirror may be on a level with a magnet laid on the platform east or west of the magnetometer. The east and west line may be laid down by drawing a line through the position of the centre of the mirror at right angles to the direction in which a long thin magnet hung by a torsionless fibre in the position of the fibre of the magnetometer places itself. [A method of more exact adjustment of this line will be described later, p. 102.]

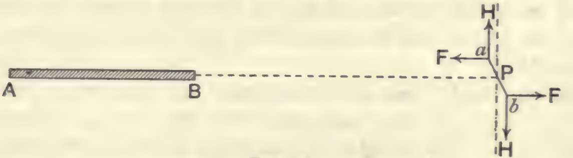

FIG. 16.

One of the magnets is placed, as shown in Fig. 16, with its length in the line thus laid down, and at such a distance that a convenient but not large deflection of the needle is produced. This deflection is noted, the deflecting magnet is then turned end for end and the deflection again noted. In the same way a pair of observations are made with the magnet at the same distance on the other side of the magnetometer, and the mean of all the observations is taken. The deflections from zero ought to be very nearly the same, and if the magnet is properly placed they will agree very exactly, so that these observations form a test of the accuracy of the arrangements. The effect of a slight error in placing the magnet is however nearly eliminated by taking the mean deflection. If there is any great discrepancy the arrangement must be carefully scrutinised to find the cause.

The same operation is gone through for each of the magnets, which are carefully kept apart from one another during the experiments by being returned to their grooves in the rack after the operations with each have been carried out. The results of each of these experiments give an equation involving the ratio of the magnetic moment of the magnet to the value of H. Thus, if M denote the magnetic moment of the magnet, M' that of the needle, r the distance of the centre of the magnet from the centre of the needle, 2λ the effective length of the magnet (for a uniformly magnetized thin bar of the dimensions stated above, the actual length), the magnetic field, in the east and west direction at the centre of the needle, and created by the magnet is

$$(M/2\lambda)\{1/(r-\lambda)^2 - 1/(r+\lambda)^2\} \quad \text{or} \quad 2Mr/(r^2-\lambda^2)^2.$$

If the needle is deflected through an angle θ from the meridian, the couple applied to it by this field is $2MM'r\cos\theta/(r^2-\lambda^2)^2$. But for equilibrium this couple must be balanced by the couple $M'H\sin\theta$, and we have the equation

$$\frac{M}{H} = \frac{(r^2-\lambda^2)^2}{2r}\tan\theta. \quad \text{.........................(21)}$$

If the arrangement of magnetometer and straight scale described above is adopted, the value of $\tan\theta$ is easily obtained, for the number of divisions of the scale which measures the deflection (whether observed by the displacement of the spot of light or by the new division brought to the vertical cross-wire of the telescope), divided by the number of such divisions in the horizontal distance of the scale from the mirror, is then equal to $\tan 2\theta$.

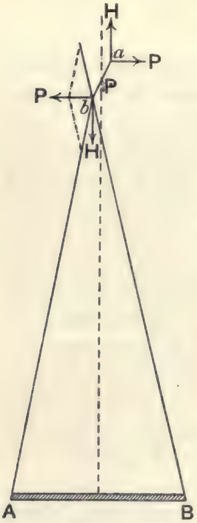

FIG. 17.

11. "End-on" and "side-on" positions A position of the magnet in the east and west line through the centre of the needle, as described above, is sometimes called an "end-on position," sometimes a "first principal position." Another position, sometimes called a "side-on position" or a "second principal position," is often employed; and it is well to make experiments in equal number for both positions. These positions are sometimes referred to as the "*A*" position and the "*B*" position. In a side-on position the magnet is placed as represented in Fig. 17 with its length east and west as before, but with its centre in the horizontal north and south (magnetic) line through the centre of the needle. [A vertical line through the centre of the needle may be used instead of this north and south line, if that is convenient.]

If we take M, M', λ, and r, to have the same meanings as before, the field produced by the magnet at the centre of the needle is east and west and of intensity $M/(r^2+\lambda^2)^{\frac{3}{2}}$. The deflecting couple exerted on the needle when the deflection is θ is thus $MM'\cos\theta/(r^2+\lambda^2)^{\frac{3}{2}}$. This is balanced as before by the restoring couple $M'H\sin\theta$, and we have the equation

$$\frac{M}{H}=(r^2+\lambda^2)^{\frac{3}{2}}\tan\theta. \qquad\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

This equation applies also to the case in which the deflecting magnet is placed side-on to the needle but above or below it.

The greatest care should be taken in all these experiments, as well as in those which follow, to make sure that there is no movable iron in the vicinity, and the instruments and magnets should be kept at a distance from any iron nails or bolts there may be in the tables on which they are placed. It is best to use tables put together without iron. No brass screws should be used, unless it has been found that they are perfectly free from iron.

12. Oscillation experiments. We come now to the second operation, the determination of the period of oscillation of the deflecting magnet when under the influence of the earth's horizontal force alone. The magnet is hung in a horizontal position in a double loop formed (by doubling twice and knotting) at the lower end of a fibre of silk or quartz, attached to the roof of a closed chamber [Fig. 18]. A box 30 cm high and 15 cm wide, having one pair of opposite sides, the bottom, and the roof of wood, put together with glue, and the remaining two sides made of glass, one of which can be slid out to give access to the inside of the chamber, answers very well. The fibre may be attached at the top to a stem which enables it to be raised or lowered by a screw as explained above (p. 93). The suspension-fibre is so placed that two vertical scratches, made along the glass sides of the box, are in the same plane with the magnet resting in its sling, and the box is turned round until the magnet is at right angles to the glass sides. The magnetometer is removed from its stand, and the box and suspended needle put in its place.

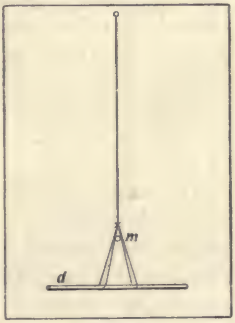

FIG. 18.

If a telescope is available it should be set up for the observation of the vibrations in the following manner. The experimenter should place it so that the line of collimation is horizontal and in the plane of the scratches. The extremity of the magnet next the telescope is then on the vertical cross-wire in the focal plane. The operator now deflects the magnet by bringing a small magnet near to it outside the box, taking care to keep this small magnet always as nearly as may be with its length in an east and west line passing through the centre of the suspended magnet. If this precaution is neglected the magnet may acquire a pendulum motion about the point of suspension, which will interfere with the vibratory motion in the horizontal plane.

When the magnet has been properly deflected and left to itself, its range of motion should be allowed to diminish to about 3° on either side of the position of equilibrium before observation of its period is begun. When the amplitude has become sufficiently small, the person observing the magnet says sharply the word "now" (or if a chronograph is used presses the key which brings into action the registering pen) when the nearer pole of the magnet is seen to pass the cross-wire in either direction, and (in the absence of a chronograph) another observer notes the time on a watch having a seconds hand. With a good watch having a centre seconds hand moving round a dial divided into quarter-seconds, the instant of time can be determined with great accuracy in this way after a little practice. [Stop watches are of little use unless they are of the "fly-back" variety, in which an indicating centre seconds hand actuated by an auxiliary movement of small inertia is started by pressing a spring. A good centre seconds arrangement of this kind may be used instead of a chronograph, in which case a second observer may be dispensed with.] The person observing the magnet again calls out sharply "now" when the magnet has just completed ten complete to and fro vibrations, again after twenty, and, if the amplitude has not become too small, again after thirty vibrations; and the other observer at each instant notes the time by the watch. If the observer at the telescope has under his control a chronograph or stop-watch, he presses the spring and stops the registering pen, or stops the indicating hand of the watch.

The observers, if there are two, then change places and repeat the same observations. In this way a very near approach to the true period is obtained by taking the mean of the results of a sufficient number of observations, and from this the value of the product MH can be calculated.

For a small angular deflection of the vibrating magnet from the position of equilibrium the equation of motion (friction neglected) is

$$\ddot{\theta}+\frac{MH}{\mu}\theta=0, \qquad \text{.............................(23)}$$

where μ is the moment of inertia of the vibrating magnet round an axis through its centre at right angles to its length. The corresponding finite equation is

$$\theta=A\sin\left(\sqrt{\frac{MH}{\mu}}\,t-B\right), \qquad \text{.......................(24)}$$

where A and B are constants, and therefore for the period of oscillation T, we have

$$T=2\pi\sqrt{\frac{\mu}{MH}}. \qquad \text{.............................(25)}$$

Hence

$$MH=\frac{4\pi^2}{T^2}\mu. \qquad \text{..................................(26)}$$

Now, if W be the mass of the magnet, $2l$ its length, and ρ its radius of cross-section, $\mu = W(l^2/3 + \rho^2/4)$, and therefore we have

$$MH = \frac{4\pi^2}{T^2} W(\tfrac{1}{3}l^2 + \tfrac{1}{4}\rho^2). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

Combining this with the equation already found for the deflection experiments made in the first or end-on position, we obtain

$$M^2 = 2\pi^2 \frac{(r^2 - \lambda^2)^2}{T^2 r} W(\tfrac{1}{3}l^2 + \tfrac{1}{4}\rho^2) \tan\theta, \quad \ldots\ldots\ldots\ldots(28)$$

and

$$H^2 = 8 \frac{\pi^2 r}{T^2(r^2 - \lambda^2)^2 \tan\theta} W(\tfrac{1}{3}l^2 + \tfrac{1}{4}\rho^2). \quad \ldots\ldots\ldots\ldots(29)$$

For a magnet 10 cm long 1 mm in diameter the term $\frac{1}{4}\rho^2$ contributes only about one part in 13,000 to the factor $\frac{1}{3}l^2 + \frac{1}{4}\rho^2$, and therefore affects the value of M or H to the extent of only one part in 26,000. For a magnet of these or similar dimensions the value of the moment of inertia may be taken as $\frac{1}{3}Wl^2$. Equations (28) and (29) may therefore be written

$$M^2 = \frac{2}{3} \frac{\pi^2(r^2 - \lambda^2)^2 l^2 W \tan\theta}{T^2 r}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(30)$$

$$H^2 = \frac{8}{3} \frac{\pi^2 l^2 r W}{T^2(r^2 - \lambda^2)^2 \tan\theta}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

These are for the end-on position. If a side-on position is chosen, we have

$$M^2 = \frac{4}{3} \frac{\pi^2 l^2}{T^2} (r^2 + \lambda^2)^{\frac{3}{2}} W \tan\theta, \ldots\ldots\ldots\ldots\ldots\ldots(32)$$

$$H^2 = \frac{4}{3} \frac{\pi^2 l^2}{T^2} \frac{W}{(r^2 + \lambda^2)^{\frac{3}{2}} \tan\theta}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(33)$$

13. Corrections in Gauss's method. Various corrections which are not here made are of course necessary in a very exact determination of H. The virtual length 2λ of the magnet should be determined, so far as that is possible as a matter of approximation, by experiment. This virtual length is only definite in an approximative sense; there are, strictly speaking, no such points as the two poles or "centres of gravity" of magnetic polarity for the two halves of a magnet. But values of λ^2 can be obtained as follows. The deflections θ, θ' of the magnetometer needle produced by the magnet, when placed in the position shown in Fig. 16 at distances r and r', are observed. We have the equations

$$\frac{M}{H} = \frac{(r^2 - \lambda^2)^2}{2r} \tan\theta = \frac{(r'^2 - \lambda^2)^2}{2r'} \tan\theta',$$

and therefore

$$\lambda^2 = \frac{r'^2\sqrt{r\tan\theta'} - r^2\sqrt{r'\tan\theta}}{\sqrt{r\tan\theta'} - \sqrt{r'\tan\theta}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(34)$$

This method is not very successful as a rule, as different pairs of distances may give different results. Other methods will be described later (see 19 below).

A small allowance only is necessary for the magnitude of the arc of vibration, if it is kept down to 6° or less, and the correction may be dispensed with unless very great accuracy is aimed at, and the observations are therefore made with an exactitude which renders such a correction justifiable. The correction for an arc of oscillation of 6° is a diminution of the observed value of T of only $\frac{1}{60}$ per cent., and for an arc of 10° of $\frac{1}{20}$ per cent.

Other sources of inaccuracy are the frictional resistance of the air to the motion of the magnet, the virtual increase of inertia of the magnet due to motion of air in the chamber, and the effect of induction and, it may be, of changes of temperature, in producing temporary changes in the moment of the magnet. The correction for induction is no doubt the most important; but its amount for a magnet of glass-hard steel, nearly saturated with magnetization, and in a field so feeble as that of the earth, may, if only a fairly accurate result is required, be neglected.

This correction arises from the fact that the magnet in the deflection experiments is placed in the magnetic east and west line, whereas in the oscillation experiments it is placed north and south, and is therefore subject in the latter case to an increase of longitudinal magnetization from the action of terrestrial magnetic force. A method of determining this increase of magnetic moment is described in 21, p. 106 below. By this it was found that the change of magnetic moment produced in hard steel bars, the length of which was 12 cm and the diameter 2 mm and previously magnetized to saturation, was found by Prof. T. Gray to be about $\frac{1}{20}$ per cent. Obviously the factor a of 5 above may be determined by such experiments. [For particulars of actual experiments see pp. 108-110 below.]

As stated above, the deflection experiments are to be performed with several magnets, and when the period of oscillation of each of these has been determined, the magnetometer should be replaced on its stand, and the deflection experiments repeated, to make sure that the magnets have not altered in moment in the meantime. The length of each magnet is then accurately determined in centimetres, and its weight in grammes; and from these data and the results of the experiments the values of M and of H can be found for each magnet by the formulae investigated above. The measurements of length may be carried out by microscopic observation of the positions of the ends of the magnet when that is placed against a finely divided standard scale of length: several determinations made with different positions of the magnet on the scale will give the length to a degree of accuracy within the limits of the unavoidable errors of the deflection and oscillation experiments. Such a series of observations is carried out for each magnet to be used as a deflector.

The object of performing the experiments with several magnets is to eliminate as far as possible errors in the determination of weight and length. The mean of the values of H, found for the several magnets, is to be taken as the value of H for the position P of the magnetometer needle. This value is to be used in calculating the values of currents from the indications of a standard galvanometer built for absolute measurement, and placed with its needle in the position P.

14. Description of actual determinations. The following account of a careful determination of H made by the method just described will form a guide to the student in arranging and carrying out the various operations. The determination was made in the Physical Laboratory of the University of Glasgow, in the summer of 1885, by the late Professor T. Gray. The apparatus and its arrangement is shown in Fig. 14. A table T supports the magnetometer M, two stands A and B for the deflecting magnets, and a lamp and scale S. The magnetometer consists of a light mirror about ·8 cm in diameter, suspended by a single silk fibre within a recess in a block of wood, and carrying attached to its back two magnets each 1 cm long and ·08 cm in diameter. Two holes cut in the wood at right angles to one another (and plugged when not in use) permit the position of the mirror and magnets to be seen and adjusted. [In later experiments a preferable form of needle was adopted. A single very small cylindrical magnet was substituted for the compound needle just described, and was carried at the lower end of a strip of aluminium, which was attached to the suspension fibre at its upper end. The mirror was cemented to the aluminium strip.] The sole plate P, made of mahogany, is supported, on three brass feet, which rest in a hole-slot-plane arrangement cut as described above, in a horizontal plate of glass cemented to the table.

15. Deflection experiments. The deflector stands A, B rest each on a base plate P, of mahogany, supported, according to the hole-slot-plane device, in precisely the same way as the magnetometer, on plates of glass p, p cemented to the table. Each stand consists of a horizontal carriage for the deflector magnet, and is constructed as follows: A strip of hard wood, about 13 cm long and 4 cm broad, has a V-shaped groove run along its length in the middle of one side. One end is faced with a plate of brass in which a brass screw works, and the piece is cemented with the groove upwards to a plate of glass g. This plate is supported on three feet of hard wood, resting on the mahogany sole plate P and is free to turn in azimuth round a closely fitting centre pivot c fixed in the sole plate. The apparatus is so adjusted that the bottom of the V-groove is just over the pivot c. The magnet when placed in the carriage lies along the groove, and the screw s serves to give a fine adjustment of one end which abuts against it. Over each carriage a wire of brass or copper bent into a semicircle serves as a support for a suspension fibre with double loop, by which the deflector can be suspended for purposes of adjustment or for the oscillation experiments. A glass shade can be placed on the plate *P* to prevent currents of air from disturbing the magnet in the oscillation experiments.

In Fig. 14 the deflecting magnets *d*, *d* are shown in positions at equal distances east and west of the magnetometer, at a distance of 70 cm between their centres. Four plates of glass are fixed to the table in two end-on positions and in two side-on positions, each pair of positions being at equal distances from the magnetometer needle, and on opposite sides of it. The scale *S*, shown at a distance from the mirror of 129 cm, is a millimetre scale carefully divided on transparent glass so that the spot of light may be observed either from the front or the back.

The first adjustment, made in setting up the apparatus, was to place the table so that the line joining the centres of *A* and *B* should be exactly at right angles to the magnetic meridian. This was done by one or other of the following two methods according as (*a*) the end-on, or (*b*) the side-on position was required. (*a*) After the adjustment had been first roughly made, a plane circuit was formed by stretching a thin wire along the line joining the centres of *A*, *B* under the magnetometer needle, and then carrying the wire back, either above the magnetometer, or below it, at a greater distance, in a vertical plane. An electric current was then sent through the wire, and the table *T*, with the apparatus, turned until the current produced no deflection of the needle. (*b*) One of the deflecting magnets was placed in its carriage, either south or north of the needle, and lifted out of the V-groove by the suspension fibre, and the table turned until the suspended magnet produced no deflection of the magnetometer needle. The magnet and needle were then in one line, and if the needle was in its proper position this line produced through the centre of the needle passed through the position of the deflector on the other side. The deflector was placed on the opposite side of the needle, and the table *T*, turned until no deflection was obtained. The position of the needle was then altered, if necessary, by the levelling screws until the positions of the table for no deflection, with the magnet first on one side then on the other of the magnetometer, were coincident. If this could not be done the plates *p* were not placed with sufficient accuracy, and their position had to be changed. This process gave the direction of the magnetic meridian with accuracy and ensured that the plates *p* in the north and south line were properly placed on the table. The two methods taken together ensured that all four plates *p* were properly placed.

16. Observation of deflections. Deflectors of different relative lengths and thicknesses, and of different degrees of hardness, were used. These were originally magnetized by placing them between the poles of a large Ruhmkorff magnet excited by a considerable current, and afterwards by the same magnet excited by a much stronger current. The relative strengths of the magnets were unchanged by the second magnetization, and their absolute strengths only very slightly. The dimensions are given in the table of results, p. 108 below. For the deflection experiments, two deflectors were used at the same time, one on each side of the magnetometer. This arrangement was more symmetrical than that of a single deflector, and, what was of very great importance, it enabled a readable deflection to be obtained with the magnets at a much greater distance from the needle, thus diminishing error due to uncertainty as to the actual magnetic distribution. As each magnet was transferred on its carriage from one glass plate to another the magnets were not handled during the experiments. One deflector A was placed (end-on) east, another B west of the magnetometer, and the plate g turned for each until their lengths were accurately in the east and west line, with their poles so pointing that each magnet gave a deflection of the needle to the same side of zero; and the deflection was then noted. The plates g were then turned through 180°, and the deflection on the opposite side of zero read off. The carriages were then turned back to the first position and the deflection again read. The difference between the mean of the first and third readings and the second reading gave twice the deflection for the position of the magnets. The same operation was then repeated with the deflectors in interchanged positions. Two similar series of observations were next made with the magnets side-on in the north and south line through the magnetometer and at equal distances on opposite sides of the needle. The mean deflection for the east and west positions, and that for the north and south positions, were calculated, and the results were used in the calculation of H in the manner described below.

17. Observation of oscillations. After the deflection observations for a particular magnet had been completed, the magnetometer was removed and the deflector stand put in its place. The magnet was suspended from the brass bow b over its carriage by a length of single cocoon fibre, in a double stirrup formed by twice doubling the lower end of the fibre and knotting. The suspension thus obtained was sufficiently fine to be practically devoid of inertia, and long enough to give a negligible moment of torsion. The magnet was deflected in the manner already described (see 12 above), and then left to oscillate. The period was observed in some cases by noting the times of the successive transits of the needle across the vertical cross wire of an observation telescope; but the method finally adopted was to attach to the stirrup as shown in Fig. 18 a light silvered mirror m (·3 cm in diameter and ·01 gramme in mass), and to use the same lamp and scale as in the deflection experiments. This latter arrangement enabled the amplitude of oscillation to be reduced to less than a degree, and so reduced to zero the correction necessary for arc. The moment of inertia of the mirror was only about $\frac{1}{40000}$ of that of the deflector, and its neglect therefore introduced an error of only $\frac{1}{400}$ per cent.

Time was observed in these experiments by means of a very accurate watch provided with a centre seconds hand moving round a dial divided into quarter seconds. When two observers were available, one counted the oscillations and called sharply "Now" at the end of every four or five periods, while the other observed the time at each call. When only one observer counted the oscillations he used a chronometer beating half seconds. Having read time, he counted the beats until he could observe a transit. He then counted the beats until he observed another transit. From the result he estimated the number of periods in one minute, and therefore observed the time of the first transit after each minute so long as there was sufficient amplitude. The fractions of half seconds were estimated from the positions of the magnet at the beat next before and the beat next after the transit. With the mirror and scale arrangement these observations could be made with great accuracy.

18. Reduction of observations. The observations were combined in the following manner * so as to give the most probable value of the period. Supposing the number of observations to have been even, $2n$ say. The interval between the nth observation and the $(n+1)$th, three times that between the $(n-1)$th and the $(n+2)$th, five times that between the $(n-2)$th and the $(n+3)$th, and so on to that between the 1st and the $2n$th were added together, the sum divided by the sum of the series $1^2+3^2+5^2+\ldots+(2n-1)^2$, and the result by the number of periods (which was the same in each case) between the observations of each successive pair. This gave the average period to a high degree of approximation. If an odd number of observations $(2n+1)$ was taken, the interval between the nth and the $(n+2)$th, twice that between the $(n-1)$th and the $(n+3)$th, three times that between the $(n-2)$th and the $(n+4)$th, and so on to the 1st and $(2n+1)$th, were added together, and the sum divided by twice the sum of the series $1^2+2^2+3^2+\ldots+n^2$. The result divided by the number of periods in each interval gave the average period. The period adopted was always the mean of those given by two closely agreeing sets of observations.

19. Correction for distribution. Assuming that the magnet has two definite poles, that is (in this connection) points at which the whole of the free magnetism in each half of the magnet may be supposed concentrated in considering the external action of the magnet (an assumption not seriously erroneous in the case of the thin magnets and the distances used); the distance between them can be calculated from the results of deflection experiments in the side-on and end-on positions obtained as described above, since the effect of the distribution is opposite in the two cases. For if r be the distance, θ the deflection, for the end-on position, and r', θ' the distance and deflection for the side-on position, we have by equating the values of M/H given by equations (21) and (22):

$$\frac{(r^2-\lambda^2)^2}{2r(r'^2+\lambda^2)^{\frac{3}{2}}}=\frac{\tan\theta'}{\tan\theta}\cdot \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35)$$

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* See any treatise (*e.g.* Merriman's) on Errors of Observation and the Combination of Experimental Results.

Expanding the numerator and denominator of each side and neglecting terms smaller than those of the second order we get :

$$\lambda^2 = \frac{r^3\theta - 2r'^3\theta'}{2r\theta + 3r'\theta'}. \qquad \text{.........................(36)}$$

By this equation the value of λ used in the calculation of H and M was found. The results for magnets of different lengths and diameters are interesting in themselves.

The moment of inertia of the bar was found by weighing the bar and carefully measuring its length and cross-section, and calculating for a vertical axis through the centre of the magnet supposed hung horizontally. The axis of suspension of the magnet in any case was not, however, that vertical, but another near it owing to the compensation for the tendency of the magnet to dip in the earth's field. The distance between these two axes can be found approximately for each magnet from the magnetic moment, mass, and length as given in the table below, and is so small that any error caused by supposing the magnet simply to vibrate round the former vertical is well within the possible limit of accuracy.

20. Theoretical results. For a cylindrical magnet of mass W, actual length $2l$ and diameter d, the moment of inertia is $W(l^2/3 + d^2/16)$. Hence (26) becomes :

$$MH = \frac{4}{3}\,\frac{\pi^2(l^2 + \frac{3}{16}d^2)W}{T^2}. \qquad \text{........................(37)}$$

Hence for a single deflector we get instead of the uncorrected equations others obtained from these by substituting, instead of l^2, $l^2 + 3d^2/16$.

If two deflectors be used, each of the actual length $2l$, and diameter d, but of masses W_1, W_2, periods T_1, T_2, and nearly equal effective lengths which give a mean, λ, we get for the end-on and side-on positions respectively :

$$H^2 = \frac{8}{3}\,\frac{\pi^2 r(l^2 + \frac{3}{16}d^2)(T_1{}^2W_2 + T_2{}^2W_1)}{(r^2 - \lambda^2)^2 T_1{}^2T_2{}^2 \tan\theta}, \qquad \text{..............(38)}$$

$$H^2 = \frac{4}{3}\,\frac{\pi^2(l^2 + \frac{3}{16}d^2)(T_1{}^2W_2 + T_2{}^2W_1)}{(r'^2 + \lambda^2)^{\frac{3}{2}} T_1{}^2T_2{}^2 \tan\theta'}. \qquad \text{..............(39)}$$

In these formulas θ and θ' are the angular deflections found from the mean readings taken as described above (p. 103) with *two* deflectors used simultaneously.

21. Corrections for alteration of moment, and for induction. There are two corrections for alteration of moment of the magnet, produced (1) by variation of temperature, (2) by induction when the magnet is in or near the magnetic meridian when oscillating. The first correction was found by placing the magnet within a bath, in one of two principal positions at such a distance from the magnetometer needle that a deflection of 1,000 divisions was obtained, and then raising the temperature through about 40° C. It was found that such a rise of temperature produced a change of deflection of only about two divisions. Thus the magnets changed in magnetic moment by only $\frac{1}{200}$ per cent. for a change of temperature of 1° C. Hence as the variation of temperature in the experiments never exceeded 2° C. or 3° C. this correction was neglected.

The correction for induction was found by immersing the deflecting magnet in an artificially produced magnetic field of known strength, and ascertaining the alteration of magnetic moment which resulted. The field was produced by surrounding the magnet with a magnetizing coil, and its intensity calculated from the number of turns of wire per unit of length of the coil and the current-strength, which was measured. The coil was sufficiently long to project beyond the magnet at each end some distance, so that the magnetic field was uniform, and equal to $4\pi nC$, where n is the number of turns per cm of length, and C the current strength in c.g.s. units. Fig. 19 shows the arrangement of apparatus for these experiments; m is the magnetometer needle, C, C' are coils each consisting of silk-covered copper wire wound on glass tubes 5 cm in external diameter, S is the lamp scale, R a box of resistance coils, G the current galvanometer, K a reversing key, and B a battery. DE represents a horizontal line through the needle and in the magnetic meridian, and AF a horizontal line at right angles to DE, and also passing through the centre of the needle. As shown in the diagram the coil C was placed with its axis parallel to AF and its centre on the line DE. C' had its axis in the line AF, and the relative distances of the coils from the magnetometer needle were so adjusted that the magnetic effect of the current passing through the coils was zero at the needle, although the current flowing was made many times greater than that used in the experiments.

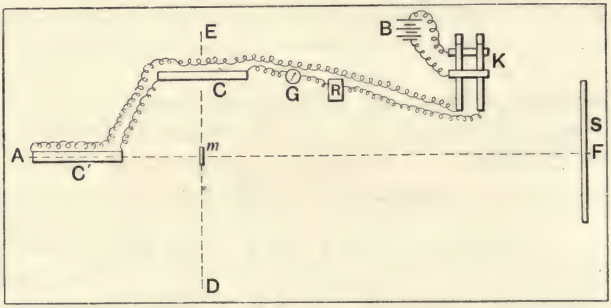

FIG. 19.

The magnet for which the induction correction was to be determined was then placed in one of the coils and the deflection read while as yet no current flowed. A field of about $\frac{1}{10}$ of a c.g.s. unit was then produced by passing a current, and the deflection was once more read. The current was then reversed, and the deflection again noted. The same operations were then repeated with greater and greater currents until a field of from 1 to 2 units had been reached. The magnet was then transferred to the other coil, and a similar series of observations made. It was found that a field of considerably greater intensity than the highest thus used is required to produce any permanent change of the magnetic moment of hard-tempered magnets. Each increase of magnetic moment being plotted as an ordinate of a curve, with the field-intensity for the corresponding abscissa, enabled the change produced by the earth's field to be obtained by interpolation in an obvious manner.

A comparison of the results obtained with the two coils showed that the percentage change of deflection produced by the field was smaller for the coil C than for the coil C'. This was undoubtedly due to change of magnetic distribution, the effect of which on the deflection is opposite in the two cases. Assuming that the magnet has an effective half-length λ, the deflection in the first case is given by (21) and in the other by (22). Thus, by using the coils in the two positions as described, the change of distribution as well as the change of moment can be approximately estimated. The plan of having two coils has also the advantage of allowing the change of magnetic moment to be obtained free from any error caused by want of exact compensation between the two coils of their direct effect upon the needle.

The results of the experiment showed that to make the effect of induction small the magnet should be hard tempered, and its length should be at least 40 times its diameter. The results are shown in the table on p. 109 below.

22. **Effects of variations of the earth's field.** The effects of variations in the intensity and direction of the earth's magnetic field were quite marked. The latter showed itself by changes of the magnetometer zero, which were eliminated by reading the zero before and after each deflection, and by reversing the magnets. The effect of change of intensity was allowed for by observing the period of a permanent magnet kept suspended for the purpose. This period was observed at the beginning of the experiment, after the deflection experiment, and again after the oscillation experiment. The necessary correction was estimated from the results and applied. It will be observed that the effect of diurnal variation is quite perceptible. The results in the table on p. 108 are tabulated in the order in which they were obtained, and it will be noticed that the earlier results of each day are generally the smaller. On some occasions on account of magnetic storms it was found impossible to obtain results at all. This was notably the case on Sept. 1, 1885.

23. **Effect in the inductive correction of varying thickness of magnet.** The results of this determination are shown in the following two tables. The variation of the effect of induction on the magnetic moment with different ratios of the length of the deflecting magnet to its diameter is shown in the curve of Fig. 20.

TABLE I.

Date 1885.		Number of deflector.	Length of deflector, in centimetres.	Diameter of deflector, in centimetres.	Weight of deflector, in grammes.	Distance of centre of deflector from the magnetometer-needle, in cm. (East and west positions.)	Distance of centre of deflector from the magnetometer-needle, in cm. (North and south positions.)	Distance of the scale from the magneto-meter mirror.	Effective length of the deflector.	Magnetic moment per gramme of the deflector.	Horizontal intensity in c.g.s. units.	Mean of each set of results.	Remarks.
May	27	1	8·03	0·25	3·054	32·06	28·75	108·7	6·91	44·9	·1520		
,,	27	2	8·05		3·063	,,	,,	,,	7·10	58·5	·1524		
,,	28	3	8·05		3·075	,,	,,	,,	6·31	54·1	·1521		
,,	29	4	8·05		3·067	,,	,,	,,	7·11	52·3	·1522	·1522	
June	5	5	4·00	0·25	1·526	30·00	25·125	,,	3·12	35·2	·1524		
,,	5	6	3·00		1.522	,,	,,	,,	3·72 ?	33·7	·1524		
,,	5	7	4·00		1·525	,,	,,	,,	2·33	31·7	·1527	·1524	
,,	10	8	14·933	0·25	5·646	51·90	38·85	,,	13·22	55·8	·1527		
,,	11	9	15·030		5·727	,,	,,	,,	12·82	64·3	·1525		
,,	11	10	15·021		5·666	,,	,,	,,	13·58	54·5	·1527	·1526	
Aug.	21	11	10·01	0·2	2·318	35·00	30·00	128·9	9·14	71·0	·1526		Double deflector method.
		12	10·01		2·336					62·7			
,,	26	11	10·01		2·318	35·00	30·00	128·9	8·98	71·0	·1527		
		12	10·01		2·336					62·7			
,,	31	13	10·000		2·318	35·00	30·00	128·9	9·14	70·0	·1526	·1526*	
		14	10·005		2·336					61·8			

* Corrected to noon for diurnal variation.

TABLE II.—SHOWING THE EFFECT OF LENGTH AND OF HARDNESS ON THE INDUCTION-COEFFICIENT OF MAGNETS.

Length of bar in centimetres.	Ratio of length to diameter.	Unit field.		Mean of numbers in columns 3 and 4.	Magnetic moment per gramme.	Remarks.
		Apparent percentage increase of moment for unit field : side-on position.	Apparent percentage increase of moment for unit field : end-on position.			
3	10	0·80	0·90	0·85	27	Glass hard.
4	16	0·67	0·73	0·70	32	,,
4	16	0·67	0·70	0·69	35	,,
6	20	0·51	0·67	0·59	36	,,
7	31	0·51	0·58	0·54	39	,,
8	32	0·51	0·58	0·54	54	,,
8	32	0·51	0·58	0·54	52	,,
10	34	0·46	0·56	0·51	40	,,
10	44	0·40	0·56	0·48	43	,,
7	47	0·46	0·51	0·49	57	,,
10	50	0·44	0·58	0·51	67	,,
10	50	0·48	0·54	0·51	60	,,
10	50	0·46	0·55	0·51	53	,,
10	50	0·46	0·52	0·49	71	,,
10	50	0·46	0·56	0·51	60	,,
10	67	0·41	0·51	0·46	65	,,
7	73	0·41	0·50	0·46	64	,,
10	105	0·42	0·45	0·44	66	,,
10	34	0·47	0·53	0·50	41·5	Glass hard.
10	34	0·63	0·67	0·65	44·5	Yellow.
10	34	0·84	0·98	0·91	54·1	Blue.
10	48	0·32	0·40	0·36	45	Glass hard.
10	48	0·43	0·55	0·49	46	Yellow.
10	48	0·53	0·67	0·60	71	Blue.

It will be observed that the effect of induction diminishes, rapidly at first, then more and more slowly, towards a constant value of about 0·4 per cent. for unit field for glass-hard magnets of the kind of steel experimented on.

CURVE ILLUSTRATING THE EFFECT OF RATIO OF LENGTH TO DIAMETER ON THE INDUCTIVE COEFFICIENT.

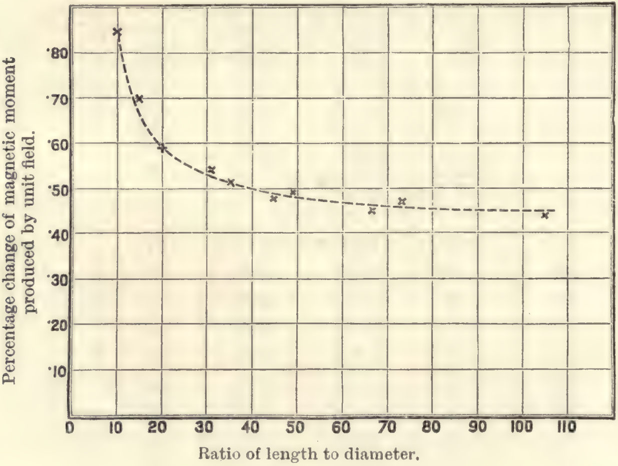

FIG. 20.

The method given above for the determination of the correction for the non-uniform magnetization of the deflecting magnet, gives of course only a first approximation to the true correction, but under the condition that the length of the bar is sufficiently small in comparison with the distance r, say from $\frac{1}{5}$ to $\frac{1}{10}$ of r, and on the supposition that the magnet is reversed at the position on either side of the needle, it is generally sufficient.

The following method eliminates to a high degree of accuracy the effect of the magnetic distribution. Let two deflections be taken by reversing the deflecting magnet at a distance r_1, on the west side of the needle, and similarly two deflections at the same distance on the east side, and let D_1 be the mean of the tangents of these four deflections. Let this process be repeated for a second distance r_2, and let D_2 be the mean tangent for this distance. It is easy to prove that, approximately,

$$\frac{2M}{H}=\frac{r_1^5 D_1 - r_2^5 D_2}{r_1^2 - r_2^2}. \quad \text{..........................(40)}$$

For if we make no particular supposition as to the distribution we may write instead of equation (21)

$$\frac{Hr^3}{2M}\tan\theta = 1 + \frac{A}{r} + \frac{B}{r^2} + \text{etc.}, \quad \text{....................(41)}$$

the series on the right converging. Therefore denoting by θ_1, θ_1', the first two deflections obtained as described above, we have

$$\frac{1}{2}\frac{Hr_1^3}{M}\tan\theta_1 = 1 + \frac{A}{r_1} + \frac{B}{r_1^2} + \frac{C}{r_1^3} + \text{etc.} \qquad \text{(42)}$$

Now reversing the magnet without altering its distance is obviously equivalent to shifting it to the same distance on the other side of the magnetometer without reversing, that is to altering the sign of r_1. Hence, by (42),

$$\frac{1}{2}\frac{Hr_1^3}{M}\tan\theta_1' = 1 - \frac{A}{r_1} + \frac{B}{r_1^2} - \frac{C}{r_1^3} + \text{etc.} \qquad \text{(43)}$$

Thus four values of $\frac{1}{2}Hr_1^3 \tan\theta/M$ are obtained which give

$$\frac{1}{2}\frac{Hr_1^3}{M}D_1 = 1 + \frac{B}{r_1^2} + \frac{D}{r_1^4} + \text{etc.} \qquad \text{(44)}$$

Similarly from the other two pairs of deflections at the distance r_2 we get

$$\frac{1}{2}\frac{Hr_2^3}{M}D_2 = 1 + \frac{B}{r_2^2} + \frac{D}{r_2^4} + \text{etc.} \qquad \text{(45)}$$

Multiplying (44) by r_1^2 and (45) by r_2^2, and subtracting, we have finally, neglecting all terms beyond the second in each equation,

$$\frac{H}{2M}(r_1^5 D_1 - r_2^5 D_2) = r_1^2 - r_2^2,$$

the relation expressed in (40).

It will be shown in the appendix on Reduction of Observations that if approximately $r_1 = 1{\cdot}32 r_2$, the effect of errors in the observed deflections on the value of M/H will be a minimum for these distances.

24. Elimination of effect of magnetic distribution. If long thin bars are used in the determination of H, their magnetic distribution could be accurately found by Rowland's method (see Index for reference) and the proper corrections applied. On the other hand, short thick bars of hard steel have the advantage of giving greater magnetic moment for a given length, and they can therefore be placed at a comparatively greater distance from the needle, so that the correction for the distribution becomes of less importance. So far, then, as the deflection experiments are concerned, it is better to use thick strong magnets of the hardest steel, and to place them at such a distance from the needle that the error, caused by neglecting the distribution, becomes vanishingly small. On the other hand, the magnets must be sufficiently long and thin to render it possible to determine with accuracy their moments of inertia, and therefore to reduce correctly the results of the vibration experiments. When the distance is so great that the effect of distribution is negligible, we may use the approximate formula

$$M=\frac{r^3}{2}H\tan\theta \quad\text{.................................}(46)$$

for the end-on position, or

$$M=r^3 H\tan\theta \quad\text{.................................}(47)$$

for the side-on position.

25. Magnetic survey. A magnetic survey of horizontal force, in the neighbourhood of a place for which H has been determined, may very readily be made with one of the magnets used in the deflection experiments, by simply observing its period of vibration at the various places for which a knowledge of H is desired. The magnetic moment M of the magnet being of course known from the previous experiments, H can be found by equation (29) or (33) above.

By keeping a magnetometer set up with lamp and scale in readiness, the magnetic moments of large magnets can be found with considerable accuracy by placing them in a marked position, at a considerable distance from the needle, and observing the deflection produced. By having a graduated series of distances for each of which the constant $\frac{1}{2}r^3H$, or r^3H, as the case may be, by which $\tan\theta$ must be multiplied to give M, has been calculated, the magnetic moments can be very quickly read off.

26. Comparison of moments of large magnets. The magnetic moments of large magnets of hard steel, well magnetized, can be compared very conveniently with considerable accuracy by hanging them horizontally in the earth's field, and determining the period of a small oscillation about the equilibrium position. They should be hung by a bundle of as few fibres of unspun silk as possible, at least six feet long, so that the effect of torsion may be neglected. The suspension thread should carry a small cradle or double loop of copper wire, on which the magnet may be laid to give it stability, and to allow of its being readily placed in position or removed. Two vertical marks are fixed in the meridian plane containing the suspension thread, and the observer placing his eye in their plane, can easily tell very exactly when the magnet is passing through the equilibrium position, and so determine the period. Or, a north and south line may be drawn on the floor or table under the magnet, and the instant at which the magnet is parallel to this line observed by the experimenter, standing opposite one end of the magnet and looking from above. An allowance for the double loop must be made, in estimating the moment of inertia. The value of M is given in terms of H by equation (26) above.

Care must of course be taken to avoid undue disturbance from currents of air, and to prevent the magnet, when being deflected from the meridian, from acquiring any pendulum swing under the action of gravity. The deflection from the meridian should be made with another magnet, brought with its length along the east and west line through the centre of the suspended magnet, near enough to produce the requisite deflection, and then withdrawn in the same manner.

27. Stroud's magnetometer for complete determination of H. A new form of magnetometer by which the determination of H is at once effected by direct observation of angular deflections, has been invented by Prof. W. Stroud. A steel ring (M of Fig. 21) is made by bending a piece of thin ribbon steel about 1 metre long, $\frac{1}{10}$ millimetre in

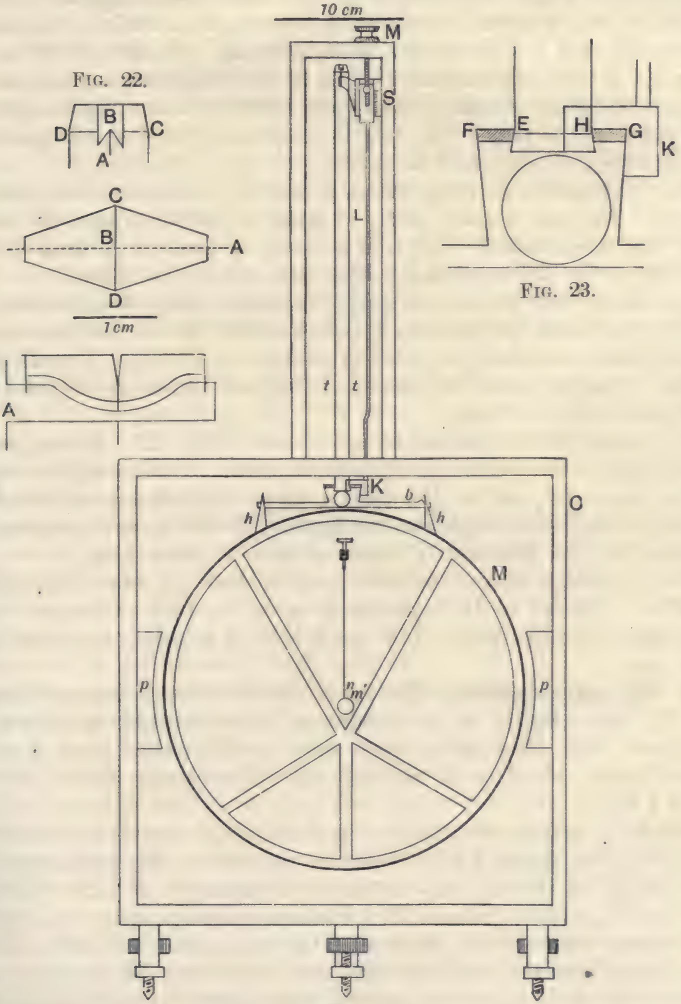

FIG. 22.

FIG. 23.

FIG. 21.

thickness, and 3 millimetres broad, into a circle, and soldering the ends together with the overlap at the top or bottom of the ring. The shape is maintained as nearly as possible a perfect circle by means of a ring of tissue paper, or, better, aluminium with connecting arms as shown in Fig. 21. This ring is hung by a bifilar suspension as described below.

When the bifilar is placed in an east and west (magnetic) vertical plane, it gives a means of measuring the couple exerted by the earth's horizontal field. That couple is proportional to MH, if M be the moment of the ring-magnet, that is, to the couple tending to turn the magnet in a field of unit intensity and of direction at right angles to the plane of the ring. The ring is magnetized, so that the poles are at the ends of a horizontal diameter.

This ring-magnet is hung within a case C, C, supported on levelling screws. The case is made partly of glass, so that the apparatus can be seen from the outside. The ring is hung by hooks h, h, from a brass crossbar b, by means of which it is attached to the bifilars t, t. The upper side of this bar is a knife-edge furnished with a V-notch near the end to receive one of the hooks h, and thus allow the wire to be removed, and replaced accurately reversed in position on the bar. A small plane mirror is carried above the centre of this bar, and serves to determine the position of the ring.

The details of the suspension are shown in Fig. 22. A is a piece of brass fixed to the wall of the instrument case. A knife-edge is worked on its upper side, and on this rests a piece of aluminium of the shape shown in the lowest diagram of the figure. To this piece is attached the bifilars, and the distance CD between them is about 1 cm.

The knife-edge bisects the distance to at least $\frac{1}{10}$ mm. The thread rests in a groove in the aluminium piece, so that the whole upper suspension arrangement is the equivalent of a pulley mounted on a knife-edge.

28. The magnetometer. The lower end of the suspension is shown in Fig. 23, and consists of an aluminium piece to which the fibres are attached. One fibre comes from above to E, passes from E to F, thence round by G to H, and then up. The distance EH is, like CD, about 1 cm.

It is to be noticed that at the top the fibres lie outside the space CD, at the bottom inside EH, so that the product of the distances of the fibres apart at the top and bottom is accurately $CD \times EH$. CD and EH are measured by means of a micrometer gauge easily to $\frac{1}{100}$ mm. Error from effect of the pressure of the gauge does not enter, as CD is measured directly, then EF, HG, and FG, giving EH by difference; so that EH is as much too great in consequence of compression produced by the gauge as CD is too small. This arrangement also eliminates error arising from the thickness and flexural rigidity of the suspending fibres.

The length of the fibres is determined as follows. A mirror *K* (Fig. 23) with a horizontal line on it is attached by a brass arm to a slider *L*, worked by a screw with milled head *M* at the top of the instrument. The screw is worked until the horizontal line on the mirror, the horizontal line given by the top of the piece *GH*, and the image of the latter in the mirror *K* behind it are in one line. By the motion of the screw, a mark on the nut at the top of the slider *L* is brought to some position on a brass scale *S* attached by brass connecting pieces to the piece *A* shown in Fig. 21. The length of the fibres is equal to the reading on the scale *S* increased by a constant quantity.

Any alteration in the length of the scale due to temperature, etc., is thus given by measurement in terms of divisions of a brass scale, so that the length can always be obtained with almost perfect accuracy. The residual temperature correction is indeed quite negligible for even large differences of temperature.

A small needle *n* is hung from an arm of brass which is attached to one side of the box, so that the needle, when in position, can hang with its centre as nearly as may be at that of the ring-magnet. A small mirror *m′* fixed at right angles to the axis of the needle is carried below it.

A forked piece of wood prevents the needle from turning round, and enables it to be placed at once very near the centre of the ring, while copper pieces *p*, *p*, on the sides of the case, damp the motion of the ring-magnet and limit the free space in which it swings to about 1 millimetre of clearance on each side.

Changes of the positions of the ring-magnet and of the small needle are read by means of a lamp and scale, or a telescope and scale, in the ordinary manner. (Of course a telescope and scale free from iron must be used.) With proper arrangement of the positions of the two mirrors a single telescope, with, if necessary, two scales, can be used to determine the deflections of both magnets.

29. Use and theory of Stroud's magnetometer. The method of using the instrument and its theory are as follows. The bifilars are adjusted so that their plane is approximately east and west, then the ring-magnet is placed in position, and the deflections of the needle and of the bar carrying the ring are read off by their mirrors. If θ be the angle which the plane of the ring makes with a vertical east and west (magnetic) plane, the magnetic couple on the ring due to H is $MH \cos\theta$. The total magnetic couple on the ring is thus $MH\cos\theta - L$, where L is a couple in the opposite direction due to the small needle at the centre of the ring. Since, if necessary, all the suspension threads may be single fibres of silk, or still better thin threads of quartz, the torsion of the bifilars may be neglected. Hence if a be the angle which the plane of the ring makes with a vertical east and west (magnetic) plane when the bifilar plane is vertical, the angle through which the bifilar has been turned is $\theta - a$, and if d, d' be the distances between the threads at top and bottom, l their length, and W the mass supported, the couple given by the bifilar is (Vol. I. p. 244) $Wgdd'\sin(\theta-\alpha)/4l$.

Hence we have

$$MH\cos\theta = g\,\frac{Wdd'}{4l}\sin(\theta-\alpha)+L. \quad\dots\dots\dots\dots(48)$$

The small needle is likewise deflected through an angle ϕ. This can be measured by observing the positions of the needle with and without the ring-magnet in the instrument.

The component of the moment M' of the small needle at right angles to the plane of the ring is $M'\cos(\theta-\phi)$. Now if we suppose a small quantity of magnetism δm of the ring to be situated at a point the radius to which makes an angle χ with the horizontal diameter through the centre, the horizontal component force due to δm will be $\delta m\cos\chi/r^2$, or $\delta m/r^3 . r\cos\chi$. It follows, if the length of the needle be taken as very small, and the breadth of the ribbon be neglected, that the moment of the couple deflecting the needle is $M'/r^3 . \cos(\theta-\phi)\,\Sigma\delta m\, r\cos\chi$, where the summation is extended throughout the whole distribution of the ring-magnet. But $\Sigma(\delta m\, r\cos\chi)$ is evidently the magnetic moment M of the ring-magnet. The couple exerted by the ring on the needle is thus $MM'\cos(\theta-\phi)/r^3$, and this is equal and opposite to the couple L exerted on the ring by the magnet.

Hence for the equilibrium of the small needle we have, neglecting the torsion of the thread,

$$\frac{M}{r^3}\cos(\theta-\phi) = H\sin\phi \quad\dots\dots\dots\dots(49)$$

or

$$\frac{M}{H} = \frac{r^3\sin\phi}{\cos(\theta-\phi)}. \quad\dots\dots\dots\dots(50)$$

But we have also

$$MH\cos\theta = \frac{Wgdd'}{4l}\sin(\theta-\alpha)+L. \quad\dots\dots\dots\dots(51)$$

If $L\,(=M'H\sin\phi)$ be small in comparison with $MH\cos\theta$, that is if $M'\sin\phi/M\cos\theta$ be a small quantity, L may be neglected in (51), and weget

$$H^2 = \frac{Wgdd'}{4lr^3}\,\frac{\sin(\theta-\alpha)\cos(\theta-\phi)}{\cos\theta\sin\phi}. \quad\dots\dots\dots\dots(52)$$

If two experiments be made with the same weight on the bifilar, but with the ring-magnet reversed, we get if $\theta'+\alpha$, ϕ', be the angular deflections of the ring and needle, respectively,

$$H^2 = \frac{Wgdd'}{4lr^3}\,\frac{\sin(\theta'+\alpha)\cos(\theta'-\phi')}{\cos\theta'\sin\phi'}. \quad\dots\dots\dots\dots(53)$$

Hence

$$\frac{\sin(\theta-\alpha)}{\sin(\theta'+\alpha)} = \frac{\cos(\theta'-\phi')\cos\theta\sin\phi}{\cos(\theta-\phi)\cos\theta'\sin\phi'}, \quad\dots\dots\dots\dots(54)$$

from which α can be found, so that H can be calculated from (52) or (53).

If the angles are all so small that they may be replaced by their sines, and the cosines may be put each equal to 1, we have

$$H^2 = \frac{Wgdd'}{4lr^3}\,\frac{\theta - a}{\phi} = \frac{Wgdd'}{4lr^3}\,\frac{\theta' + a}{\phi'} \quad \text{..................(55)}$$

or

$$H^2 = \frac{Wgdd}{4lr^3}\,\frac{\theta + \theta'}{\phi + \phi'}. \quad \text{........................(56)}$$

Hence all that is necessary is to take the angular readings before and after the reversal of the ring. The differences of the readings in the two cases are $\theta + \theta'$ and $\phi + \phi'$.

30. Order of magnitude of errors. The errors due to neglect of the couples, caused by torsion of the fibres, the couple exerted on the ring by the small needle, and the error due to uncertainty of magnetic distribution in the thickness of the wire of the ring, are not all of the same order of magnitude. The first may be made quite negligible even with silk fibres; the couple due to the small needle produced in Prof. Stroud's first instrument, which had a ring of pianoforte steel wire, gave an effect of about 1 to 700, and the thickness of the wire gave a possible extreme error of about 1 in 300. The two latter couples are made negligibly small by increasing M sufficiently, and making the ring of thin steel strip instead of wire. Of course the couple due to the small needle can always be approximately determined and allowed for.

31. Results obtained with trial instrument. The following table (p. 118) contains examples of determinations of H made by Prof. Stroud with his first trial instrument.

Other methods of determining H which depend on current induction will be explained in a later chapter.

32. Errors in usual magnetometer arrangements. In the usual form of magnetometer the magnetizing solenoid is placed with its axis in the magnetic east and west line passing through the magnetometer needle. The effect of the current is balanced at the needle by means of a compensating coil connected up in the circuit. This latter coil has its axis coincident, or nearly so, with that of the solenoid. When a feeble magnetic specimen is under examination the solenoid, and consequently the compensating coil, must of necessity be brought up close to the needle. If large magnetizing currents are employed, any small shift of the coils from their correct positions may be sufficient seriously to impair the balance. In consequence of this the operation of adjusting the position of the compensating coil (the solenoid is usually clamped once for all in a convenient position) is a difficult one, especially as the slight inevitable movement of the coil which results from clamping it in position generally results in disturbance of the balance.

Even if this adjustment be accomplished with the requisite accuracy for the undisturbed position of the magnetometer needle, it does not necessarily follow that the compensation is complete for the needle in its deflected position. In practice, the axes of the solenoid and

	I. May 17, 1890.	II. May 20, 1890.	III. May 20, 1890.
Length of scale	100 cm	200 cm	Same constants as in II.
Distance from centre of inst.	97 ,,	200 ,,	
Length of bifilars	30·07 ,,	27·81 ,,	
Distance apart above	1·293 ,,	1·293 ,,	
Distance apart below	1·314 ,,	1·302 ,,	
Diameter of ring	27·50 ,,	27·50 ,,	
Reading for ring before reversal	10·33 ,,	153·66 ,,	153·69 cm
Reading for ring after reversal	52·59 ,,	77·34 ,,	77·72 ,,
Difference of readings	42·26 ,,	76·32 ,,	75·97 ,,
Reading for needle before reversal	6·21 ,,	191·76 ,,	192·51 ,,
Reading for needle after reversal	88·69 ,,	25·40 ,,	26·67 ,,
Difference of readings	82·48 ,,	166·36 ,,	165·84 ,,
Mass suspended from bifilars	11·37 grammes	11·89 grammes	11·89 grammes
Value of *H* in c.g.s. units, from these data	0·1803	0·1805	0·1803

compensating coils are in general slightly inclined to one another and to the east and west line passing through the needle. The effect of this is to increase the directive force on the needle for one direction of the current and to diminish it for the other. That this is the case will be seen from Fig. 24, in which the want of alignment of the coil and solenoid has been greatly exaggerated. The magnetometer needle is situated at the point *P*, and it has been assumed that the solenoid and coil are so placed that they produce fields at *P* in the directions *PS* and *PC* respectively. If the intensity of the field due to the solenoid be denoted by F_s, and that due to the coil by F_c, then, since the coils balance for the undisturbed position of the needle it follows that $F_s \cos \theta_1 + F_c \cos \theta_2 = 0$. There are left, however, the components of the intensities in the north and south direction, and it is evident from the figure that if *H* is the horizontal component of the earth's magnetic field at *P*, the total directive force at the needle is

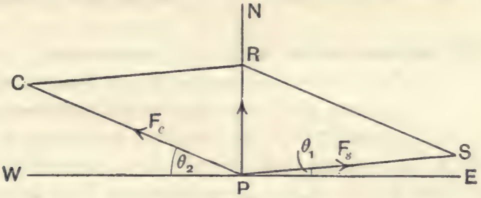

FIG. 24.

$$H + (F_s \sin \theta_1 + F_c \sin \theta_2).$$

If the current is reversed in the circuit the directions of F_s and F_c change, and the directive force at the needle becomes

$$H - (F_s \sin \theta_1 + F_c \sin \theta_2).$$

The presence of the effect referred to may be made apparent by placing a permanent magnet close to the magnetometer, and thus deflecting the needle. On reversing a current in the circuit, a change in the deflexion will in general be observed. The magnitude of the errors introduced may be determined in this way for various parts of the scale and allowed for in the results, or the coils may be rotated until the effect disappears. If the former method is adopted, the labour of computing the results is much increased, and, further, it is difficult to make a proper correction, since the allowance to be made is a function both of the angle of deflexion and the strength of the current. The second method can only be used if the coils are capable of being rotated on their stands, and the adjustments would be difficult and troublesome to carry out.

The necessity for attending to this source of inaccuracy was first pointed out by Erhard,* who investigated the magnitude of the errors which were caused by neglecting it. In a magnetometer of the usual type examined by him, it was found that, with a magnetizing field of 128·3 c.g.s. units in the solenoid, there was a change of 6·8 per cent. in the directive force on the needle when the current was reversed. Erhard advised that the magnitudes of the errors introduced should be determined for various parts of the scale and allowed for in the results.

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* "Eine Fehlerquelle bei magnetometrischen Messungen," *Ann. der Phys.* 1902, p. 724.

33. Improved magnetometer table and accessories. While carrying out a research on certain feebly magnetic alloys Messrs. Gray and Ross found that the elimination of these sources of error caused very considerable delay in the progress of the work. An attempt was therefore made to design a form of magnetometer which should be free from the objections common to the usual instruments. The following requirements were kept in view: (1) capability of accurate and rapid adjustment; (2) zero Erhard effect; (3) suitability for testing specimens at all temperatures; (4) applicability to the testing of strongly magnetic and feebly magnetic specimens alike; (5) a solenoid capable of furnishing fields up to at least 400 c.g.s. units; (6) rigidity (and for this all the parts were fitted on one bed-plate); (7) clamping arrangements without influence on the compensation.

The general principle of the instrument made will be seen from Fig. 25. *ns* represents the magnetometer needle provided with a concave mirror, by means of which and a source of light L, its movements are observed on the scale SS. H is the magnetizing solenoid placed due east or west of the magnetometer needle and clamped in a convenient position. C_1 and C_2 are compensating coils placed with their axes approximately in coincidence with that of the solenoid. In adjusting the apparatus the effect of the current in H on the needle *ns* is first approximately annulled by means of C_1, which is then clamped in position. The final adjustment of the compensation, so far as the undisturbed position of the needle is concerned, is carried out by means of C_2, which on account of its great distance from the needle contributes only a small fraction of the balancing field, and thus provides adjustment. The position of C_2 necessary for balance having been obtained, it is clamped in position; obviously, since the distance of C_2 from the needle is great, any slight movement caused by doing so produces no perceptible effect on the compensation.

34. Arrangement of compensating coils. If the axes of C_1, H, and C_2 were coincident and passed through the magnetometer needle, the adjustment would now be complete. If, however, the needle *ns* is deflected by means of a permanent magnet, and a large current is reversed in the circuit, in general an alteration in the scale-reading on SS will be observed. A coil C_3, placed with its axis in the magnetic north and south line passing through the needle, is now included in the circuit. By properly adjusting the direction of the current in C_3, and altering the distance of C_3 from the needle, the compensation can be made perfect for all positions of *ns*.[*] In a magnetometer where *ns*, C_1, H, and C_2 are all carried on stands moving in one channel in the bed-plate, there should be little departure of the axes of the coils from coincidence. Accordingly the resultant magnetic field, due to the coils and solenoid, in the north and south direction will be small. The coil C_3 is therefore made of little power, and a small change in its position brings about only a very slight alteration in its effect upon the needle. It can therefore be clamped without any risk of upsetting the balance. The manner of making the adjustments will be fully explained later.

The instrument with its fittings is shown in plan in Fig. 26 (p. 122). The bed-plate is in the form of a cross, and is built of well-seasoned mahogany planks 22 cm broad and 2·5 cm thick. The length over all is 350 cm, and the breadth from end to end of the arms 135 cm.

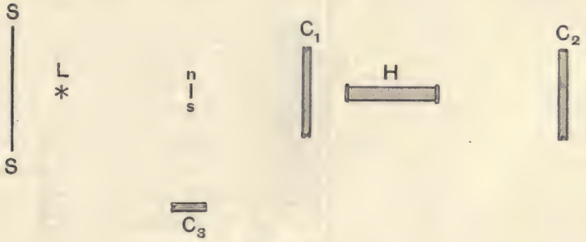

FIG. 25.

The cross-piece is at a distance of 100 cm from one end of the main length. Like the main portion of the bed-plate, it is formed from one piece of wood, and the two lengths are set accurately at right angles and half checked into one another. The junction is made rigid by means of glue and brass screws. A channel 11·5 cm broad is formed over the entire length of the cross-piece by means of two mahogany strips which are square in section and fixed parallel to the edges of the arms. A similar channel runs down the main length of the bed-plate, being discontinued where it is crossed by the channel already mentioned. The wooden strips forming these channels are permanently fixed by glueing and by brass screws driven in from the under side of the base-board. After they have been constructed they are made of perfectly uniform width by sand-papering, the width being tested from time to time during the process by means of a wooden gauge.

A is a mahogany box consisting of bottom, sides, and top, with the ends which face east and west left open. In the bottom is a slot running parallel to the cross-piece of the bed-plate. A brass screw projecting upwards from the base-board of the magnetometer passes through this slot and is provided with a brass washer and locking-nut. By this means the box can be moved through a small distance in the north and

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[*] A side coil has been used by Dr. G. E. Allan in his magnetometric work for giving compensation throughout the scale.

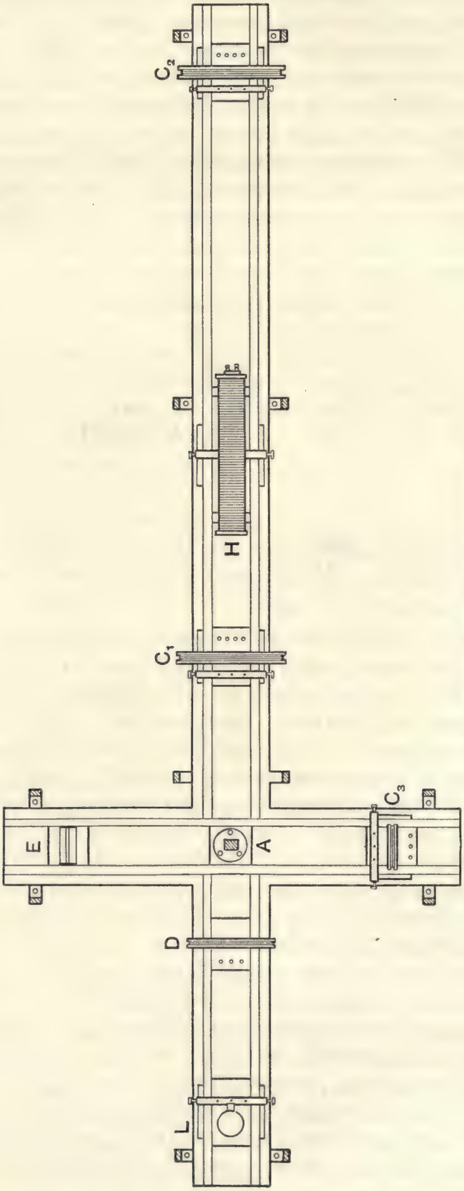

FIG. 26.—Plan of Magnetometer.

south direction, and securely clamped in position. On the upper surface of the box is fastened a plate of glass on which stands the magnetometer proper. This part of the instrument is also constructed of mahogany. A wooden pillar 20 cm in height has a narrow hole drilled longitudinally down through it. This hole terminates in a small cell with a glass window in front. The cell is just large enough to contain the mirror of the magnetometer—a concave mirror, 1 cm in diameter, having a focal length of 50 cm. The mirror has attached to its back a small piece of magnetized watch-spring about 8 mm in length. The needle and mirror are suspended by a fine quartz fibre from a screw at the top of the upright pillar of the magnetometer. By means of this screw, the axis of which is vertical, all torsion can be removed from the fibre when the needle is hanging in its equilibrium position; and by giving the screw an observed number of complete turns a determination of the torsional rigidity of the fibre can be made. The pillar of the magnetometer is attached to a circular base provided with three small brass levelling-screws. The position of these feet on the glass top of the box-stand is defined by the hole, slot, and plane method.

AA (Fig. 29) is the magnetizing solenoid. Two coaxial brass tubes 45 cm in length are connected at their ends by brass rings so as to form a water-jacket *BB* measuring 4 cm in internal and 6 cm in external diameter. On the outside of this are wound 868 turns of No. 15 s.w.g. copper wire in four layers, of which only one is shown in the figure. The wire is double silk-covered, and each layer is varnished over after it is wound. The terminals of the coil are mounted on an ebonite block at one end of the solenoid. *D* and *C* are the inlet and outlet tubes of the water-jacket. Although the water-jacket is somewhat narrow it is found to be effective in keeping the helix of wire cool, even though the interior is raised to a temperature of over 1000° C. by means of an electric furnace. The water-jacket is made small in capacity in order to keep down the mean radius of the solenoid, and hence maintain the end effect of the solenoid small. The field at the centre of a coil of length $2l$ and radius a is less than $0{\cdot}4\pi nC$ in the ratio $l/\sqrt{l^2+a^2}$, where n is the number of turns in the coil per unit length and C is the magnetizing current in amperes. In the case of the solenoid now described the reduction in the field from the value $0{\cdot}4\pi nC$ due to the finite length of the coil is 1·14 per cent. The solenoid is carried on a mahogany base-board provided with two vertical supports terminating in V-shaped grooves to receive the coil. The position of the solenoid carrier in the channel of the magnetometer board may be fixed by means of a brass clamp (shown in Fig. 26). This friction clamp is furnished with two screws which press mahogany blocks against the outer surface of the wooden strips forming the channel of the magnetometer bed-plate.

C_1 and C_2 (Figs. 25 and 26) are circular coils of 15 cm radius erected on wooden stands provided with brass clamps as in the case of the solenoid. Each coil is wound in three sections, the terminals of which are screwed into the base of the stand. The sections in the case of C_1 contain 5, 7, and 9 turns of wire respectively, and in the case of C_2 6, 8, and 10 turns. These sections may be used singly or in combination, and accordingly there is a wide range of variability in the powers of the coils. C_3 is a coil of similar construction, but has a radius of only 6 cm, and is built in two sections of 1 and 3 turns of wire respectively.

D is a coil having a radius of 12 cm, and its function is to prevent loss of time due to vibrations of the needle about its position of equilibrium. It is connected up in series with a single cell and a reversing key ; and by properly tapping the key a series of impulses is communicated to the needle, which is thus quickly brought to rest.

L is a sliding stand carrying the object screen, which consists of a vertical wire placed in front of a window of obscured glass fitted in a metal box containing an electric lamp. By altering the position of this stand, the image of the cross-wire formed by the mirror of the magnetometer can be produced at any distance from 110 cm upwards. From 150 cm to 200 cm is in most cases a suitable value. At this distance it is received on an engine-divided glass scale of the usual type.

E is a deflector stand on which a small permanent magnet may be mounted in the " side-on " position of Gauss. The construction of the stand is similar to that of the stand which carries the magnetometer proper. On the top of it is fixed a rectangular block of wood provided with a groove for receiving the magnet.

The bed-plate of the magnetometer is mounted on six pairs of mahogany feet, which are fastened to a rigid table by means of brass screws.

35. Adjustment of the instrument. The process of setting up the apparatus is as follows. The centre of the magnetometer needle has first to be placed on the axis of the solenoid. To accomplish this, coil C_1 (Fig. 26) is removed, and the solenoid *H* is moved along the bed-plate towards *A* until its inner end is almost in contact with the back of the magnetometer casing. The stand *A* is then moved in its channel until the needle is brought exactly on the axis of the helix, and is then permanently fastened in this position by means of the clamping screw already mentioned. The table carrying the magnetometer is now placed so that the long channel of the bed-plate lies due east and west, the adjustments being carried out and tested by means of the following method. A wire is stretched out vertically beneath the needle, and accurately parallel to the short channel of the bench. On passing a current through this wire a deflexion of the needle is produced. If the current is reversed in direction the deflexion will have the same numerical value as before, provided that the wire lies exactly north and south magnetic. The table is so placed that this condition is fulfilled, and its feet are then clamped to the floor by means of L-shaped brass brackets. The scale is erected on a separate table in order that the movements of the observer may not set up oscillations of the needle. The coils C_1, H, and C_2 are now connected up in series with the storage-battery, ammeter, and variable resistances, etc., care being taken that the direction of the current in C_1 and C_2 is opposite to that in H. The permanent adjustments of the instrument are now complete.

When a specimen is tested the solenoid H is moved to a convenient distance from the magnetometer needle and firmly clamped. The coil C_2 is placed at the farther end of the magnetometer table, and a current two or three times greater than the maximum to be used in the subsequent test is sent through the complete circuit. Coil C_1 is then moved until it just falls short of balancing the effect of the solenoid on the needle. It is then securely clamped. Coil C_2 is next slowly moved up towards the magnetometer needle until the deflexion of the latter is brought exactly to zero ; C_2 is now clamped, and the accuracy of the compensation verified by suddenly reversing the current in the coils. No measurable change in the scale-reading should result. The current having been interrupted, a small permanent magnet is next placed east and west on the stand E, and the stand moved along the cross channel in the magnetometer bed-plate until the spot rests near one extremity of the scale. The current is again made and reversed, and, if any appreciable deflexion of the spot on the scale is observed, coil C_3 is included in the circuit, the current through it so directed that the deviations of the needle from its equilibrium position are diminished. The coil is gradually moved closer to the magnetometer until the Erhard effect is completely annulled, and is then clamped in position. The compensation now holds for all parts of the scale, and the apparatus is ready for carrying out magnetic tests.

The several sections in which the three compensating coils are built allow the adjustment to be completely made with the coils in several different positions. This is a great advantage, as it always affords a means of escape from any arrangements of the coils which might prove awkward when specimens are in the solenoid.

The method of adjustment of the coils for balance, in the manner described above, is systematic, delicate, and accurate, and the operations can be carried out with great rapidity ; unless the solenoid is very close up to the magnetometer, the changing over of the apparatus from one degree of sensibility to another can be completed in about two minutes.

The magnitude of the directive force at the needle is easily determined by passing a measured current through one of the balancing coils and noting the deflexion of the magnetometer needle produced. The value of the directive force is then easily calculated.

Fig. 27 is a photograph of the apparatus when adjusted for the examination of a strongly magnetic specimen ; Fig. 28 shows the arrangement when a feebly magnetic specimen is being dealt with. When the solenoid has to be placed very close to the magnetometer needle to allow of a very feebly magnetic specimen being examined, the coil C_1 is placed on the opposite side of the needle to the solenoid. For general use, however, it is convenient to have the solenoid and coil on the same side. It is worthy of remark, in passing, that even if C_1 is placed as close up as possible to the end of the solenoid, it cannot alter the field at the centre of the specimen by so much as $\frac{1}{4}$ per cent.

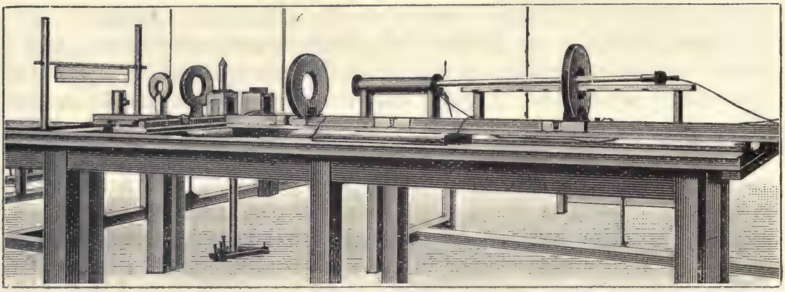

FIG. 27.

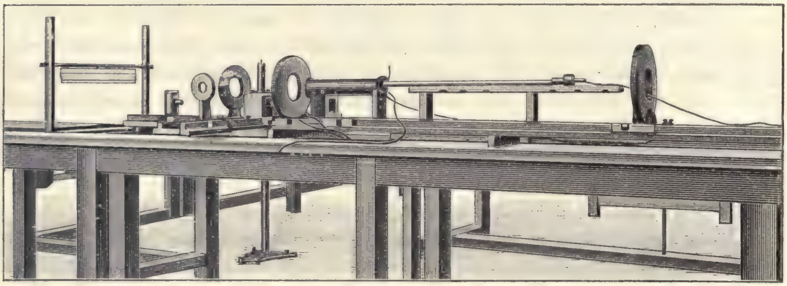

FIG. 28.

36. Testing of specimens at different temperatures. When used for testing specimens at temperatures higher than that of the room, an electric furnace of a type similar to that devised by Dr. G. E. Allan,* is placed within the helix. In Fig. 29 it is shown in position. A tube *E* of unglazed porcelain of about the same length as the solenoid, having an internal diameter of 23·5 mm and a thickness of about 2 mm, is wound non-inductively with fine platinum wire; the ends of this wire are brought out to two terminals mounted on a slate frame at *F*. The tube is enclosed in a tube *G* of Jena glass, which fits as a cartridge within the magnetizing solenoid. The space *HH* between the glass and procelain tubes is packed with dry kaolin clay, which performs the double duty of supporting the furnace and preventing the coils of the platinum wire from changing their positions when expanded by heat.

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* *Phil. Mag.* 1904, vol. vii. p. 46.

A cylinder of electrolytic sheet-copper is placed within the tube *E*, and assists in maintaining a very uniform temperature over the space occupied by the specimens.

In the figure the platinum wire is shown equally spaced over the porcelain tube. In reality this is far from being the case. The proper winding of the tube is an exceedingly troublesome operation, and can only be accomplished by repeated trial.

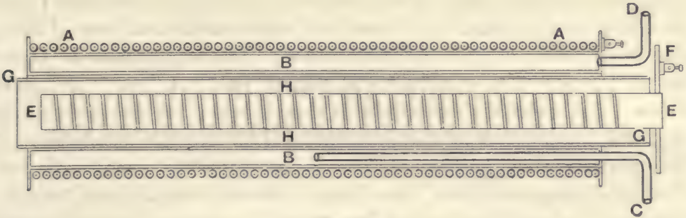

Fig. 29.

The temperature of the furnace is measured by means of the ordinary thermo-element or a platinum resistance thermometer. The two wooden stands used for the pyrometer are shown in position in Figs. 27 and 28. As will be seen at once, the several slots in the horizontal carrier for fitting on the tops of the stands permits of these latter being placed clear of the sliding bases of the compensating coils.

For tests at the temperature of liquid air the arrangement shown in Fig. 30 is employed. The specimen *A* is enclosed in a glass tube *BCD*, of which the end *B* is closed and the end *D* is open and bent up. Cork stoppers *F*, *F* are fitted on the tube so as to bring the axis of the specimen into coincidence with that of the solenoid. A third stopper *F* or a pad of cotton-wool is used to prevent access of warm air into the interior of the solenoid, and a covering of cotton-wool on the portion *CD* prevents it from warming up and conducting heat to the specimen. Instead of closing the glass tube at *B*, a cork may be used to stop up the opening. The cork, however, if dry, is apt to loosen and permit the liquid air to leak out, or if it is at all damp it expands and fractures the tube.

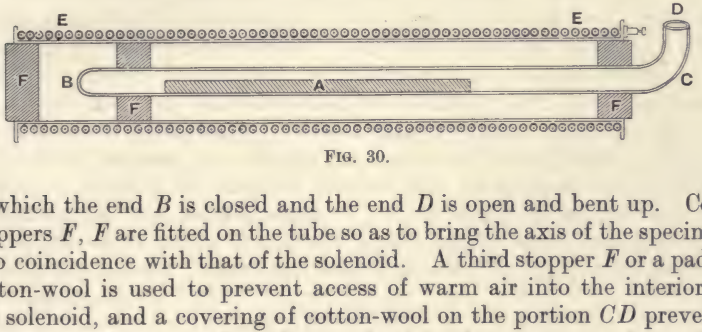

Fig. 30.

Where tests have to be made as the specimen slowly warms up from the temperature of liquid air, a Dewar tube is used, with its mouth closed by a cork which has two bent tubes passed through it—one for the pouring in of the liquid air, and the other for the bringing out of the leads from one or more thermo-elements in contact with the specimen.

The dimensions given above for the internal diameter of the solenoid will be found sufficient for receiving a double vacuum Dewar tube for tests at – 252° C. on specimens immersed in liquid hydrogen.

A slightly modified form of the stand supporting the solenoid permits of the latter being carried in an east and west position on one of the arms of the cross-piece of the magnetometer. The apparatus is therefore available for use with specimens in either the "*A*" or "*B*" position of Gauss; the methods described in this chapter for the determination of the effective lengths of the specimens thus become available.

The considerable height of the magnetometer needle above the level of the magnetometer base-board (18 cm) would also permit the apparatus to be readily adapted for testing by the "one-pole" method.*

Several instruments of the above type have been built in the Physical Institute of the University of Glasgow, and very greatly facilitate accurate magnetic testing.

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* See Ewing's *Magnetic Induction in Iron and other Metals*, p. 39.

CHAPTER IV.

CURRENTS IN DERIVED CIRCUITS AND IN A NETWORK OF LINEAR CONDUCTORS.

1. Steady flow in linear conductors. It is proposed to give in this chapter some general results of theory which will be useful in the discussions which more immediately follow. A special chapter on Comparison of Resistances will be given later.

We suppose that by means of a battery, or thermopile, or some form of magneto-electric machine or dynamo, an electromotive force (e.m.f.) E is maintained in a circuit, two points of which A, B are joined by a network of linear conductors. In one or more of these conductors we may assume that electromotive forces of specified amount have their seat. If the conductors are in a magnetic field it is to be understood that they are not anywhere in motion relatively to the field, unless it is explicitly stated that they are.

It is outside the purpose of this book to describe batteries or other arrangements for the production of electromotive forces, or to discuss the origin of electromotive force: only such matters as relate to the theory or practice of electrical measurements can be dealt with.

2. Ohm's law. At the foundation of the theory of flow of electricity in a network of conductors is the law given by Ohm. This law applies in the first instance to a linear conductor of homogeneous material (a wire of uniform or variable section) and at uniform temperature throughout, at two cross-sections, S_1, S_2, in which potentials V_1, V_2 are maintained by means of an e.m.f. which has its seat, or origin, elsewhere in the circuit. Ohm's law asserts that if γ be the current flowing in the conductor

$$\gamma = \frac{1}{R}(V_1 - V_2). \qquad \text{.............................}(1)$$

Here $1/R$ is a coefficient of proportionality, in other words, a constant multiplier; for unless the physical properties of the conductor are altered, for example by the influence of heat produced in the conductor by the current, the value of R remains unaltered. Ohm's law asserts therefore that in the circumstances stated the current is proportional to the difference $V_1 - V_2$ of potential maintained between the two cross-sections. The quantity R is called the resistance of the conductor.

3. Ohm's law in a heterogeneous circuit. Equation (1) is not fulfilled in general by a conductor made up of homogeneous parts of different materials placed end to end, or by a conductor moving in a magnetic field. For such cases the equation is

$$\gamma = \frac{V_1 - V_2 + e}{R}, \qquad (2)$$

where V_1, V_2 have the same meaning as before, and R is the sum of the resistances of the homogeneous portions of the conductor, in the former case, or the actual resistance of the conductor in the latter.

The conductor in such cases is said to contain, or to be the seat of, an e.m.f. e, or (as frequently in what follows) an e.m.f. e is said to be in the conductor. Since in a heterogeneous conductor (1) applies in the first case to every part, except any, however small, which includes a surface of discontinuity, the e.m.f. has its seat at the surface or surfaces of discontinuity. In the case of a conductor between which and a magnetic field there is action due to relative motion, the e.m.f. has its seat in every part of the conductor moving in the field, or across which lines of induction are passing; the laws of this action are set forth in treatises on electricity, and we shall have occasion to touch on it only in connection with certain problems of determination of electrical constants. We have already done so in Chapter I. above, on Units and Dimensions.

4. Arrangement of a battery. We now consider a battery consisting of a number N of cells each of e.m.f. E and internal resistance r, and made to send a current through an external resistance R. If N be the product of two whole numbers m, n, the cells can be joined in m parallel rows, each made up of n cells joined in series, all facing of course the same way, and if the m terminals at each extremity of the set of rows be joined by thick wires or bars of negligible resistance, each column of m cells will be equivalent to a single cell of internal resistance r/m. No current will flow in the arrangement, and the electromotive force of the set of m rows will be nE. Thus we have a battery of e.m.f. nE and internal resistance nr/m. If then R_1, R_2 be the resistances of the connections joining the battery to the conductor of resistance R, we get for the current in this latter conductor

$$\gamma = \frac{nE}{R + R_1 + R_2 + \frac{n}{m} r},$$

or, as we may write the equation,

$$\gamma = \frac{mnE}{m(R + R_1 + R_2) + nr}. \qquad (3)$$

The numerator of the expression on the right of (3) does not change with alteration of the mode of joining the cells, and there are twice as many ways of doing so as there are different pairs of factors in N. The total external resistance is $R+R_1+R_2$, which, for brevity, we shall denote by the single symbol R, so that (3) becomes

$$\gamma=\frac{mnE}{mR+nr}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

If it be desired to join the battery in such a way as to produce the greatest possible current in the external part of the circuit, we have a problem which in general can be solved practically, but for the solution of which the theory of maxima and minima values of a function of given variables is not directly available. The variables here are m, n, and these, being restricted to be whole numbers, do not vary continuously. If, however, N be resoluble into different pairs of factors, we can find which pair more nearly fulfils the condition obtained when continuous variation is assumed.

Since mnE is invariable the value of γ is a maximum when $mR+nr$ is least, subject to the condition that $mn=N$. Now

$$mR+nr=(\sqrt{mR}-\sqrt{nr})^2+2\sqrt{mn\,Rr}.$$

Thus clearly, since $2\sqrt{mn\,Rr}$ is invariable, $mR+nr$ is least when $mR=nr$, that is when $R=nr/m$. When this condition is fulfilled the external resistance R is equal to the internal resistance nr/m of the battery as arranged. It may not be possible in practice to join a given battery so as to fulfil this condition, but if the strongest possible current is required it should be fulfilled as nearly as possible by choosing the most favourable pair of factors of N.

The arrangement which gives the strongest possible current is not, however, an economical one. The whole activity in the circuit is $nE\gamma$, and this, since no work is done against back e.m.f., is the rate at which heat is produced in the circuit. The rate of working in the external part of the circuit is $mnE\gamma R/(mR+nr)$, which is a maximum under the same condition as obtains for a maximum of γ. Thus the rate of evolution of heat in the external resistance is a maximum when the relation $mR=nr$, is as nearly as possible fulfilled. But if this relation is exactly fulfilled just as much energy is spent in heat within the battery itself as in the external resistance; and it is plain that for economy as little as possible of the energy of the battery must be spent in the battery itself, and as much as possible in the working part of the circuit. Hence for economical working (without regard to cost of conductors) the internal resistance of the battery and the resistance of the wires connecting the battery with the part of the circuit (electric lamps for example) in which useful work is done, must be made as small as possible. The cost of conductors limits the application of this principle of purely electrical efficiency; and it is obvious that when this cost is taken into account the most economical arrangement is obtained when the annual cost of the connecting cables (that is provision for deterioration and interest on capital expended) is just equal to the cost of the energy wasted in heat in these connections.

The theorem discussed above applies only to the case (not at all common) in which we have a *given* battery, and are obliged to arrange it so as to produce the *greatest* current through a given external resistance R. It is a fallacy to suppose, as is sometimes done, that of two batteries of equal e.m.f., but one of which has a high, the other a low resistance, the former is better adapted for working through a high external resistance.

5. Networks of linear conductors. We now consider the theory of a system of linear conductors in which steady currents are flowing. When a steady current flows across any cross-section of a conductor, the current strength is the same across every other cross-section; in other words, at any instant the rate of flow of electricity into any portion of the conductor is equal to the rate of flow out of that portion. This is the principle of *continuity* as applied to the case of a *steady* flow of electricity. By the same principle we have, in the case in which steady currents are maintained in the various parts of a network of conductors, the theorem that the total rate of flow of electricity *towards* a point at which several wires meet is equal to the total rate of flow *from* that point. Thus in Fig. 31 the current arriving at A by the main conductor is equal to the sum of the currents which flow from A by the conductors which connect it with B.

If, as we here suppose is the case, the wires connected at A and B (Fig. 31) be of the same material, no question of difference of potential due to contact of dissimilar substances arises. If then A, B have maintained between them a difference of potential V, while connected by, say, two wires of resistances r_1, r_2, the current through the wire of resistance r_1 will be V/r_1, and that through the other wire will be V/r_2. Hence if γ be the whole current arriving, say at A, and flowing away from B, we have, by the principle of continuity,

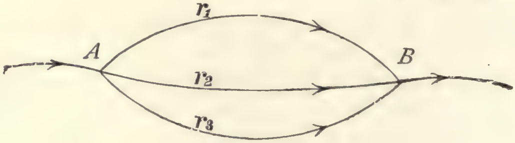

FIG. 31.

$$\gamma=\frac{V}{r_1}+\frac{V}{r_2}=\frac{V}{R}, \quad \text{......................(5)}$$

where R is the resistance of a single wire which might be substituted for the two wires between A and B without altering the current γ. Hence

$$\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}, \text{ or } R=\frac{r_1 r_2}{r_1+r_2}. \quad \text{......................(6)}$$

The reciprocal of the resistance R of a wire, that is, $1/R$, is called the conductance of the wire. Equation (6) therefore affirms that the conductance of a wire, the substitution of which between A and B would give with the same difference of potential the same current γ between these points, is the sum of the conductances $1/r_1$, $1/r_2$ of the two wires. It follows, as also stated in (6), that the resistance R of this equivalent wire is equal to the product of the resistances r_1, r_2 divided by their sum.

This result is easily generalized by adding wires (Fig. 31) of resistances $r_3, r_4, \ldots$, one at a time, between A and B, and finding the conductance and resistance in each case of the multiple connection. Thus we get finally for n wires,

$$\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\ldots+\frac{1}{r_n} \quad \ldots\ldots\ldots\ldots(7)$$

and

$$R=\frac{r_1 r_2 r_3 \ldots r_n}{r_1 r_2 \ldots r_{n-1}+r_2 r_3 \ldots r_n+r_3 r_4 \ldots r_1+\ldots}. \quad \ldots\ldots\ldots\ldots(8)$$

In the last equation the numerator is the product of all the resistances, the denominator the sum of all the products of the resistances taken $n-1$ at a time.

A simple example is the case of a number n of incandescent lamps, each of resistance r, placed in parallel across the mains of an electric lighting installation. If the resistance of the mains between each lamp and the next be neglected, we get $R=r^n/nr^{n-1}=r/n$.

6. E.m.f. in a circuit in a network. The considerations stated above lead to the following important theorem.* In any closed circuit of conductors forming part of any linear system, the sum of the products obtained by multiplying the current in each part, taken in order round the circuit (taking account of the sign of the current in each case) is equal to the sum of the electromotive forces in that circuit. This follows at once by an application of Ohm's law as stated in (2) to each part of the circuit.

As an example of a circuit containing no electromotive forces, consider the circuit formed by the two wires (Fig. 31) of resistances r_1, r_2 joining A, B. The current γ_1 flows we suppose from A to B, the current $-\gamma_2$ from B to A. Hence by the theorem,

$$\gamma_1 r_1-\gamma_2 r_2=0, \quad \ldots\ldots\ldots\ldots(9)$$

since the wires are not the seat of any e.m.f.

As another example consider the diagram (Fig. 32) of resistances r_1, r_2, r_3, r_4, r_5, r_6, which connect the points A, B and the points

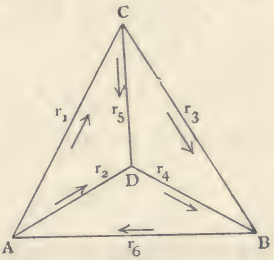

FIG. 32.

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* This theorem and the application of the principle of continuity referred to above were first stated explicitly by Kirchhoff, *Pogg. Ann.* Bd. 72, 1847, also *Ges. Abhand.* p. 22.

C, *D*. Take the circuit *ACD*, which we shall suppose contains no e.m.f. If V_a, V_c, V_d be the potentials at *A*, *C*, *D* respectively, we have identically

$$V_a - V_c + V_c - V_d + V_d - V_a = 0.$$

But this is the same thing as

$$\gamma_1 r_1 + \gamma_5 r_5 - \gamma_2 r_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

if γ_1, γ_5, γ_2 be the currents in the resistances marked by the same suffixes, taken as flowing in the direction of the arrows.

7. Principle of continuity. Current through a galvanometer in a bridge. We extend this notation for currents to the other conductors of this network, and suppose that the directions of flow are as shown by the arrows. Also we shall assume that the wire of resistance r_6 contains an e.m.f. *E*, so that r_6 is the resistance of the battery (or other electrical generator) and the wires joining it to *A*, *B*. The arrangement is then that of a Wheatstone bridge or balance, and the chief problem that arises is the determination of the current which flows in the conductor of resistance r_5 joining the points *C*, *D*. To solve this we apply first the principle of continuity to the currents at the points *A*, *C*, *D*. We get

$$\gamma_6 = \gamma_1 + \gamma_2, \quad \gamma_3 = \gamma_1 - \gamma_5, \quad \gamma_4 = \gamma_2 + \gamma_5. \quad \ldots\ldots\ldots\ldots(11)$$

Applying the circuital theorem stated above to the circuits *BACB*, *ACDA*, *CBDC* we obtain, using (11), the three equations

$$\left.\begin{aligned} (r_1 + r_3 + r_6)\gamma_1 + r_6\gamma_2 - {}& r_3\gamma_5 = E, \\ r_1\gamma_1 - r_2\gamma_2 + {}& r_5\gamma_5 = 0, \\ r_3\gamma_1 - r_4\gamma_2 - {}& (r_3 + r_4 + r_5)\gamma_5 = 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(12)$$

Eliminating γ_1 and γ_2 we find

$$\gamma_5 = \frac{r_2 r_3 - r_1 r_4}{D} E, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

where
$$\begin{aligned} D = {}& r_5 r_6 (r_1 + r_2 + r_3 + r_4) + r_5 (r_1 + r_3)(r_2 + r_4) \\ & + r_6 (r_1 + r_2)(r_3 + r_4) + r_1 r_3 (r_2 + r_4) + r_2 r_4 (r_1 + r_3). \quad \ldots(14) \end{aligned}$$

By substituting for γ_2 in the second and third of (12) its value $\gamma_6 - \gamma_1$, we get

$$\left.\begin{aligned} (r_1 + r_2)\gamma_1 + {}& r_5\gamma_5 - r_2\gamma_6 = 0, \\ (r_3 + r_4)\gamma_1 - {}& (r_3 + r_4 + r_5)\gamma_5 - r_4\gamma_6 = 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(15)$$

From these, by eliminating γ_1, we obtain

$$\gamma_5 = \frac{(r_2 r_3 - r_1 r_4)\gamma_6}{r_5 (r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)}. \quad \ldots\ldots\ldots\ldots(16)$$

8. Resistance of a bridge network. By means of (13) and (16) we can solve the problem of finding the equivalent resistance of the system of five wires which, external to the battery, lie between the points *A*, *B*. For let *R* be this equivalent resistance; since γ_6 is the current through the battery we have $\gamma_6 = E/(r_6 + R)$. Substituting this value of γ_6 in (16), and equating the values of γ_5 given by (13) and the modified form of (16), and solving for R, we obtain

$$R = \frac{r_5(r_1+r_3)(r_2+r_4) + r_1 r_3(r_2+r_4) + r_2 r_4(r_1+r_3)}{r_5(r_1+r_2+r_3+r_4) + (r_1+r_2)(r_3+r_4)}. \qquad \text{........(17)}$$

The solution of this problem is very easily obtained directly. Consider the three modes of passing from A to B, ACB, ADB, $ACDB$. From these, writing γ for γ_6, we get

$$r_1\gamma_1 + r_3\gamma_3 - R\gamma = 0, \;\; r_2\gamma_2 + r_4\gamma_4 - R\gamma = 0, \;\; r_1\gamma_1 + r_5\gamma_5 + r_4\gamma_4 = R\gamma, \; \text{...(18)}$$

or, by (11),

$$\left.\begin{array}{rrr} r_1\gamma_1 + & r_3\gamma_3 - & R\gamma = 0, \\ -r_2\gamma_1 - & r_4\gamma_3 - & (R - r_2 - r_4)\,\gamma = 0, \\ (r_1 + r_5)\,\gamma_1 - & (r_4 + r_5)\,\gamma_3 - & (R - r_4)\,\gamma = 0. \end{array}\right\} \qquad \text{..........(19)}$$

These give the determinantal equation

$$\left|\begin{array}{ccc} r_1, & r_3, & -R \\ -r_2, & r_4, & -(R - r_2 - r_4) \\ r_1 + r_5, & -(r_4 + r_5), & -(R - r_4) \end{array}\right| = 0, \qquad \text{............(20)}$$

which expanded and solved for R gives (17).

9. Addition of conductors to network without change of flow. It follows from Ohm's law and the theorems which have been deduced from it, that any two states of a system of conductors may be superimposed; that is the resulting potential at any point is the sum of the potentials at the point, the current in any conductor is the sum of the currents in the conductor, and the electromotive force in any circuit is the sum of the electromotive forces in the circuit, in the two states of the system.

The following result, which is a direct inference from the foregoing principles, and can be verified by experiment, will be of use in what immediately follows. Any two points in a linear circuit which are at different potentials may be joined by a wire without altering in any way the state of the system, provided the wire contains an e.m.f. equal and opposite as regards the production of a current in the conductor to the difference of potential between the two points. For the wire, before being joined to the circuit, will, in consequence of the e.m.f., have the same difference of potential between its extremities as there is between the two points, and if the end of the wire which is at the lower potential be joined to the point of lower potential, it will have the potential of that point, and no change will take place in the system. The other end will then be at the potential of the other point, and may be supposed coincident with that point, without change in the system. The new system satisfies the principle of continuity, and the circuital theorem, and is therefore possible; and it can be proved that it is the only possible arrangement under the condition that the state of the original system shall remain unaltered.

As a particular case of this theorem we see that any two points in a linear system which are at the same potential may be connected, either directly or by a wire of any chosen resistance, without alteration of the state of the system.

10. Conjugate conductors in a network. Further, it follows that if an e.m.f. in one conductor, *A*, of a linear system can produce no current in another, *B*, of the system, either conductor may be removed without altering the current in the other. For let *A* be removed: the potentials at the points of the system at which it was attached will in general thereby be altered. And since any two points in a linear system between which there is a difference of potential may, without altering the state of the system in any way, be joined by a wire which contains an e.m.f. equal and opposite to the difference of potential, we may suppose the conductor replaced with an e.m.f. in it equal to the difference of potential now existing between the two points, and its presence or removal will not now affect the current in any part of the system. But the same result may be attained, of course, without removing the conductor, by simply placing within it the required e.m.f., and this by hypothesis does not affect the current in the other conductor. Hence the removal of the conductor, *A*, does not affect the current in *B*. Again, by the first reciprocal relation below, p. 137, if an e.m.f. in *A* can produce no current in *B*, an e.m.f. in *B* can produce no current in *A*. Hence *B* may be removed without affecting the current in *A*.

If *A*, *B*, *C*, *D* be any four points of meeting in a network of linear conductors, in one wire of which joining *AB* there is an e.m.f., while *CD* is connected by one or more separate wires, the network can be reduced to a system of six conductors arranged as in Fig. 32, and such that the wires *AB*, *CD*, the currents in them and the potentials at their extremities remain unchanged. For currents must enter any one mesh of the network at certain points, and leave it at certain other points. One of the former must be the point of maximum potential in the mesh, one of the latter the point of minimum potential. The circuit of the mesh, therefore, consists of two parts joining these two points, and to any point in one of the parts will correspond a point of the same potential in the other part. We may therefore suppose any point in one in coincidence with the point of the same potential in the other. If this coincidence exist for a sufficient number of points we may as exactly as may be desired replace the mesh by a single wire joining the two points, and such that the currents entering or leaving it by wires joining it to the rest of the system, and the potentials at the points of junction, are not altered.

Since the only e.m.f. is in the wire *AB*, the current must enter the network at one of its extremities, *A* say, and leave at the other extremity *B*. *A* and *B* are therefore the points of maximum and minimum potential of the network. Hence we can replace the meshes of the system one by one by single wires, keeping CD unaltered until we have reduced the network to two meshes, one on each side of CD, connected by single wires to A and B respectively. Each mesh and connecting wire can be replaced by two wires joining A and B respectively with CD, and thus the whole system is reduced to an equivalent system of the form shown in Fig. 32. We can now deduce from this simple system relations for the currents and potentials in the conductors AB, CD, which will hold for these conductors in the more complex system.

Let the e.m.f. hitherto supposed to act in AB be transferred to CD, while the resistances r_5, r_6 are maintained unaltered. The value of γ_6 will be obtained from (13) by retaining the numerator unaltered and interchanging r_5 and r_6, r_1+r_2 and r_1+r_3, r_3+r_4 and r_2+r_4 in D. But these interchanges will not cause any alteration in the value of D, and hence the new value of γ_6 is equal to the former value of γ_5. Hence the theorem : *An e.m.f. which, placed in any conductor C_l of a linear system, causes a current to flow in any other C_m, would, if placed in C_m, cause an equal current to flow in C_l.*

If the arrangement is such that when the e.m.f. is in C_l the current in C_m is zero, the current in C_l will be zero when the e.m.f. is in C_m; and no e.m.f. in one will produce a current in the other. The two conductors are in this case said to be *conjugate.*

11. Reciprocal relation of conductors in a network. We can easily obtain another important theorem. The five conductors AC, AD, BC, BD, CD, in Fig. 32, may be regarded as the reduced equivalent of a network of conductors, at one point of which, A, a current of amount γ_6 enters, and at another point of which, B, the same current leaves. We suppose for the moment that no e.m.f. has its seat in any conductor of the system considered. The current γ_5 which flows in the conductor CD is given by (16). Multiplying by r_5 we get for the difference of potential between C and D due to the current γ_6 the value

$$\gamma_5 r_5 = \frac{\gamma_6 r_5 (r_2 r_3 - r_1 r_4)}{r_5(r_1+r_2+r_3+r_4)+(r_1+r_2)(r_3+r_4)}. \qquad \text{..........(21)}$$

But the resistance r of this system of five conductors (the equivalent of the network referred to standing alone, that is, without r_6), between the points C and D is given by

$$r = \frac{r_5(r_1+r_2)(r_3+r_4)}{r_1(r_1+r_2+r_3+r_4)+(r_1+r_2)(r_3+r_4)}, \qquad \text{..........(22)}$$

and if a current of amount γ_6 enter this system at C and leave it at D, the difference of potential between C and D will be equal to this expression multiplied by γ_6. The product multiplied by $r_1/(r_1+r_2)$ is the excess of the potential of C above that of A, and multiplied by $r_3/(r_3+r_4)$ it is the excess of the potential of C above that of B. Hence the excess e of the potential at A above that of B caused by the current γ_6 in CD is given by the equation

$$e=\frac{\gamma_6 r_5(r_2 r_3-r_1 r_4)}{r_1(r_1+r_2+r_3+r_4)+(r_1+r_2)(r_1+r_4)}. \quad \text{..........(23)}$$

Now let the network be part of any more general system, and let any e.m.f.s exist anywhere and produce any system of currents whatever consistent with zero current in the conductor CD. The superposition of the state just considered for the given network will be consistent with the previously existent state, and the difference of potential between A and B will be altered by the amount just found. Hence we have the theorem:

If by a current entering at one point A of a linear system of conductors and leaving at another point B there is caused a certain difference of potential between two other points CD, then, by an equal current entering the system at C and leaving at D, there is caused the same difference of potential between A and B.

12. Derived circuits in a network. The following result is easily proved and is frequently useful. If the potentials at two points A, B, of a linear system of conductors containing any e.m.f.s, be V, V' respectively and R be the equivalent resistance between these, then if a wire of resistance r be added, joining AB, the current in the wire will be $(V-V')/(R+r)$. In other words, the linear system, so far as the production of a current in the added wire is concerned, may be regarded as a single conductor of resistance R connecting the points AB, and containing an e.m.f. of amount $V-V'$. For, let the points A, B be connected by a wire of resistance r, containing an e.m.f. of amount $V-V'$ opposed to the difference of potential between A and B, no current will be produced in the wire, and no change will take place in the system of conductors. Now imagine another state of this latter system of conductors in which an equal and opposite e.m.f. acts in the wire between A and B, and there is no e.m.f. in any other part of the system. A current of strength $(V-V')/(R+r)$ will flow in the wire. Now, let this state be superimposed on the former state: the two e.m.f.s in the wire will annul one another, and the current will be unchanged. The potentials at different points, and the currents in different parts, of the system, will be the sums of the corresponding potentials and currents in the two states, and will therefore, in general, differ from those which existed before the addition of the wire.

13. Sensitiveness of a bridge arrangement. The arrangement shown in Fig. 33 represents the Christie or Wheatstone balance used for the comparison of resistances. A battery is included between A and B in the conductor of resistance r_6 (so that the total resistance between A and B through the battery is r_6), and a galvanometer is included in CD. One of the remaining four resistances, say r_4, is given: it is required to find the conditions which must be fulfilled for maximum current through the galvanometer.

It is clear that the current γ_5 is zero when $r_1 r_4 = r_2 r_3$, or $r_1/r_3 = r_2/r_4$. Let us suppose that instead of fulfilment of this relation we have

$$c\frac{r_1}{r_3} = \frac{r_2}{r_4}, \quad \text{.............................}(24)$$

where c is a constant multiplier. Then we have by (13)

$$\gamma_5 = E(c-1)\frac{r_1 r_4}{D}, \quad \text{.........................}(25)$$

where E is the e.m.f. of the battery and D has the value given in (14), but is modified in form by substitution of the value $cr_1 r_4/r_3$ for r_2, so that r_2 no longer appears in the expression for γ. For example, the relation $r_1/r_3 = r_2/r_4$, once fulfilled, might be deviated from by a small change of r_4 to the value $r_4 + dr_4$. We should then have $cr_1(r_4 + dr_4) = r_2 r_3$ or $(c-1)r_1 r_4 + cr_1 dr_4 = 0$, and $-Er_1 dr_4/D$ on the right in (25).

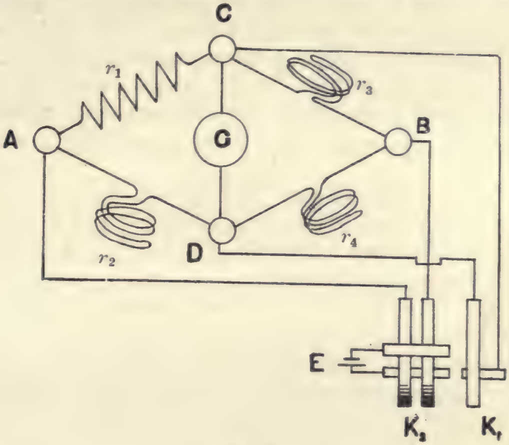

FIG. 33.

We have, then, to find when γ_5 is a maximum on the supposition that r_4, r_5, r_6, E and c are constants, or, which is the same but more convenient, when $D/r_1 r_4$ is a minimum under the same conditions. Writing u for $D/r_1 r_4$ and calculating the values of

$$\frac{du}{dr_1}, \quad \frac{du}{dr_3}, \quad \frac{d^2u}{dr_1^2}, \quad \frac{d^2u}{dr_3^2}, \quad \frac{d^2u}{dr_1\,dr_3},$$

we find, by equating the first two differential coefficients to zero, that either $r_1 = 0$ and $r_3 = 0$, or $r_1 = \sqrt{r_5 r_6}$ and $r_3 = \sqrt{r_4 r_6 (r_4 + r_5)/(r_4 + r_6)}$. Substituting these values of r_1 and r_3 in the expressions for the three second differential coefficients, we find that the latter pair of corresponding values gives positive values to each of the expressions

$$\frac{d^2u}{dr_1^2}, \quad \frac{d^2u}{dr_3^2}, \quad \frac{d^2u}{dr_1^2}\frac{d^2u}{dr_3^2} - \left(\frac{d^2u}{dr_1\,dr_3}\right)^2,$$

which is the condition for a minimum. The first pair of values, $r_1=0$ and $r_3=0$, gives neither a maximum nor a minimum.

When the battery and galvanometer resistances are capable of modification, we have along with the equations found above,

$$r_1=\sqrt{r_5 r_6},\quad r_3=\sqrt{r_4 r_6\frac{r_4+r_5}{r_4+r_6}},$$

the conditions $\quad r_5=r_1\dfrac{r_3+r_4}{r_1+r_3},\quad r_6=\dfrac{(r_1+r_3)(r_2+r_4)}{r_1+r_2+r_3+r_4}.$

The first of these last two conditions is proved as follows. We assume that the mass of wire in the galvanometer coil and the channel in which it is wound are the same in different coils, which we here consider. When this is the case the electromagnetic force at the needle is (see XII. 22 below) proportional to the square root of the resistance of the coil. Hence for a given value of γ_5 the deflection of the needle may be put equal to $a\gamma_5\sqrt{r_5}$, where a is a constant. Hence we may write

$$\text{deflection}=\frac{aE(c-1)r_1 r_4\sqrt{r_5}}{D}.$$

In this case we have to find when $D/r_1 r_4\sqrt{r_5}$ is a minimum. This expression can be put into the form $(m+nr_5)/k\sqrt{r_5}$, where m, n, k do not involve r_5. Equating to zero the first differential coefficient of this quantity with respect to r_5, we get for a minimum $r_5=m/n$. It will be found that though m/n appears to involve r_6, the battery resistance, the relation $r_1/r_3=r_2/r_4$, *renders it independent of* r_6, and so we obtain

$$r_5=\frac{r_1(r_3+r_4)}{r_1+r_3}=\frac{(r_1+r_2)(r_3+r_4)}{r_1+r_2+r_3+r_4}, \quad \text{..................(26)}$$

which is the best resistance of the galvanometer, subject to the conditions stated.

Next, let the total area of the acting surfaces in the battery be given, while the resistance may be varied by the mode adopted for combining the cells in series and parallel. We have (see p. 131 above) the greatest current when the battery is so arranged that its resistance is equal or nearly equal to the external resistance. When balance is nearly obtained, we may take as the external resistance between the points A, B (Fig. 32), the value $(r_1+r_3)(r_2+r_4)/(r_1+r_2+r_3+r_4)$. If r_6 may be taken as the resistance of the battery alone (that is, if the electrodes joining the battery to A and B be made so massive that their resistance may be neglected), we have to arrange the battery so that

$$r_6=\frac{(r_1+r_3)(r_2+r_4)}{r_1+r_2+r_3+r_4}. \quad \text{......................(27)}$$

When everything except r_4 is a matter of choice and arrangement, it follows that we should make $r_1=r_2=r_3=r_4=r_5=r_6$. This however is almost never a practical arrangement, and the statement of it is apt to be misunderstood. It is to be observed that for a given available electromotive force in the circuit, not susceptible of change, as supposed here, by a choice, say, of battery arrangement, the sensibility is greater the smaller r_6. The greater the electromotive force the better, if overheating is avoided.

If a deviation from fulfilment of the relation $r_1/r_3 = r_2/r_4$ (see Fig. 32) is brought about by the change of r_4 to $r_4 + dr_4$, where dr_4 is small, we get, by (13),

$$\gamma_5 = -\frac{E}{D} r_1 \, dr_4. \quad \text{.........................(25')}$$

Now let us suppose that the resistance of the battery, with the leads to A, B, is infinitesimal, so that we may put $r_6 = 0$, then, by (14),

$$D = (r_1 + r_3)(r_2 + r_4)\left(r_5 + \frac{r_1 r_3}{r_1 + r_3} + \frac{r_2 r_4}{r_2 + r_4}\right).$$

Also if γ be the current (only slightly changed by the change in r_4) through AD and DB, we have now $E = \gamma(r_2 + r_4)$. Thus we get, since $r_1/r_3 = r_2/r_4$,

$$\gamma_5 = -\frac{\gamma \, dr_4}{r_5 + \dfrac{(r_1 + r_2)(r_3 + r_4)}{r_1 + r_2 + r_3 + r_4}} \frac{r_1 + r_2}{r_1 + r_2 + r_3 + r_4}. \quad \text{........ (26)}$$

But we have seen that the best value of r_5 is (irrespective of the battery resistance) given by

$$r_5 = r_1 \frac{r_3 + r_4}{r_1 + r_3} = \frac{(r_1 + r_2)(r_3 + r_4)}{r_1 + r_2 + r_3 + r_4}.$$

Using this value of r_5 in the last equation, we get

$$\gamma_5 = -\frac{\gamma \, dr_4}{2(r_3 + r_4)}. \quad \text{.........................(27')}$$

If we suppose that r_5 is practically all in the galvanometer coil, we know that the deflection is proportional to $\gamma_5 \sqrt{r_5}$. But by the first form of r_5 given above, we get

$$\gamma_5 \sqrt{r_5} = -\tfrac{1}{2} \gamma \frac{dr_4}{\sqrt{r_4}} \left\{\frac{r_1 r_4}{(r_1 + r_3)(r_3 + r_4)}\right\}^{\frac{1}{2}}. \quad \text{..............(28)}$$

We shall obtain applications of this result in the discussion of methods of comparing resistances.

14. Flow of electricity in three dimensions. So far we have considered only cases of steady flow in *linear* conductors. It is of importance however, for the correction of certain measurements with respect to flow in linear conductors, to consider the distribution of the flow and the forms of the equipotential surfaces in different cases of conductors which cannot be considered as linear. For this purpose it is necessary to find the differential equation of the potential for the case of steady flow in a given medium, and from one medium to another. We shall consider only media in which the conductivity is the same in all directions.

By the fundamental principle of the theory of Ohm the rate of flow of electricity at any point (x, y, z) in any direction is directly proportional to the gradient of the potential V at that point and in that direction, we have for the rate of flow per unit of time per unit of area, in three mutually rectangular directions,

$$-k\frac{dV}{dx}, \quad -k\frac{dV}{dy}, \quad -k\frac{dV}{dz},$$

since the flow takes place in the direction in which V diminishes. The multiplier k is the *specific conductance* (conductivity) of the medium, and is the conductance (the reciprocal of the resistance) between two opposite faces of a centimetre cube of the substance.

15. Differential equations of flow. Considering an element in the shape of a rectangular prism taken in the medium with its centre at the point (x, y, z) and containing within it no seat of e.m.f., we find that the excess of the rate of inflow over outflow for the element is

$$-k\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)dx\,dy\,dz.$$

Since the flow is steady this is zero, and we have the differential equation

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0. \quad \text{.........................(29)}$$

This is known as Laplace's equation of the potential, which thus holds for flow of electricity in a uniform isotropic conducting medium. The theory of solutions of this equation under certain conditions constitutes the department of analysis called *Spherical Harmonics*, some results of which we shall have to use in later chapters.

It follows from this equation that the density of electricity at all points within a medium, in which the flow is steady, is zero, that is there is no electric charge on an element of the medium.

At any point of a surface at which a medium of conductivity k_1 is in contact with a medium of conductivity k_2 we have the equation

$$k_1\frac{dV}{dn_1}+k_2\frac{dV}{dn_2}=0, \quad \text{.............................(30)}$$

where dV/dn_1, dV/dn_2 are the rates of variation *from* the surface along a normal in each case towards the medium in which the flow is considered. This follows from the fact that the normal component of flow in the first medium up to an element dS of the surface is $k_1(dV/dn_1)dS$, while the normal component of flow in the second medium away from dS is $-k_2(dV/dn_2)dS$, and these must be equal to one another.

Putting $k_2=0$, we get for the equation at a surface separating a conducting medium from one of zero conductivity

$$\frac{dV}{dn}=0, \quad \text{....................................(30')}$$

or the component of flow at right angles to the surface is zero at every point of the surface.

If, on the surface of separation between the media, there be an e.m.f. E acting from the second medium to the first, we have besides (30) the equation

$$V_1 - V_2 + E = 0,$$

where V_1, V_2 are the potentials at the point but on opposite sides of the surface of contact.

These differential equations are precisely similar to equations which hold for flow of heat and for electrostatic phenomena. Solutions obtained for electrostatic problems are at once interpretable for flow of electricity, if conductivity be substituted for specific inductivity, flow of electricity per unit of area per unit of time for electrostatic induction, and line or tube of flow for line or tube of electric induction.

16. Examples of flow. We consider here only a few particular cases of practical interest. 1. An annular space contained within two coaxial right cylindrical surfaces is filled with a conducting liquid (or other homogeneous conductor): it is required to find the resistance of the arrangement for conduction from one surface to the other. This is the case of the column of liquid between two coaxial cylindric plates in a voltaic cell.

Equation (29) gives for radial flow

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0, \quad \text{.................................}(31)$$

where r is the distance of any point from the common axis. Integrating we get

$$V = A \log r + B.$$

Hence, if at the inner and outer surfaces (radii r_1, r_2) the potentials be V_1, V_2, we get

$$V_1 - V_2 = A \log \frac{r_1}{r_2}.$$

But if l be the length of the cylinder, V_1 the greater potential, and k the specific conductance of the substance, the total current across the coaxial cylinder of radius r is $-2\pi klr \,.\, dV/dr$, that is $-2\pi klA$. Hence we obtain

$$\frac{V_1 - V_2}{-2\pi klA} = \frac{1}{2\pi kl}\log\frac{r_2}{r_1}. \quad \text{........................}(32)$$

The expression on the left is the difference of potential between the two surfaces divided by the total current from one to the other, and this of course is the resistance to flow. The resistance is therefore

$$(\log r_2 - \log r_1)/2\pi kl,$$

and depends on the *ratio* of the radii, and is inversely proportional to the length of the cylinder.

17. Resistance between electrodes buried in a large mass of conductor. 2. Two small highly conducting spherical electrodes kept at different potentials are buried in an infinitely extended uniform conductor of comparatively much lower specific conductivity k: it is required to find the resistance between the spheres.

The potential of each sphere may be taken as nearly the same throughout its mass, and if the distance of the spheres apart be great in comparison with the radius of either, the potential at any point near the surface of one of the spheres, and at a distance r from the centre, will be approximately in inverse proportion to the distance r. This follows from the equation of potential, (29) above, which becomes in the case of a single sphere,

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0.$$

Integrating, we get
$$V = \frac{A}{r} + B, \quad \text{.................................(33)}$$

where A and B are constants. B must be zero since, for large values of r, V must be small.

Thus, if V_1, V_2 be the potentials of the two spherical electrodes $(V_1 > V_2)$, and r_1, r_2 the radii, the potential at such a point as that specified is V_1r_1/r or V_2r_2/r according to the sphere to which it is adjacent; and the corresponding outward gradients of potential will be

$$-V_1r_1/r^2, \quad -V_2r_2/r^2.$$

This gives at the surfaces of the electrodes the values

$$-V_1r_1/r_1^2 \quad \text{and} \quad -V_2r_2/r_2^2.$$

The outward rate of flow from the sphere of higher potential is therefore $4\pi kV_1r_1$, and the inward rate of flow over the other is $-4\pi kV_2r_2$. Hence if γ be the total current, we have

$$\gamma = 2\pi k(V_1r_1 - V_2r_2).$$

The total resistance R to conduction from one sphere to the other through the infinitely extended medium is therefore given by

$$R = \frac{V_1 - V_2}{\gamma} = \frac{V_1 - V_2}{2\pi k(V_1r_1 - V_2r_2)} \left[= \frac{1}{2\pi kr}, \quad \text{if} \quad r_1 = r_2 = r \right], \text{......(34)}$$

a result pending on the radii of the spheres and not at all on the distance between them. The result is of interest in connection with the "earthing" of telegraph wires and other conductors; for we infer that the resistance between two electrodes buried in a large mass of conducting material, such as that afforded by a good "earth," is practically independent of their distance apart.

If the conductor were separated into two parts by a plane passing through the centres of the spheres the resistance between the hemispherical electrodes in each part would be double that given by (34), or $1/\pi kr$.

18. End corrections of bar terminating in large conducting masses. 3. The same case as in 2, except that the electrodes are circular disks of highly conducting material.

Supposing, as before, that the electrodes are at a distance apart great in comparison with either disk-radius, the distribution of potential in the medium surrounding either is the same approximately as that in the field of a single charged disk. Let r_1, r_2 be the radii of the disks, V_1, V_2 their potentials in order of magnitude. We shall take first the electrostatic analogue, and consider the surface density of the charge on each disk as constant, $=\sigma$, say. The outward component of electric force along a normal is $-dV/dn=4\pi\sigma$. Hence, integrating over both faces of the disk, and putting Q for the whole charge, we have

$$-\int dS\,.\,dV/dn=4\pi Q.$$

But the total outward flow is $\gamma=-k\int dS\,.\,dV/dn$, and we write here for Q, by the theory of a charged disk [see Appendix, *Notes*] $2r_1V_1/\pi$, so that we get $\gamma=8kr_1V_1$. Similarly we get $\gamma=-8kr_2V_2$. Hence

$$\gamma=4k(V_1r_1-V_2r_2)=\frac{V_1-V_2}{R}.$$

Since $V_1r_1=-V_2r_2$, this gives

$$R=\frac{r_1+r_2}{8kr_1r_2}. \qquad\text{(35)}$$

We infer that the parts of R due to the respective disks are $1/8kr_1$ and $1/8kr_2$. If the disks lie in the bounding surface conduction takes place from only one face of each, and the value of R is twice that just obtained.

The result gives an inferior limit to the correction to be made on the resistance of a cylindrical wire which is joined to a large mass of metal. Let the junction be made by a thin disk of very highly conducting matter. The flat surface of the end of the wire will be brought to one potential, and therefore its conducting power will be fully made use of right up to the disk. Hence an inferior limit to the correction is an addition of $1/4kr_1$ to the resistance, or if k' be the specific conductance of the wire, of $\pi k'r_1/4k$ to the length. The late Lord Rayleigh gave $\cdot 8242k'r_1/k$ as a *superior* limit to the addition to be made to the length for each end. If $k'=k$, as in the case in which a glass tube filled with mercury opens at each end into a large mass of mercury, we have as an approximation to a lower limit for the correction $\pi r/4$ or $\cdot 7854r$; that is this addition at least must be made to the length for the effect of the spreading out of the lines of flow at each end from the tube into the large volume of mercury.

This is also the approximate correction to be made on the length of a resonating cylindrical column of air which responds to a tuning fork vibrating above it.

CHAPTER V.

THEORY OF ELECTROMAGNETIC ACTION.

I. Actions between Currents and Magnets.

1. Oersted's experiment. The action of a current on a magnet, discovered by Oersted in 1820, is the foundation of the modern science of electromagnetism, for from it has come by a steady process of discovery, at once inductive and deductive, the whole theory of the mutual action of magnets and currents, and of currents on one another, of the induction of currents by the motion of conductors in a magnetic field, and the great modern applications of electricity to telegraphy and telephony, lighting and transmission of power, and electric traction. We shall follow to a certain extent the historical order of development of this part of the subject, making use freely, however, for brevity and clearness, of the theorems contained in the digest of magnetic theory already given, and of the ideas and methods suggested by later writers, such as Thomson and Maxwell.

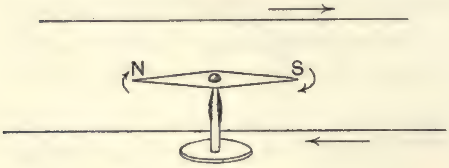

FIG. 34.

In Oersted's experiment, as commonly performed, a magnet is suspended horizontally in the magnetic meridian, and a conductor carrying a current is stretched parallel to the needle, above it or below it. The magnet is acted on by a couple which turns it round towards the position at right angles to the conductor, and it finally rests in equilibrium in a position in which this deflecting couple is balanced by the return couple due to the terrestrial magnetic field. The deflecting couple is reversed in direction by turning round in azimuth through 180°, or "end for end," the conductor carrying the current, so that, for example, the current flows from south to north instead of from north to south; and it is likewise reversed when the conductor is transferred from a position above the needle to a position below the needle, and *vice versâ*. Thus the direction of the deflecting couple is not reversed when the conductor is both turned end for end and transferred from above to below, or from below to above; and we see therefore that if the current flow, say from north to south above the magnet and back from south to north below the magnet, the deflecting couples due to both currents are in the same direction. By multiplying the number of conductors or portions of one conductor thus carrying currents, the effect on the needle is also enhanced. Hence by winding the conductor into a coil of a large number of turns, one part of each of which is above the other below the magnet, the actions of the various turns on the magnet are given all the same direction, and the magnet is acted on by a resultant couple round a vertical axis, made up of the component couples round such an axis which are furnished by the turns of wire in the coil. This is the construction and mode of action of the old form of "galvanic multiplier," and of the modern galvanometer.

2. Equivalence of a current and a distribution of magnetism stated. Since the needle is deflected by the current just as it would be by bringing another magnet into its neighbourhood, we are led to regard the current as producing a magnetic field, which is superimposed on the terrestrial magnetic field so as to give a resultant field, parallel to a line of force of which the needle, if short, places its magnetic axis. In fact, the current produces the same effect as would a certain distribution of magnetism, and we have to inquire what is the nature of this distribution. This is set forth in the following general theorem given by Ampère: *Every linear conductor carrying a current is equivalent to a simple magnetic shell, the bounding edge of which coincides with the conductor, and the moment of which per unit of area, that is, the strength of the shell, is proportional to the strength of the current.* The direction of magnetization of the shell is reversed when the current is reversed, and may be found in any given case as follows. Supposing an observer to be standing on the edge of the shell with its surface on his left hand, and to be looking in the direction in which the current is flowing,* the side of the shell towards the observer will be covered with northern magnetism. This may also be remembered by the rule, that the magnetism of the earth coincides in direction with that of a needle placed within it, and turned into position by currents circulating round the earth in the direction of the sun's apparent motion.

The theorem of Ampère just stated depends on another theorem which we shall consider first. *The magnetic field produced by the current in a plane closed circuit is the same at all points, the distances of which from every part of the conductor are great in comparison with every dimension of the circuit, as that produced by a small magnet placed anywhere within the circuit, with its axis at right angles to the plane of the current, and having a magnetic moment proportional to the current flowing, and to the area of the circuit.*

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* From copper to zinc in the external part of the circuit of a voltaic cell, according to the ordinary convention.

The truth of this theorem may be demonstrated by a simple experiment which has become a common laboratory exercise. A plane circuit of convenient form, for example circular, is arranged in a vertical position parallel to the magnetic meridian, by connecting to a circular coil, of one turn or more, the terminals of a battery placed at a considerable distance from every part of the apparatus used in the experiment. It is easy to prove by separate experiments that the current in the part of the circuit consisting of the battery itself and the wires connecting it to the circular conductor, produces no appreciable effect if the wires are twisted together, and are both joined as nearly as may be at the same point to the coil. The effective part of the circuit is then only the coil, and it is this only we mean when we refer in what follows to the "circuit." A magnetometer is placed with the centre of its needle on a horizontal magnetic east and west line passing through the centre of the circular conductor, which is so arranged that the distance of its centre from the magnetometer needle can be altered at pleasure. It is found by observing the deflections of the magnetometer needle that the magnetic forces produced at the centre of the needle are very nearly in the inverse ratio of the cubes of the distances of the centre of the needle from the centre of the coil, when these distances are great in comparison with the dimensions of the circular conductor. The same result may be obtained for a plane conductor of any other form by so placing it that the east and west line through the centre of the needle passes through the plane of the conductor within or near the circuit, and taking the distance as that between the plane and the needle's centre. Now, by equations (9), Chapter V., this is precisely the result that we should have obtained for a small magnet placed as specified above with regard to the circuit; and it is possible to adjust the moment of the magnet so that its action and that of the current may be identical.

It is further found experimentally that if we have a magnet and a current which produce the same magnetic force at distant* points upon an east and west line passing through the circuit, the magnet and the current produce the same magnetic effect at all other distant points. Finally, by altering the area of the circuit in any ratio, we find the magnetic force at every point altered in the same ratio. Hence the equivalence is completely proved.

3. Definition of current strength and unit current. We define the current strength in a given circuit as proportional to the intensity of the magnetic field which the current produces at a given point; and hence it is not necessary to prove that the moment of the equivalent magnet must be proportional to the current, since we know that the magnetic field due to a magnet at a given point so distant that the effect of distribution of magnetism does not enter into account, is proportional to the magnetic moment of the magnet. We shall find that this mode of measuring current-strength gives results consistent with those obtained from the definition based on the electrostatic system of units, viz. the quantity of electricity which passes across an equipotential surface in the circuit per unit of time.

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* "Distant" here, as elsewhere in a similar connection, means that the points are at distances from the circuit great in comparison with any of its dimensions.

We formally define unit current as that current which flowing in a circuit of unit area can be replaced by a magnet of unit magnetic moment. This definition depends on the unit of magnetism already defined, and, when the latter unit is 1 c.g.s. unit of magnetism, we have by the definition 1 c.g.s. unit of current. We shall find other, but equivalent, definitions of unit current.

The magnet equivalent at distant points to the plane circuit may be supposed broken up into an infinite number of equal short magnets uniformly distributed over the circuit with their centres in and their lengths at right angles to its plane. If the aggregate magnetic moment be the same as before, the same effect will be produced, since the position of the equivalent magnet within the circuit and its form do not affect the force which it produces at distant points. But this converts the equivalent magnet into a uniform magnetic shell, the strength of which is, by the definition of unit current just given, simply the strength of the current.

4. Proof of general theorem of equivalence of a linear current and a magnetic shell. Ampère's further proposition, that any finite linear circuit carrying a current is equivalent to a magnetic shell, can now be proved at once. For let *ABC* be the circuit, in which we shall suppose a current of strength γ to be flowing. We may construct, as indicated in the figure, a network of conductors of which the circuit is the bounding edge, having each mesh so small that it may be considered plane. Round each of these meshes a current γ may be supposed to flow in the same direction as that of the current in the boundary. It is clear that this will give two equal and opposite currents in every conductor which is common to two meshes, and thus the system reduces simply to the current in the original conductor which forms the boundary. Each of these small circuits may, however, by the proposition just proved, be replaced by a small magnet, or by an infinite number of equal infinitely small magnets uniformly distributed over it, and the aggregate of these small magnets gives a magnetic shell bounded by the circuit.

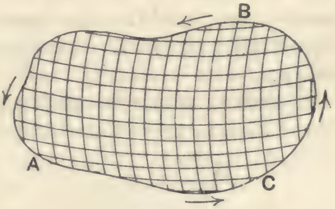

FIG. 35.

It is important to notice that the meshes may have any continuous succession of positions, provided the boundary be undisturbed. Thus the shell is geometrically defined only by its boundary the conductor. It should also be observed that there is not here any restriction of the equivalence to the action at distant points ; only, since the conductor must always in practice be a wire of finite thickness, the points at which the action is considered must be at a distance of several diameters of the wire from the boundary.

5. Work done in carrying a pole in a closed path round a current. We can now at once show that the work done in carrying a unit pole from any point P in the field of a current round a closed path to the point P again is zero, if the path do not embrace the circuit, and is $4\pi\gamma$ if the path embrace the circuit once. For, let a position of the equivalent shell be chosen which does not intersect the closed path, if the latter does not embrace the circuit, and one close to the point P, if the closed path does pass round the circuit. In both cases the work done is equal to the total change of potential in passing round the path. In the former case this is zero. In the latter case let the pole be carried first from the point P to a point Q infinitely near to P on the opposite side of the shell. The change of solid angle in passing from P to Q is, as proved in II. 21 above, 4π, and therefore by the definition of current strength the work done is $4\pi\gamma$. Now although the shell was fixed in position in estimating the work done in carrying the unit pole from P to Q, it is not necessary to suppose it fixed in the same position in finding the work done in carrying the pole along the infinitely small part of the closed path which lies between Q and P. We may therefore suppose the shell in any other position clear of the element QP of path. The work done in carrying the pole from Q to P is therefore infinitely nearly zero, that is, the work done in carrying the pole round the closed path is $4\pi\gamma$. Another proof of this theorem is given in 12 below.

6. Case in which the path and circuit interlace any number of times. If the path be laced round the circuit any number, n, of times, the whole work done in carrying the pole round the path will be $4\pi n\gamma$. To see this we have only to join P to the points R, T, etc. (Fig. 36.) The work done in carrying the pole round the path $PQRS \ldots P$, is equal to the work done in carrying the pole round the n closed paths $PQRP$, $RSTR$, ... , $VWPRTV$, since the portions PR, RT, etc., are each traversed twice but in opposite directions, so that the work done in traversing them in one direction cancels the work done in traversing them in the other.

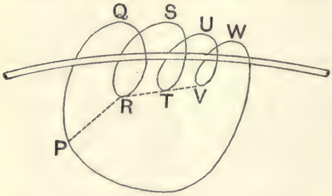

FIG. 36.

In the same way, we can prove that if the circuit pass n times through the path, the work done in carrying the pole round the path is $4\pi n\gamma$. For, consider the case represented in Fig. 37, in which the circuit passes twice through the path, and join the two points Q, S of the path by the line QS passing between the two portions of the circuit. The work done in carrying a unit pole round the path is plainly equal to the work done in carrying it round the two closed paths *PQSP*, *QRSQ*, since *SQ* is traversed in opposite directions in the two cases. But in each case the work done is $4\pi\gamma$, and hence the whole work done is $2 \times 4\pi\gamma$. Hence, proceeding in the same way for further interlacing of the circuit with the path, we obtain the general result stated above.

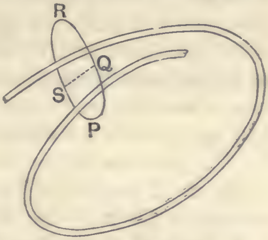

FIG. 37.

Any combination of the two kinds of interlacing will give a result which can be calculated according to the circumstances of the case by combining the two results just found.

7. Magnetic field of a long straight conductor carrying a current. Ampère's theorem is confirmed by quantitative experiments on the magnetic effects of a long straight conductor carrying a current. With the arrangement of horizontal conductor and horizontal needle, it is found as has been stated that, according as the conductor is above or below the level of the needle, the latter is deflected in one direction or the other, and hence when the conductor is in the same horizontal plane with the needle, no deflection is produced. If the free period of oscillation of the needle be observed when the conductor is present in the same horizontal plane it is found to be the same as when no current is flowing.

These results show that the current produces then no component force in the horizontal direction on a magnetic pole, and it follows that the resultant force is in the vertical direction. The force on a magnetic pole is therefore at right angles to the plane through the conductor and the pole.

The same thing is shown by the fact observed by Ampère, that the position of the needle is at right angles to the conductor in a plane parallel to it when there is no force acting on the needle except that due to the current; for this proves that there is no component in the plane through the current and a magnetic pole on which the current acts.

8. Lines of force circles round a conductor. It is found that the magnitude of the magnetic force due to the current in a straight conductor, at points not opposite the ends, and at distances from the conductor small in comparison with its length, varies inversely as the distance of the point considered from the conductor. Its direction is, as we have seen, at right angles to the plane through the conductor and the point considered. A magnetic pole free to move in a circular groove with the conductor for its axis would move round the groove in the same direction and would be acted on by the same force, which would be everywhere tangential to the groove. In fact the lines of magnetic force round the conductor, except near its ends, are circles having the conductor for their common axis.

These results for a straight conductor are proved by a number of simple experiments. That the intensity of the magnetic field varies inversely as the distance from a thin conductor was shown by Biot and Savart,[*] who placed a horizontal conductor at right angles to the magnetic meridian, and at different distances above and below the centre of a horizontally suspended needle, and observed the periods of oscillation when the needle was under the influence of the earth's force alone, and again when a current was made to flow in the conductor. If T, T' be the periods in the two cases, M the magnetic moment, and μ the moment of inertia of the magnet, the intensity of the field due to the current is given by the expression $4\pi^2\mu/M.(1/T'^2 - 1/T^2)$.

9. Law of force found experimentally: method of Maxwell. The law of variation of force with distance is also shown by the following elegant experiment apparently suggested by Maxwell.[†] A conductor is placed in a vertical position and a light carriage of non-magnetic material is suspended so as to be free to turn round the conductor as an axis. It is found that when a magnet is fixed on this carriage there is no couple tending to turn the carriage round the conductor. Consider a thin uniformly magnetized bar-magnet attached to the carriage. It may be regarded as composed of two equal and opposite magnetic poles at its extremities. The moment round the axis on one pole must be equal and opposite to the moment exerted on the other, whatever the position of the magnet on the carriage may be. Let F_1, F_2 be the forces on the poles at right angles to the planes through them and the conductor, r_1, r_2 the distances of the poles from the conductor supposed to be a thin wire. The moments round the conductor give

$$F_1 r_1 + F_2 r_2 = 0,$$

and therefore

$$-\frac{F_1}{F_2} = \frac{r_2}{r_1}, \quad \text{.................................(1)}$$

or the forces have opposite moments and are inversely as the distances from the axis.

10. Deduction of law of force from equivalent magnetic shell. We may deduce the results stated above for a long straight conductor from Ampère's theorem of the equivalence of a current and a magnetic shell. We have seen that the shell is defined only by its bounding edge and the strength of the current. If we consider an infinitely long straight conductor carrying a current γ, the equivalent shell is geometrically defined only by its edge, and we may take the shell as a plane surface, otherwise in any position we please. Let the shell be at right angles to the plane of the paper, A (Fig. 38) the projection of the conductor, AB of the shell, P the position of the magnetic pole, CP $(=a)$ its distance from the plane of the shell, and AC $(=b)$ the distance of C from A. Let E be the projection of an element of the shell, the distance $CE = y$, the distance of the element from the plane of the paper z, and its area $dy\,dz$. The radius vector from P to the element has for length $(y^2+z^2+a^2)^{\frac{1}{2}}$, and the projection of the element at right angles to the radius vector is $a\,dy\,dz/(y^2+z^2+a^2)^{\frac{1}{2}}$. Hence the solid angle subtended by the element at P is $a\,dy\,dz/(y^2+z^2+a^2)^{\frac{3}{2}}$. The total solid angle ω

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[*] *Ann. de Chim. et de Phys.* t. xv. 1820.

[†] *El. and Mag.* vol. ii. p. 130 (2nd ed.).

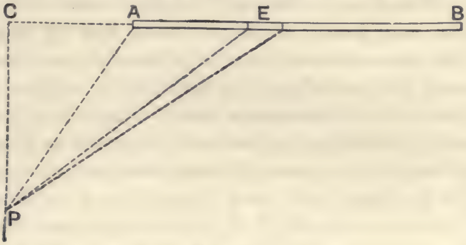

FIG. 38.

subtended at P by the shell, supposing the positive side turned towards P, is given by the equation

$$\omega = \int_b^\infty \int_{-\infty}^\infty \frac{a\,dy\,dz}{(y^2+z^2+a^2)^{\frac{3}{2}}} = \pi - 2\tan^{-1}\frac{b}{a}. \quad \text{..............(2)}$$

Hence for the potential V at P of the magnetic shell we have

$$V = \gamma\left(\pi - 2\tan^{-1}\frac{b}{a}\right), \quad \text{..........................(3)}$$

where for $\tan^{-1} b/a$ is to be taken the angle between 0 and $\pi/2$ which has b/a for its tangent.

The same result may be obtained geometrically with great ease thus: The solid angle subtended at P by a plane rectangle, of finite breadth and infinite length, is the area of the lune cut out of the unit sphere (centre P) by planes drawn through P and the edges of the rectangle. If θ be the angle between these planes the area is $4\pi \times \theta/2\pi = 2\theta$. Thus, if by the addition of a rectangular strip the edge of the shell were brought to C, the solid angle would be $2 \times \pi/2$ or π. But for this strip, $\theta = \tan^{-1} b/a$. Hence the actual solid angle is $\pi - 2\tan^{-1} b/a$.

The components of the magnetic force at P are $-dV/da$, $-dV/db$ along CP and parallel to AC respectively. Hence the resultant at P is

$$\{(dV/da)^2 + (dV/db)^2\}^{\frac{1}{2}} = 2\gamma/(a^2+b^2)^{\frac{1}{2}},$$

or if r be the distance of P from A it is $2\gamma/r$. The direction of the force is therefore in the plane of the paper, and at right angles to PA, and *from* that side of the plane through P and the conductor on which C lies, for we have

$$-dV/da = -2\gamma b/(a^2+b^2), \quad -dV/db = 2\gamma a/(a^2+b^2),$$

and the equation of the plane the projection of which is PA, is $bx - ay = 0$, if x be taken from P in the direction PC. The x and y direction cosines of a normal to this plane are respectively proportional to $-b$ and a, as are also the component forces parallel to x and y. By experiment it is found that the direction which the current must have in order that a positive or north-seeking pole should move as here specified is from below upwards through the paper. This agrees with the rule near the foot of p. 147. P is thus on the positive side of the shell.

11. Expression for potential found from law of force. We may proceed from the experimental fact, that the intensity of the magnetic field at any point is inversely as the distance, r, of the point from the straight conductor, to determine whether the current has a magnetic potential or not. First defining the unit of current so that the magnetic force is $2\gamma/r$, taking the origin at A, and the axes of x and y parallel to CP and along AB respectively, and putting x, y, z for the coordinates of the point P, we have for X, Y, Z, the components of magnetic force at P, the values $X = -2\gamma y/r^2$, $Y = 2\gamma x/r^2$, $Z = 0$, and hence

$$X\,dx + Y\,dy + Z\,dz = -2\gamma \frac{d(y/x)}{1 + y^2/x^2},$$

that is, the expression in the left is a perfect differential of the function $-\gamma \tan^{-1} y/x + C$, which is therefore the potential at P. This is a many valued function of x, y, z; but since we have to deal only with the difference of potential between two points, that is with the work done in carrying a unit pole from one to the other, there is no ambiguity.

We have here to take into account, as pointed out above, the difference in the work done in any closed path according as the path does or does not pass round the conductor. The work done in any closed path is zero, if the path can be supposed shrunk in upon any point within it without cutting the conductor, for, clearly, the work done in carrying a unit pole from any point P to another point Q is equal and opposite to the work done from Q to P along the remaining part of the path.

12. Theorem of work done in carrying unit pole round current: second proof. On the other hand, if the path embrace the conductor this reasoning does not hold. It is clear that the work done in carrying a unit pole once round in a circle of which the conductor is the axis, say $TUVWT$ in Fig. 39, is $4\pi\gamma$. For the force at each point is tangential to the circle, and has the value $2\gamma/r$, while the length of path is $2\pi r$, and these give the product $4\pi\gamma$. Let now the given closed path, which may or may not be in a plane, for example $PQRSP$ in Fig. 39, be connected with the circle by the lines QT and SV. The work done in each of the closed paths $SRQTUVS$, $SVWTQPS$ is zero, since neither embraces the conductor. Hence the whole work done in these two paths is zero. But if the pole be carried round these paths in the order stated

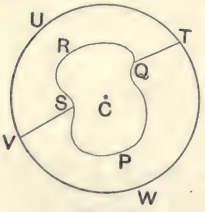

FIG. 39.

above, the whole work done is the sum of the work done in carrying the pole round the circle *TUVWT*, and round the given closed path in the direction *PQRS*, since the work done in the paths *SV* and *QT* is zero, these being traversed twice in opposite directions. Hence the work done in the path *PQRS* embracing the conductor is also numerically $4\pi\gamma$.

This method of proof leads also to the result, already proved in 5 above, that the work done in carrying a pole round a conductor whether straight or not is $4\pi\gamma$. For if we suppose the conductor infinitely thin and to have finite curvature, and take a closed circular path infinitely near it, the pole will be acted on only by the portion of the conductor which is near it as compared with the rest of the circuit, and this may be considered as a long straight conductor. The work done in carrying a unit pole round the circular path is $4\pi\gamma$. Then by connecting with the circular path any other path embracing the conductor, the work done in carrying a pole round it is found to be $4\pi\gamma$.

If the circuit be not infinitely thin the actual conductor may be supposed made up of an infinite number of filamental conductors coinciding with the lines of flow, and for each of these the work done in carrying a unit pole round a path embracing it is $4\pi \times$ the current in the filament. Hence in a closed path embracing the whole current γ, the work done upon a unit pole traversing it is $4\pi\gamma$. Thus the theorem is extended to non-linear conductors. The case of interlacing of the path and the conductor may be dealt with as in 6.

The *external* magnetic field of a long straight conductor of circular section, carrying a current symmetrical about the axis, coincides there with that of an axial filament carrying the same current. For the lines of force are coaxial circles and the line integral for each is $4\pi\gamma$, so that at distance r from the axis the force is $2\gamma/r$. A long tubular conductor has no internal field.

13. Relation of current to line integral of magnetic force round conductor. If the current strength per unit area at right angles to the direction of flow at any point be denoted by q, and l, m, n be the direction cosines of that direction, then we may call lq, mq, nq the components of the current along the axes. Denoting these by u, v, w, we have for the component of flow in any direction of which the cosines are λ, μ, ν, the expression $\lambda u + \mu v + \nu w$.

If now we take any closed path round a conductor, or portion of a conductor, carrying current, and take the line-integral of the magnetic force round the path, and the surface integral of the current across the surface, the theorem just discussed may be thus expressed,

$$4\pi\int(\lambda u + \mu v + \nu w)\,dS = \int\left(a\frac{dx}{ds} + \beta\frac{dy}{ds} + \gamma\frac{dz}{ds}\right)ds. \quad \text{.........(4)}$$

The second integral may be transformed by the following process, which may also be employed to transform the expression on the right of (49), p. 62, and so give the values of a, b, c, in terms of the components of vector potential. Let ABC, Fig. 40, be one face of a tetrahedron, the other three faces of which are OAB, OBC, OCA, and have edges OA, OB, OC in the direction of the axes of x, y, z.

Taking $\int\left(\alpha\frac{dx}{ds}+\beta\frac{dy}{ds}+\gamma\frac{dz}{ds}\right)ds$ round the closed path AB, we see at once that it can be converted into the corresponding integrals round the three paths $OABO$, $OBCO$, $OCAO$, since the integral along each of the lines OA, OB, OC is thus taken twice in opposite directions. Thus we obtain

$$\int_{ABC}\left(\alpha\frac{dx}{ds}+\beta\frac{dy}{ds}+\gamma\frac{dz}{ds}\right)ds=\int_{OAB}\left(\alpha\frac{dx}{ds}+\beta\frac{dy}{ds}\right)ds+\int_{OBC}\left(\beta\frac{dy}{ds}+\gamma\frac{dz}{ds}\right)ds$$
$$+\int_{OCA}\left(\gamma\frac{dz}{ds}+\alpha\frac{dx}{ds}\right)ds.$$

Now consider any one of these three integrals, say the first taken round $OABO$. Let α, β, γ be the components of the magnetic force at O. Then the values of the first two components at any point distant δx, δy from O in the plane xy are

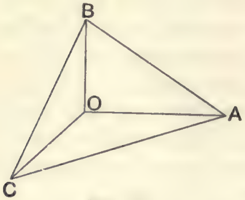

FIG. 40.

$$\alpha+\frac{d\alpha}{dx}\delta x+\frac{d\alpha}{dy}\delta y,\quad \beta+\frac{d\beta}{dx}\delta x+\frac{d\beta}{dy}\delta y.$$

If we take the tetrahedron so small that its edges OA, OB, OC are δx, δy, δz, the integral round $OABO$ may be found by taking the values of the components at the middle points of OA, AB, BO as the mean values over these distances. Thus we get for the integral

$$\left(\alpha+\frac{1}{2}\frac{d\alpha}{dx}\delta x\right)\delta x-\left(\alpha+\frac{1}{2}\frac{d\alpha}{dx}\delta x+\frac{1}{2}\frac{d\alpha}{dy}\delta y\right)\delta x$$
$$+\left(\beta+\frac{1}{2}\frac{d\beta}{dx}\delta x+\frac{1}{2}\frac{d\beta}{dy}\delta y\right)\delta y-\left(\beta+\frac{1}{2}\frac{d\beta}{dy}\delta y\right)\delta y,$$

which reduces to

$$\frac{1}{2}\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)\delta x\,\delta y=\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)\times\text{area } AOB.$$

In the same way we obtain corresponding results for $OBCO$ and $OCAO$. But if ABC be taken as dS we have area $OBC=\lambda\,dS$, $OCA=\mu\,dS$, $OAB=\nu\,dS$.

Hence

$$4\pi(\lambda u+\mu v+\nu w)=\lambda\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right)+\mu\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right)+\nu\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right),$$

and therefore

$$\left.\begin{aligned} u&=\frac{1}{4\pi}\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right),\\ v&=\frac{1}{4\pi}\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right),\\ w&=\frac{1}{4\pi}\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right),\end{aligned}\right\}\qquad(5)$$

equations which will be found of great importance in the sequel.

14. A circuit and magnet equivalent in one medium not necessarily so in another. There is one remark on the equivalence of a current and a magnetic distribution which ought to be made here, though we have not space to deal fully with the matter. The mutual action between a current flowing in a conductor and a distribution of magnetism is independent of the nature of the medium in which they are placed, if that medium be the same throughout, but this does not hold for the mutual action between two distributions of magnetism.*

The value of the magnetic force at any point, by definition, does not depend on the nature of the medium *at* the point in question, but only on the magnetization elsewhere. In a uniform medium, which has imbedded in it a conductor carrying a current, the potential at any point may be taken as made up of two parts, that which would be produced by the circuit alone, in a medium of unit inductive capacity, and that due to the magnetization which the medium receives in consequence of its specific inductive capacity differing from unity. Now the second part of the potential is single valued, and hence the line-integral of its variation round a closed curve is zero. If the induced magnetization of the medium is solenoidal (as it always is when κ is uniform) and the medium extends indefinitely in all directions, no force due to the magnetization of the medium is experienced by a magnetic pole placed anywhere; but the action is precisely the same as if the circuit and pole were situated in air. Of course if the medium is different in different parts, as for example when it consists partly of iron, partly of air, the magnetization of the different parts must be taken into account in assigning the value of the magnetic force at any point. In the case of solenoidal distribution this is effected by taking into account the virtual surface distribution, resulting from the discontinuity of the magnetization at the separating surfaces.

15. Action of a magnetic system on a current. We come now to the action of a magnetic system upon a current. The theorem of the equivalence of a current to a magnetic distribution established above leads of course to the conclusion that whatever process, or function, is available for the calculation of the forces acting on the magnetic shell, is also available for the calculation of the action on the current when in the field. This will be manifested as certain forces acting on the conductor which we have now to investigate. Effects of the electromagnetic action on the current itself will be discussed later.

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* Neglect of this difference in the two cases has led to the assignment of wrong dimensions to unit quantity of magnetism in electrostatic units. See a discussion in the *Phil. Mag.* 1882.

The function from which we determine the force acting on a magnetic distribution in a magnetic field is the expression for the potential energy which the system possesses in virtue of its being in the field. We have found for this in the case of a shell of strength ϕ,

$$\mathsf{E} = \phi \iint \left(l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz} \right) dS, \quad \text{..................}(6)$$

where V is the magnetic potential (due to the distribution producing the field and not at all to the shell itself) at the element dS, the coordinates of which are x, y, z, and the integral is taken over the surface of the shell. But, if there be none of the magnetism producing the field at the shell itself, we have for the components of magnetic induction at (x, y, z) $a, b, c = -dV/dx,\ -dV/dy,\ -dV/dz$; and therefore, writing for ϕ the current strength γ, we get instead of (6),

$$\mathsf{E} = -\gamma \iint (la + mb + nc)\, dS. \quad \text{.....................}(6')$$

If the surface, as supposed here, do not pass through magnetized matter, α, β, γ coincide in value with a, b, c; but it is easy to see that a, b, c ought to be used in the integral in the general case. For let the surface bounded by the circuit be taken so as to pass through a portion of another medium. Then since

$$\iint (la + mb + nc)\, dS$$

has the same value for all surfaces having the same bounding edge, it is an expression which gives the same value of E for all positions of the surface.

The integral in this equation is the value of the magnetic induction through the shell, and here and in what follows we denote it by N. It is to be taken positive or negative according as it passes through the shell from the negative to the positive or from the positive to the negative side, that is, according as its direction agrees with or is opposite to that in which a right-handed screw would move through the circuit if the handle were turned round in the direction of the current. Hence

$$\mathsf{E} = -\gamma N. \quad \text{................................}(7)$$

If the circuit is imbedded in a medium of magnetic permeability differing from unity, the magnetization of the medium must be taken into account in finding the potential energy of the system. We have simply as above to calculate the value of N for the circuit. It will not be necessary however to deal practically here with any such case. Those in which movable coils containing iron cores have to be dealt with do not cause any difficulty, since the magnetism of the core forms in each case part of the distribution producing the field.

16. Force on element of circuit. Now the magnetic forces acting on the shell are such as to diminish its potential energy; and hence, if $d\psi$ be any small change of position or configuration of the shell, and Ψ the corresponding force producing it, we have for the work done by this force $\Psi\, d\psi$. The sum of this and the change in the value of the potential energy is zero, that is

$$\Psi\, d\psi + d\mathsf{E} = 0, \quad \text{..............................(8)}$$

or, γ remaining constant,

$$\Psi = \gamma \frac{dN}{d\psi}. \quad \text{..............................(8')}$$

The direction of the electromagnetic force is therefore to increase N; that is, the circuit if free to move as a rigid whole will change its position so as to increase N, and, what is here of great importance, if flexible, will alter its form so as to include a greater value of N. It is clear, then, that no force acts on an element of the circuit in the direction parallel to the magnetic force, for a displacement in that direction would not alter the value of E, and the resultant electromagnetic force on each element is therefore at right angles to the magnetic force.

But the element itself, in the general case, is inclined to the direction of the magnetic induction. Let the angle between the latter direction (taken as that in which a north-seeking pole tends to move through the circuit) and that of the current in an element ds of the circuit be θ; and let the element be moved through any displacement $d\psi$ at right angles to the line of magnetic induction at its centre. The change in N is $\gamma \mathbf{B} \sin\theta\, ds\, d\psi$. Thus we have for the force on the element

$$\Psi = \gamma \mathbf{B} \sin\theta\, ds. \quad \text{..........................(9)}$$

The direction in which the element tends to move may be remembered by the following rule. Let, as supposed above, a human figure stand on the magnetic shell which replaces the circuit, so that, when the face of the figure is turned in the direction in which the current is flowing, the positive direction of the magnetic induction is from the feet of the figure towards the head. Then the element, if free to move, will do so towards the figure's right hand. Or, if the figure swim in the circuit so that the current enters at the feet and leaves at the head, and look in the positive direction of magnetic induction, the element will tend to move towards the *left* hand.

17. Equations of electromagnetic force. The direction of the force on an element of the circuit is shown in Fig. 41. The corresponding reaction is discussed below (Section II.).

Denoting by l, m, n, the direction cosines of ds, we have

$$\sin\theta = \{(mc - nb)^2 + (na - lc)^2 + (ma - lb)^2\}^{\frac{1}{2}}/\mathbf{B}.$$

Hence (9) becomes

$$\Psi = \gamma\{(mc - nb)^2 + (na - lc)^2 + (ma - lb)^2\}^{\frac{1}{2}} ds. \quad \text{........(9')}$$

If σ denote the area of cross-section of the conductor at the element ds, taken at right angles to the direction of γ, then u, v, w, the components of current, are defined by the equations

$$l\gamma/\sigma, \quad m\gamma/\sigma, \quad n\gamma/\sigma = u, \quad v, \quad w.$$

Substituting in (9′) and resolving Ψ/σ along the axes, denoting the components by X, Y, Z, we find instead of (9′),

$$\left.\begin{aligned} X &= vc - wb, \\ Y &= wa - uc, \\ Z &= ub - va. \end{aligned}\right\} \qquad (9'')$$

X, Y, Z, are the component electromagnetic forces per unit of volume acting on the conductor: we shall find them useful in considering action on non-linear conductors.

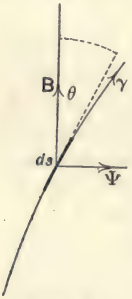

FIG. 41.

With regard to the potential energy of the shell and field, care must be taken, while using this expression for the calculation of the force on the circuit (a procedure the legitimacy of which follows from the theorem of equivalence as regards forces), not to allow it to cause any misconception as to the energy of the current in the field. It is not the case that there is any sensible mutual potential energy of the current and the magnetic distribution, such that, when the circuit moves in the field in obedience to magnetic force, exhaustion of this potential energy takes place in the same way as when the shell moves in the field. The shell and field remaining each unchanged, the magnets are set in relative motion, and kinetic energy is acquired, or work is done against external resistance at the expense of potential energy, which so far as our knowledge goes at present may be regarded as a function of the configuration of the system. On the other hand the fact, as illustrated by the experiments of Joule referred to below (Chapter V.), and all experience of the motion of conductors in magnetic fields, is that the kinetic energy acquired, or external work done, in the case of motion of the circuit, is obtained at the expense of the battery or electrical generator maintaining the current, and no available energy is gained or lost in virtue of geometrical displacement *per se*.

II. Action of Currents on Currents.

18. Mutual action of two circuits. It is a result of experiment that the equivalence of a current and a magnetic shell which enables the action of a current on a magnet, or of a magnet on a current, to be calculated, is also available for the determination of the action of currents on one another.* Experiments which prove this were made first by

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* It is to be clearly understood that electrostatic action due to difference of potential between adjacent conductors is not here taken into account. We shall have examples later of combined electrostatic and electrodynamic action.

Ampère, Weber, and others; but the best experimental proof of the truth of this proposition is to be found in the uniformly consistent results obtained by means of measuring instruments made and graduated to give absolute determinations by applying it. Weber's electrodynamometer was the first instrument of this kind constructed, and with it the inventor accurately verified the laws of electromagnetic action which had previously been announced by Ampère, as a deduction from his celebrated series of four experiments.

Ampère however, besides giving the theorem of the equivalence of currents and magnetic shells, took another view of the subject, in which he regarded every element of a conductor carrying a current as acted on by every element of the other conductor, and the law of action which he gave was a law for the mutual action between two elements. This law agrees with experiment in so far as it gives when applied over the whole circuit of each conductor exactly the electromagnetic action observed; but it is only one of several laws of action between elements which do the same thing. The actions in all cases which have been investigated have been actions between parts of different closed circuits, or between different parts of one closed circuit, and no difference in result has been found between these two cases. We are in ignorance of how two unclosed conductors, or two parts of an unclosed circuit, carrying currents (if such an arrangement can really be obtained) act upon one another, but, though this be true, it is allowable in the case of closed circuits to establish and use any formula for the mutual action of each pair of elements, which is mathematically true in the sense of giving the actual forces observed between the circuits. A simple expression of this kind is that found by Ampère. We shall here give first some account of Ampère's experiments, and show how by means of a certain assumption the law given by him can be deduced.

19. Ampère's experiments. These experiments were made by means of apparatus invented by Ampère himself, copies of which are now to be found in almost all collections of apparatus. The chief piece is one for enabling a part of a closed circuit (in itself generally nearly a closed circuit) to turn freely round a vertical axis. The arrangement with the movable conductor in position is shown in the diagram (Fig. 42).

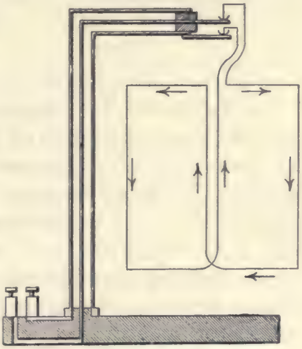

FIG. 42.

Two metallic cups containing mercury are arranged close together in the same vertical line at the extremities of two projecting arms, and in these rest the turned down extremities of the movable conductor. This has different forms according to the effect to be tested or measured. The two arms carrying the cups are in conducting contact with the mercury, and one of them is generally attached to a vertical metallic tube fixed to a heavy sole plate, the other is a continuation of a wire or rod which, insulated from the tube, passes up within it from the sole plate. The current is thus led to one cup and from the other without the conveying wires themselves producing any sensible action.

The portion of the circuit suspended in the cups in the first two experiments was (as shown in Fig. 42) a double rectangular frame of wire, the wires of which are insulated from one another at the points of crossing. This frame gives two nearly closed circuits of equal area; and round these the current flows in opposite directions, so that the suspended conductor does not experience any action in the earth's magnetic field.

Fig. 43 shows Ampère s apparatus as improved by M. Nodot. A vertical platinum wire is hung by a silk thread as shown, passes down through a mercury cup, the bottom of which is a plate of mica perforated by a hole just large enough to give the wire clearance. A rectangular frame is attached as shown, and a point at its lower end dips into a mercury cup, vertically under the platinum line above. The current is let in and out at the cups.

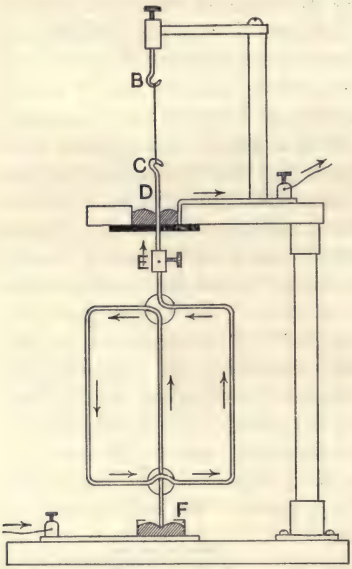

FIG. 43.

In Ampère's first experiment a wire (Fig. 44) carrying a current was doubled on itself, and the two portions were kept from touching by insulating material between them. This double wire being brought near and parallel to one side of the suspended frame, the latter did not experience any sensible deflecting force, showing that the effect of the current in one direction in one portion of the doubled conductor neutralized almost exactly the effect of the opposite current in the other part. Exact experiments show that this neutralization is complete, if one conductor be a tube containing the other.

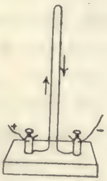

FIG. 44.

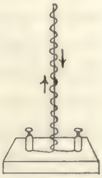

FIG. 45.

In the second experiment one of the two portions of the doubled wire was not straight, but (Fig. 45) contained a series of small and rather sharp bends, no part of any one of which was far from the straight conductor. The suspended conductor was still found unaffected. The conclusion from this experiment is that the effect of an element of a straight conductor may be replaced by that of a small crooked conductor having the same beginning and end as the element has, if the same current flow in both cases. In other words the effect of any element may be considered as the resultant in the ordinary sense of any number of component elements at the same place.

In Ampère's third experiment a conductor which formed an arc of a horizontal circle was made movable round a vertical axis through the centre of the circle. This was done by supporting the arc of wire at its ends on the convex surface of mercury projecting above a horizontal plane from troughs, and attaching it to a light radial arm of insulating material moving about the vertical axis. The current passed through the arc from one trough to the other. It was found that no magnet, or circuit carrying a current, produced any effect in moving the conductor in the direction of its length, that is the resultant force upon it was normal to the element.

In the fourth experiment currents were made to pass through three similar and nearly closed conductors *A*, *B*, *C* (Fig. 46), the middle one of which *B* was attached to the stand and was movable round a vertical axis. The currents in *A* and *C* were of equal strength and in the same direction; the direction and strength of the current in *B* were indifferent. The three circuits were similar in form, and the two, *A*, *C*, which were on opposite sides of the movable conductor *B*, were of very different dimensions, but so chosen that each dimension of the circuit *B* was *n* times the corresponding dimension of *A*, and $1/n$ of the corresponding dimension of *C*. The position of the conductor *B* relative to *C* was similar to that of *A* relative to *B*, and therefore the distance of any element of *C* from any element of *B* was *n* times the distance of the corresponding elements in *B* and *A*.

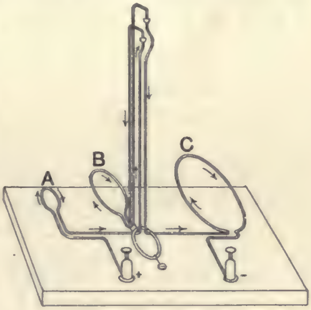

FIG. 46.

20. The action between two elements varies inversely as distance². The movable circuit *B* was thus subjected to two opposite force-systems from *A* and *C*, and was found to remain in equilibrium under that action. From this it follows that if we assume the action on the whole of the movable conductor to be made up of the actions on each of its elements of all the elements of the other two conductors, the action between any pair of elements varies inversely as the square of the distance between them. To prove this let r_1 be the distance between an element b_1 in *B* and an element a_2 in *A*, and r_2 the distance between two similarly situated elements, c_1 and b_2, in C and B; and let $f(r_1)$, $f(r_2)$ be the forces between the elements of the respective pairs per unit of length and per unit of current in each case. Now if ds be the length of each of the elements of B chosen, those of the elements a_2 and c_1 of A and C are respectively ds/n, $n\,ds$.

From the equilibrium of B it is clear that the forces for corresponding pairs of elements are equal, and therefore we have, if γ be the current in A and C and γ_1 that in B,

$$\frac{ds^2}{n}\gamma\gamma_1 f(r_1) = n\,ds^2\gamma\gamma_1 f(r_2)$$

or

$$\frac{f(r_2)}{f(r_1)} = \frac{1}{n^2}, \quad \text{.................................} (10)$$

that is the law of force is the inverse square of the distance.

21. Theoretical results of Ampère's experiments. Now, since by the second experiment each element can be replaced by its components, we may first resolve each into two components parallel to and at right angles to the line joining the centres of the elements. Also by the first experiment the forces are as the lengths of the elements and as the strengths of the currents. Let ds, ds' be the lengths of the elements AB, $A'B'$, θ, θ', the angles which they make with the line joining their centres, as shown in Fig. 47, then the components are $ds\cos\theta$, $ds'\cos\theta'$ along the line, and $ds\sin\theta$, $ds'\sin\theta'$ perpendicular to it. The last ($B'E'$ in the figure) is not in the same plane with $ds\sin\theta$, and gives, if η be the angle $B'E'F'$ between these two, the components $ds'\sin\theta'\cos\eta$, $ds'\sin\theta'\sin\eta$, parallel to and at right angles to $ds\sin\theta$. We now consider the actions (couples excluded) between the different pairs of these elements.

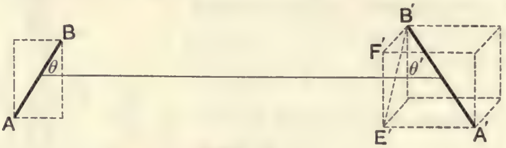

FIG. 47.

In the first place we have the two elements $ds\cos\theta$, $ds'\cos\theta'$ in the same straight line. The only determinate direction of any action between these two elements is the straight line in which they lie. We suppose, therefore, that when θ, θ' are both acute the force between the elements is an attraction. It has the value $A\gamma\gamma'\cos\theta\cos\theta'\,ds\,ds'/r^2$, where A is a constant, and r is the distance between the centres of the elements.

We take next the two elements $ds\sin\theta$, $ds'\sin\theta'\cos\eta$, which are parallel to one another and at right angles to the line joining their centres. The force here has for value $B\gamma\gamma'\sin\theta\sin\theta'\cos\eta\,ds\,ds'/r^2$, where B is a constant, and must act in the plane of the elements; for there is no reason why a component at right angles to this plane should act towards one side or the other of it. Further it must act in the line joining the elements, for to change the sign of one of the currents reverses the action, and to change the sign of both must leave the action unchanged. We shall suppose that it is also an attraction.

Lastly we have the four pairs of elements at right angles to one another. Of these the pair $ds \sin\theta$, $ds' \sin\theta' \sin\eta$ are at right angles to one another and likewise to the line joining their centres. Now if we make the assumption that the force between two components at right angles to one another is in the line joining their centres (an assumption necessary for Ampère's theory in the cases of the other two pairs of elements), we can easily prove that the force between the pair of elements now being considered is zero. Suppose that we have such a pair of elements α, β, at right angles to one another, and let the force act as shown in Fig. 48 from left to right. Now if the whole system be turned through 180° round the direction of α as an axis (or be looked at from the other side of the paper), the direction of β will be reversed, and the force will now act from right to left. The system then turned from its new position through 90° about the line joining α, β gives the original arrangement of the elements, with the force between them reversed. Hence no such force can exist.

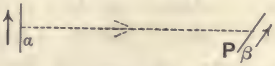

FIG. 48.

The assumption made above is not really necessary for this case, since, if there exist a component at right angles to the line of centres, it must act in the plane of one of the elements and the line of centres, or in a plane bisecting the angle between the planes of the elements and the line of centres, and there is nothing to determine in which plane it must act.

There remain the three pairs of elements $ds \cos\theta$, $ds' \sin\theta' \cos\eta$, $ds \cos\theta$, $ds' \sin\theta' \sin\eta$, and $ds \sin\theta$, $ds' \cos\theta'$, and the constituents of each pair are in one plane. Making the assumption stated above for these, we see that between the elements of each pair there can be no force, since if a force did exist it would not be reversed by the reversal of the current in $ds \sin\theta$, $ds' \sin\theta' \sin\eta$, or $ds' \sin\theta' \cos\eta$, for this would merely be equivalent to turning the whole system through 180° round the line joining the elements. We find therefore, collecting these results, a total force of attraction between the two elements ds, ds' of amount

$$dF = \gamma\gamma' ds\, ds' \frac{1}{r^2}(A \cos\theta \cos\theta' + B \sin\theta \sin\theta' \cos\eta), \quad \text{.......(11)}$$

and it remains to determine the coefficients A and B.

Now applying the result of Ampère's third experiment we resolve the force on ds into two components, one along ds, the other at right angles to it, and equate the integral of the former, taken round the circuit of ds', to zero. Hence

$$\gamma\gamma' ds\left\{A\int\frac{1}{r_2}\cos^2\theta\cos\theta' ds' + B\int\frac{1}{r^2}\sin\theta\cos\theta\sin\theta'\cos\eta\, ds'\right\} = 0. \quad (11')$$

This expression can be transformed as follows. We have by geometry, if the coordinates of the centres of the elements be x, y, z, x', y', z',

$$-\cos\theta = \frac{dr}{ds}; \quad -\cos\theta' = \frac{dr}{ds'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

and

$$r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2.$$

The last gives

$$r\frac{dr}{ds} = -r\cos\theta = (x-x')\frac{dx}{ds} + (y-y')\frac{dy}{ds} + (z-z')\frac{dz}{ds};$$

and differentiating this with respect to s' we get

$$\frac{dr}{ds}\frac{dr}{ds'} + r\frac{d^2r}{ds\,ds'} = -\frac{dx}{ds}\frac{dx'}{ds'} - \frac{dy}{ds}\frac{dy'}{ds'} - \frac{dz}{ds}\frac{dz'}{ds'} = -\cos\epsilon, \quad \ldots\ldots(13)$$

where ϵ is the angle between the elements ds, ds'.

But by (12)

$$r\frac{d^2r}{ds\,ds'} = r\sin\theta\frac{d\theta}{ds'} = r\sin\theta'\frac{d\theta'}{ds} = -\sin\theta\sin\theta'\cos\eta, \quad \ldots\ldots(14)$$

since by geometry

$$-r\frac{d\theta}{ds'} = r\frac{d\theta'}{ds'}\cos\eta = \sin\theta'\cos\eta.$$

Therefore by (12) and (13)

$$\cos\epsilon = -\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\eta. \quad \ldots\ldots\ldots\ldots(15)$$

Substituting from (12) and (14) in (11′) and rearranging, we find

$$\gamma\gamma' ds\left[(A-\tfrac{1}{2}B)\int\frac{1}{r^2}\left(\frac{dr}{ds}\right)^2 dr - B\int\frac{d}{ds'}\left\{\frac{1}{2r}\left(\frac{dr}{ds}\right)^2\right\}ds'\right] = 0. \quad \ldots(16)$$

22. Second proof of equation (16). Discussion of pairs of elements may be avoided and equation (16) proved as follows. We have seen that the mutual action of ds and ds' is equal to

$$\gamma\gamma'\, ds\, ds'\, f(\theta, \theta', \epsilon)/r^2,$$

where $f(\theta, \theta', \epsilon)$ is a function of the relative positions to be determined. But by Ampère's second experiment it is the sum of the forces between the element ds' and the three components dx, dy, dz, into which ds may be resolved. Hence

$$f(\theta, \theta', \epsilon)\, ds\, ds' = P\, dx\, ds' + Q\, dy\, ds' + R\, dx\, ds',$$

or

$$f(\theta, \theta', \epsilon) = P\frac{dx}{ds} + Q\frac{dy}{ds} + R\frac{dz}{ds},$$

where P, Q, R depend only on the position of ds'. Thus $f(\theta, \theta', \epsilon)$ is a linear homogeneous function of the direction cosines of ds; and similarly it must be a linear homogeneous function of the direction cosines of ds'. To fulfil these conditions and involve ϵ it must be made up of two parts, $A\, dr/ds\,.\,dr/ds'$, $Br\, d^2r/ds\, ds'$, where A and B are constants. Hence Ampère's third experiment gives

$$\gamma\gamma' ds \int \frac{1}{r^2} f(\theta, \theta', \epsilon) \frac{dr}{ds} ds'$$
$$= \gamma\gamma' ds \int \frac{1}{r^2} \left\{ -A \left(\frac{dr}{ds}\right)^2 \frac{dr}{ds'} + Br \frac{d^2r}{ds\, ds} \frac{dr}{ds} \right\} ds' = 0,$$

which is equivalent to (16).

The second integral in (16) vanishes when taken round the circuit of s', and we are left with the equation

$$\gamma\gamma' ds \left(A - \tfrac{1}{2}B\right) \int \frac{1}{r^2} \left(\frac{dr}{ds}\right)^2 dr = 0. \quad \text{..................(16')}$$

Now the integral in this equation does not in general vanish, and therefore $A = \frac{1}{2}B$. This may be seen by considering the particular case of a circuit, formed by two perpendicular straight lines, and a circular arc joining their extremities, acting on an element ds at the centre of the circular arc and in line with one of the straight lines. If the radius of the circle be c, and the distance of ds from the straight line perpendicular to it be a, the portion of the integral contributed by the latter straight line is $1/3a - a^2/3c^3$, by the other straight line $1/c - 1/a$, and by the circle zero. Hence the total integral is not zero. Hence substituting in (14), and using (15), we get

$$dF = B\gamma\gamma' ds\, ds' \frac{1}{r^2} \left(\cos\epsilon + \tfrac{3}{2}\cos\theta\cos\theta'\right), \quad \text{..............(17)}$$

Ampère's expression for the action between the elements.

Ampère assumed B to be equal to 1, which amounted to defining the unit of current as that current which flowing in the same direction in each of two parallel elements at unit distance apart gives unit force of attraction between them. We shall show that for agreement with the definition of unit current adopted above the value 2 must be given to B. Thus Ampère's unit of current is $1/\sqrt{2}$ of the electromagnetic unit of current now in ordinary use.

23. Ampère's expression deduced from the magnetic shell theory. Returning to equation (74), p. 72, for the mutual potential energy of two magnetic shells we are led, by the theorem of equivalence of currents and magnetic shells, to write for the mutual potential energy of two closed circuits

$$\mathsf{E} = -\gamma\gamma' \int \frac{\cos\epsilon}{r} ds\, ds'.^{*} \quad \text{.....................(18)}$$

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* F. Neumann gave $-\gamma\gamma' ds\, ds' \cos\epsilon/r$ as the mutual potential of the two elements. The corresponding expression in (19) is due to Weber. Either gives the same result for closed circuits as does Ampère's formula. Thus the forces between the circuits may be found from (17), (18), or (19). The energy of the system is calculated for particular cases in Chapters VI. and XIII. below.

We shall inquire what expression for the action between two elements can be deduced from the quantity on the right, and compare it with that given in (17). Substituting for $\cos\epsilon$ its value given by (13) and noticing that the integral of the complete differential $d^2r/ds\,ds'.ds$ is zero, we get

$$\mathsf{E}=\gamma\gamma'\iint\frac{1}{r}\frac{dr}{ds}\frac{dr}{ds'}ds\,ds'. \quad\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

Now let the circuit of ds be slightly deformed in any way while that of ds' is kept unchanged: r, dr/ds, dr/ds', ds will be affected by the deformation. The change in E is $\delta\mathsf{E}$, and by taking the variation of the right-hand side of (19), remembering that

$$\delta\frac{dr}{ds}=\frac{d\delta r}{ds}-\frac{dr}{ds}\frac{d\delta s}{ds},\quad \delta\frac{dr}{ds'}=\frac{d\delta r}{ds'},$$

we find
$$\delta\mathsf{E}=-\gamma\gamma'\left\{\iint\frac{1}{r}\frac{dr}{ds}\left(\frac{1}{r}\frac{dr}{ds'}\delta r-\frac{d\delta r}{ds'}+\frac{dr}{ds'}\frac{d\delta s}{ds}\right)ds\,ds'\right.$$
$$\left.-\iint\frac{1}{r}\frac{dr}{ds'}\frac{d\delta r}{ds}ds\,ds'-\iint\frac{1}{r}\frac{dr}{ds}\frac{dr}{ds'}d\delta s\,.\,ds'\right\}.$$

The last integral and the last term of the first integral cancel one another, and we have

$$\delta\mathsf{E}=-\gamma\gamma'\iint\left(\frac{1}{r^2}\frac{dr}{ds}\frac{dr}{ds'}\delta r-\frac{1}{r}\frac{dr}{ds}\frac{d\delta r}{ds'}-\frac{1}{r}\frac{dr}{ds'}\frac{d\delta r}{ds}\right)ds\,ds'. \quad\ldots..(20)$$

Integrating the last two terms by parts and rejecting the integrals, round the circuits, of perfect differentials, we get

$$2\gamma\gamma'\iint\frac{1}{r^2}\left(\frac{dr}{ds}\frac{dr}{ds'}-r\frac{d^2r}{ds\,ds'}\right)\delta r\,ds\,ds'$$

for the corresponding part of δE. Hence finally, by (20),

$$\delta\mathsf{E}=\gamma\gamma'\iint\frac{1}{r^2}\left(\frac{dr}{ds}\frac{dr}{ds'}-2r\frac{d^2r}{ds\,ds'}\right)\delta r\,ds\,ds', \quad\ldots\ldots\ldots\ldots(21)$$

which by (15) becomes

$$\delta\mathsf{E}=2\gamma\gamma'\iint\frac{1}{r^2}(\cos\epsilon+\tfrac{3}{2}\cos\theta\cos\theta')\,\delta r\,ds\,ds'. \quad\ldots\ldots\ldots..(22)$$

The interpretation of this result is an attraction of amount $2\gamma\gamma'(\cos\epsilon+\frac{3}{2}\cos\theta\cos\theta')/r^2$ between the elements ds, ds' in the line joining them. This agrees with Ampère's result and shows that the value of B in (17) is 2.

Having thus shown the equivalence of the two modes of regarding the mutual action of currents, we now give a very short account of the apparatus and experiments by which Weber investigated the subject.

24. Weber's experiments. Weber made his measurements of electromagnetic action by means of his electrodynamometer. This consisted of two circular coils, one suspended by bifilar wires (which also conveyed the current) so as to be free to turn round a vertical axis, the other coil fixed and arranged so that by levelling the planes of its windings could be made vertical. The apparatus was in two forms: (1) with the movable coil suspended within the fixed coil, with the centres as nearly as might be coincident; (2) with the fixed and movable coils distinct so that they could be placed at any required distance from one another, and in any relative positions. Deflections of the movable coil were measured by the mirror and telescope method described above (II. 8).

By the first experiment made by Weber it was proved that the electromagnetic action between the two currents varied as the square of the current strength. Apparatus (1) was used, and the fixed coil was set up with its axis perpendicular to, while that of the suspended coil was in, the magnetic meridian. Currents of different strengths were sent through the coils, and to prevent too great a deflection, the current through the suspended coil was reduced to 1/246·26 of the whole current by a shunt of thick wire inserted between the terminals to which the bifilar wires were attached. A magnetometer with magnetized steel mirror in a damping covering of copper was set up north of the fixed coil, at a distance of 58·3 centimetres, and the tangents of the deflections of this mirror (read by a telescope as in the other case) gave a comparative measure of the different currents used. The results shown in the following table were obtained; and from these it will be seen that the mutual action between the systems was proportional to the square of the current, that is, to the product of the strengths of the two (equal) magnetic shells.

No. of cells used.	Comparative values of force between coils $= A$.	Force on magnetometer needle in arbitrary units $= B$.	Force on needle found by formula $5{\cdot}15534\sqrt{A}$.	Diff. $B - 5{\cdot}15534\sqrt{A}$.
3	440·038	108·426	108·144	+0·282
2	198·255	72·398	72·389	+0·009
1	50·915	36·332	36·786	−0·454

In another series of experiments Weber used the apparatus (2). The axis of the suspended coil was placed horizontal and parallel to the magnetic meridian, while the fixed coil was placed with its axis at right angles to the magnetic meridian, and its centre (1) in the magnetic north and south horizontal line, (2) in the magnetic east and west line through that of the suspended coil. Experiments were made in each case with distances between the centres, of respectively 0, 30, 40, 50, 60 centimetres. The current from eight Bunsen's cells was sent through both coils, and also through a coil set up about 8 metres from the fixed coil so as to act on the magnetometer referred to above, and through a reversing key, so arranged that the current through the suspended coil could be sent first in one and then in the opposite direction without altering its direction in the rest of the circuit. The object of thus reversing the current was to determine and allow for the turning moment of the earth's magnetic field, when the axis of the suspended coil was deflected from the magnetic meridian. The corrected results of the experiments are shown in the table below, in which the second column for each series of positions gives the corresponding numerical values calculated by Ampère's formula, (17) above.

Distance of centres of coils apart.	Position of centres of coils.			
	In magnetic east and west line.		In magnetic north and south line.	
	Couple observed.	Couple calculated.	Couple observed.	Couple calculated.
cm 0	22960	22680	22960	22680
30	189·93	189·03	−77·11	−77·17
40	77·45	77·79	−34·77	−34·74
50	39·27	39·37	−18·24	−18·31
60	22·46	22·64	—	—

Here the results for the greater distances agree very fairly well with calculation from Ampère's formula, and we have shown that Ampère's formula and the magnetic shell theory give identical results.

It is to be remarked that in these experiments the two coils are not independent circuits; but that they may be so regarded is plain from the fact that the remaining portion of the circuit, if the wires are close or twisted together, is of no effect since it can be altered at pleasure without affecting the action between the coils, provided the current be maintained constant.

But the deflections θ, θ' in the two cases agree closely for the greater distances with the formulae

$$\tan\theta = \frac{2MM'}{d^3}\left(1+\frac{a}{d^2}\right), \quad \tan\theta' = \frac{MM'}{d^3}\left(1+\frac{b}{d^2}\right),$$

which express the action between two magnets of moments M, M', in the "end-on" and "side-on" positions and at distances d apart, great in comparison with the dimensions of the magnets.

Elaborate experiments have also been made by Cazin, Boltzmann, and others in verification of the theory. For these the student should consult Wiedemann, *Elektricität*, vol. iii.

25. Forces between straight conductors. It is an experimental fact that the action between two long parallel conductors carrying currents is an attraction when the currents are in the same direction, and a repulsion when the currents are in opposite directions, and that if the conductors are not parallel there is attraction between them if the directions of the currents in the portions forming equal acute angles with one another are both towards or both from the shortest line joining the conductors, and repulsion if the direction of one is towards that line, and of the other from it. We have not space here to go into calculations regarding such cases, but their general nature may easily be seen by considering the magnetic fields produced by the currents, and the consequent motions of the conductors according to the rules given above. In both cases the lines of force are closed curves surrounding each conductor, and it is obvious that if we consider each circuit completed by a return wire at a great distance, the magnetic induction through each will be increased by the approach of the conductors if the currents are in the same direction, and diminished if they are in the opposite direction. The same will clearly be the case if the two conductors considered be parts of the same circuit. The action of a current in a straight conductor, on an element of a parallel conductor, is shown in Fig. 49, which with the statements made above explains itself.

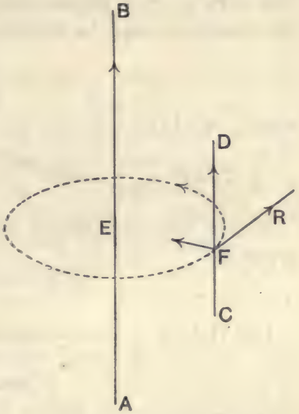

FIG. 49.

26. Ampère's formula applied to find the action of a thin solenoid. We give here the application made by Ampère* of his formula to the calculation of the force on an element of a conductor, and the turning moment on a given finite conductor produced by a simple solenoidal electromagnet, that is, a succession of infinitely small circuits arranged equidistantly at infinitely short distances apart with their centres on, and their planes at right angles to a given curve, and carrying currents such that the product of the area of the circuit and the current strength is the same in each case. The solution of this problem is of the greatest importance in Ampère's theory of magnetism, in which he supposes all effects of magnets to be produced by currents flowing in molecular circuits within the body. We shall see that the arrangement specified above is equivalent to a uniformly magnetized magnet, having a strength of magnetic pole equal to the sum of the products of current and area for the circuits round unit length of the given curve forming their common axis.

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* "Théorie des phénomènes électro-dynamiques," *Mémoires de l'Institut*, vi. 1823. The proof of Ampère's formula, and the applications here given, have been very elegantly treated by quaternion methods by Professor Tait: see his *Quaternions*, 2nd edition, p. 249.

Let *ds* be an element of the closed circuit, and consider its action on an element *ds'* of another conductor, the current being unity in each case. If the coordinates of *ds* be x, y, z, and the origin be taken at the centre of *ds'*, the direction cosines of *ds* and r are dx/ds, dy/ds, dz/ds, and x/r, y/r, z/r respectively. The expression for the action between the elements may be written

$$2ds\,ds'\frac{1}{r^2}\left(-r\frac{d^2r}{ds\,ds'}+\frac{1}{2}\frac{dr}{ds}\frac{dr}{ds'}\right),$$

which by (14) becomes

$$2ds\,ds'\frac{1}{r^2}\left(-r\sin\theta'\frac{d\theta'}{ds}-\tfrac{1}{2}\cos\theta\frac{dr}{ds'}\right)=2ds\,ds'r^{-\frac{1}{2}}\frac{d}{ds}(r^{-\frac{1}{2}}\cos\theta').$$

Hence the component of this action along the axis of x, or dX, is given by

$$dX=2ds\,ds'xr^{-\frac{3}{2}}\frac{d}{ds}(r^{-\frac{1}{2}}\cos\theta'). \quad \ldots\ldots\ldots\ldots(23)$$

But if λ, μ, ν be the direction cosines of *ds'*,

$$\cos\theta'=\lambda\frac{x}{r}+\mu\frac{y}{r}+\nu\frac{z}{r}.$$

Hence (23) becomes

$$\begin{aligned}dX&=2ds\,ds'\frac{x}{r^{\frac{3}{2}}}\left[\frac{d}{ds}\left\{\frac{1}{r^{\frac{3}{2}}}(\lambda x+\mu y+\nu z)\right\}\right]\\&=ds\,ds'\left\{\lambda\frac{d}{ds}\left(\frac{x^2}{r^3}\right)+\mu\frac{x}{y}\frac{d}{ds}\left(\frac{y^2}{r^3}\right)+\nu\frac{x}{z}\frac{d}{ds}\left(\frac{z^2}{r^3}\right)\right\}. \quad \ldots\ldots(24)\end{aligned}$$

But

$$\frac{x}{y}\frac{d}{ds}\left(\frac{y^2}{r^3}\right)=\frac{d}{ds}\left(\frac{xy}{r^3}\right)+\frac{1}{r^3}\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right),$$

and

$$\frac{x}{z}\frac{d}{ds}\left(\frac{z^2}{r^3}\right)=\frac{d}{ds}\left(\frac{xz}{r^3}\right)-\frac{1}{r^3}\left(z\frac{dx}{ds}-x\frac{dz}{ds}\right).$$

Substituting in (24) we find

$$\begin{aligned}dX=ds\,ds'\bigg[\frac{d}{ds}\left\{\frac{x}{r^3}(\lambda x+\mu y+\nu z)\right\}&+\frac{\mu}{r^3}\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)\\&-\frac{\nu}{r^3}\left(z\frac{dx}{ds}-x\frac{dz}{ds}\right)\bigg]. \quad \ldots\ldots(25)\end{aligned}$$

The first term disappears when integrated round the circuit of *ds*. Hence

$$X=ds'\left\{\int\frac{\mu}{r^3}\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)ds-\int\frac{\nu}{r^3}\left(z\frac{dx}{ds}-x\frac{dz}{ds}\right)ds\right\}.$$

Similarly we obtain

$$\left.\begin{aligned}Y&=ds'\left\{\int\frac{\nu}{r^3}\left(y\frac{dz}{ds}-z\frac{dy}{ds}\right)ds-\int\frac{\lambda}{r^3}\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)ds\right\},\\Z&=ds'\left\{\int\frac{\lambda}{r^3}\left(z\frac{dx}{ds}-x\frac{dz}{ds}\right)ds-\int\frac{\mu}{r^3}\left(y\frac{dz}{ds}-z\frac{dy}{ds}\right)ds\right\}.\end{aligned}\right\} \quad \ldots\ldots(26)$$

27. Ampère's directrix of electrodynamic action. Denoting the integrals (divested of the multipliers λ, μ, ν) in these expressions by A, B, C, we have

$$\left.\begin{aligned} X &= ds'(\mu C - \nu B), \\ Y &= ds'(\nu A - \lambda C), \\ Z &= ds'(\lambda B - \mu A). \end{aligned}\right\} \qquad (27)$$

These equations give $\lambda X + \mu Y + \nu Z = 0$, as they ought, since the component force along ds' is zero. Their form also shows that the resultant force on ds' is at right angles to the line the direction cosines of which are proportional to A, B, C, that is, its direction is at right angles to the plane through ds' and that line. The resultant of A, B, C, Ampère called the *directrix*. By comparison with (9) above we see that it is the magnetic induction at ds' produced by the circuit.

Equations of precisely the same form as (27) hold of course for any assemblage of circuits. In that case however A, B, C are sums of integrals of the form given in (26).

The component force in any plane may be found as follows. Let ϕ be the angle between the given plane and the plane containing ds' and the directrix. Then clearly the angle which the resultant force, R, makes with the given plane is $\pi/2 - \phi$, and the component is $R \sin \phi$. Squaring equations (27) and adding we find $R = ds'D \sin \omega$, where ω is the angle between ds' and the directrix, and $D = \sqrt{(A^2 + B^2 + C^2)}$. If ψ be the angle between the directrix and the given plane, we get, by projecting unit distance along the directrix on a line at right angles to ds', and then at right angles to the given plane, for the final projection the length $\sin \omega \sin \phi$. But the same line projected directly gives $\sin \psi$. Hence $\sin \omega \sin \phi = \sin \psi$. The component force in the given plane is therefore $ds'D \sin \omega \sin \phi = ds'D \sin \psi$. If a, b, c be the direction cosines of the normal to the given plane, $\sin \psi = aA/D + bB/D + cC/D$, and the component is $ds'(aA + bB + cC)$, or $ds'U$, where

$$U = aA + bB + cC. \qquad (28)$$

From this we obtain the remarkable result that the action in the given plane is independent of the direction of ds' if only the element lie in that plane.

28. Calculation of result for a small circuit. To apply the results found above to the problem of the solenoid, let the circuit be small and plane. The values of the components A, B, C can be calculated approximately for this case as follows. Let $MPQN$, Fig. 50, represent the circuit, and let it be cut by planes passing through the axis of z. Let two of these planes meet the circuit in MN, PQ, and Omn, Opq be their traces on the plane of x, y, meeting the projection of the circuit in mn, pq, we have for C the equation

$$C = \int \frac{1}{r^3}\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right) ds \qquad (29)$$

taken round the circuit. Clearly this may be written

$$C=\int\frac{u^2da}{r^3} \quad\text{..............................(30)}$$

if a be the angle which Omn makes with Ox, and u the distance from O of the element of the projection corresponding to ds.

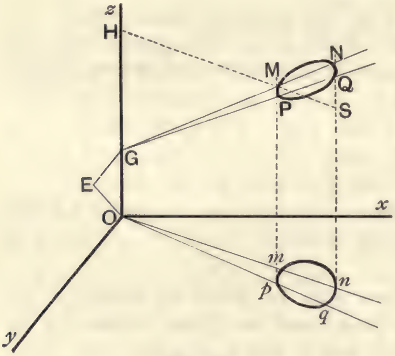

FIG. 50.

Now since the circuit is small we may suppose the elements mp, nq, intercepted by the planes, to be at a small distance δu apart, corresponding to a small distance δr between the actual elements MP, NQ. We then find the value of C by calculating the sum of the contributions to it corresponding to the pairs of elements mp, nq of the projection on the plane of xy. Now taking the area swept out by the radius vector as positive when the end of the radius moves from m to p, and therefore negative when it moves from q to n, and taking a between the extreme tangents drawn from O to the projection, we get

$$C=\int\left\{\frac{u^2}{r^3}-\frac{(u+\delta u)^2}{(r+\delta r)^3}\right\}da=\int\frac{1}{r^3}\left(\frac{3u^2}{r}\,\delta r-2u\,\delta u\right)da. \quad\text{........(31)}$$

Now $r^2=u^2+z^2$, and therefore $\delta r=(u\,\delta u+z\,\delta z)/r$. Letting fall a perpendicular OE from O, on the plane of the circuit, and calling its length h, and its direction cosines l, m, n, we have $OG=h/n$. But if HMS be drawn parallel to the plane of x, y, we have by the similar triangles MSN, MHG, $\delta z/(z-OG)=\delta u/u$. Hence

$$\delta r=\{u^2+z(z-h/n)\}\,\delta u/ur=(r^2-zh/n)\,\delta u/ur.$$

Substituting in (31), and taking mean values of r and z, which since the circuit is small, may be those for the mean point of the area, we have, if S be the area of the circuit,

and similarly,

$$\left.\begin{aligned} C&=\frac{1}{r^3}\left(1-\frac{3zh}{nr^2}\right)nS,\\ A&=\frac{1}{r^3}\left(1-\frac{3xh}{lr^2}\right)lS,\\ B&=\frac{1}{r^3}\left(1-\frac{3yh}{mr^2}\right)mS.\end{aligned}\right\} \quad\text{........................(32)}$$

29. Application of the result to a solenoid. Now consider a solenoid, as defined above (p. 171), made up of such circuits uniformly arranged along a common axis, and such that the current and the area in each case have a constant infinitely small product; then taking the circuits as infinitely close, denoting by g the sum of these products per unit of length of the axis, and by ds an element of the axis, we have $g\,ds$ for the sum of the products for the element ds. Hence, to find A, B, C for this part of the assemblage of circuits, we have to substitute $g\,ds$ for S in (32). Doing this, and denoting by A', B', C' the values of A, B, C, for the whole assemblage of circuits, we have

$$\left.\begin{aligned} A' &= g\int \frac{1}{r^3}\left(l - \frac{3hx}{r^2}\right)ds, \\ B' &= g\int \frac{1}{r^3}\left(m - \frac{3hy}{r^2}\right)ds, \\ C' &= g\int \frac{1}{r^3}\left(n - \frac{3hz}{r^2}\right)ds. \end{aligned}\right\} \quad \text{......................(33)}$$

Now we have here

$$l,\ m,\ n = dx/ds,\ dy/ds,\ dz/ds, \quad \text{and} \quad h = x\,dx/ds + y\,dy/ds + z\,dz/ds.$$

Hence substituting and integrating from one end of the axis of the solenoid to the other, we find

$$A' = g\left(\frac{x_2}{{r_2}^3} - \frac{x_1}{{r_1}^3}\right), \quad B' = g\left(\frac{y_2}{{r_2}^3} - \frac{y_1}{{r_1}^3}\right), \quad C' = g\left(\frac{z_2}{{r_2}^3} - \frac{z_1}{{r_1}^3}\right), \quad \text{....(34)}$$

where the suffixes distinguish the values of the quantities for the two ends of the solenoid.

These values of A', B', C' are proportional to the direction cosines of the directrix for this case, and substituted in (27) give the components of the force on ds'. It is evident that the force on ds' depends only on g and the positions of the ends of the solenoid.

If the axis be a closed curve or extend to infinity in both directions, the values of A', B', C' are zero, and hence by (27) the solenoid exerts no force on ds'.

If the axis extend to infinity in the direction of integration the first terms in (34) are zero, and we have

$$A' = -gx_1/{r_1}^3, \quad B' = -gy_1/{r_1}^3, \quad C' = -gz_1/{r_1}^3. \quad \text{.........(35)}$$

30. A singly infinite solenoid equivalent to a magnetic pole. Substituting these values in (27) we see at once that the action is at right angles to the plane through ds' and the extremity of the solenoid. Let now the conductor be straight and infinitely extended in both directions. Then changing the origin to the extremity of the solenoid, taking as the plane of xy the plane through the conductor and the end of the solenoid, and the direction of the conductor as that of x, we find $A' = gx/r^3$, $B' = gy/r^3$, $C' = 0$, $\lambda = 1$, $\mu = \nu = 0$. Hence the resultant force is $Z = g\,ds'a/(a^2 + x^2)^{\frac{3}{2}}$, where a is the constant value of y. Writing dx for ds', and integrating over the whole conductor from $-\infty$ to $+\infty$, we find the value $2g/a$ for the total force. This result shows that the action between the conductor and the extremity of the solenoid is the same as that (see 10 above) between the conductor and a magnetic pole of strength g.

Returning to (35) it is clear that if we had another solenoid with value of g numerically the same but opposite in sign, and extending to infinity from the point x_2, y_2, z_2, we should have for it

$$A' = gx_2/r_2^3, \quad B' = gy_2/r_2^3, \quad C' = gz_2/r_2^3, \quad \text{..............(36)}$$

and these two solenoids acting together would be equivalent to the finite solenoid already discussed.

31. Action of a solenoid on a finite conductor. We can now consider the action of a finite solenoid on a conductor of finite length s' carrying a current of unit strength. The component forces on the element ds' are by (27) and (34),

$$\left.\begin{array}{l} X = g\,ds'\left\{\mu\left(\dfrac{z_2}{r_2^3} - \dfrac{z_1}{r_1^3}\right) - \nu\left(\dfrac{y_2}{r_2^3} - \dfrac{y_1}{r_1^3}\right)\right\}, \\ Y = \qquad \text{etc.} \qquad \text{etc.}, \\ Z = \qquad \text{etc.} \qquad \text{etc.} \end{array}\right\} \quad \text{............(37)}$$

If we no longer take the origin at ds', but transfer it to the extremity x_1, y_1, z_1 of the solenoid, and take the line joining its ends as axis of x, the calculation of the action on the conductor will be simplified. The coordinates of ds' are now $-x_1$, $-y_1$, $-z_1$, which we shall write x, y, z. If l be the numerical value of the distance between the ends of the solenoid, $x_2 = -x + l$. Substituting these in (37) with the values dx, dy, dz for $\lambda\,ds'$, $\mu\,ds'$, $\nu\,ds'$, we get equations adapted to the calculation of the force components for the whole conductor.

We shall apply these to find the moment tending to turn the conductor round the line joining the extremities of the solenoid. The moment, dM, of the forces on ds' is $Zy - Yz$. Calculating this by equations (37), modified as just described, we find after reduction

$$dM = g\left\{\frac{d}{ds'}\left(\frac{x-l}{r_2}\right) - \frac{d}{ds'}\left(\frac{x}{r_1}\right)\right\}ds' = g\frac{d}{ds'}(\cos\theta_2 - \cos\theta_1)\,ds', \quad \text{...(38)}$$

where θ_2, θ_1 are the angles which the radii drawn from the extremities of the solenoid make with the axis of x. Integrating from one end of the conductor to the other, and distinguishing by accents the angles for the end where the integration terminates from the angles for the end at which it begins, we get finally

$$M = g(\cos\theta'_2 - \cos\theta'_1 - \cos\theta_2 + \cos\theta_1). \quad \text{..............(39)}$$

32. A solenoid compared with a uniform magnet. This result, derived from Ampère's formula, agrees with that which we should obtain from equation (8) above, by considering the solenoid as an infinitely thin uniformly magnetized bar-magnet. The turning moment of such a magnet on the conductor may be obtained most simply as follows.

The magnetic field of the magnet may be regarded as produced by equal and opposite quantities of magnetism at its extremities. Take the line joining these as axis, and draw lines from the ends to an element ds of the conductor making with the positive direction of the axis the angles θ_1, θ_2, and let the element make angles ϕ_1, ϕ_2 with these lines. First suppose the conductor wholly in a plane through the axis. Let the strength of each pole be m, then considering the action first of the positive pole (distant r_1 from the element), the force on the element is $m\,ds \sin\phi_1/r_1^2$, and its direction is at right angles to the plane in which the conductor lies. The moment of this force round the axis is therefore $m\,ds\sin\theta_1\sin\phi_1/r_1$. Similarly the other pole gives a moment $-m\,ds\sin\theta_2\sin\phi_2/r_2$. The total moment is therefore

$$m\,ds(\sin\theta_1\sin\phi_1/r_1 - \sin\theta_2\sin\phi_2/r_2).$$

But $r_1\,d\theta_1/ds = \sin\phi_1$, $r_2\,d\theta_2/ds = \sin\phi_2$. Hence the moment may be written $m(\sin\theta_1\,d\theta_1 - \sin\theta_2\,d\theta_2)$, and the total moment is therefore

$$m(\cos\theta'_2 - \cos\theta'_1 - \cos\theta_2 + \cos\theta_1), \quad \text{..................}(40)$$

which, if m be taken equal to g, agrees with the value given in (39).

If the conductor be not in a plane through the axis, we may resolve any element ds into two components—one in such a plane, the other at right angles to it. The latter component will be acted on by a force passing through the axis, and therefore having no moment round it. Each element having been thus dealt with, all the components in planes through the axis may by rotation round the axis be transferred without alteration of turning moment to one such plane. They will therefore give a continuous curve in that plane, the values of θ_1, θ_2, θ'_1, θ'_2, for which are the same as for the actual conductor.

By (40) and the equation of the lines of force of such a magnet, (18) Chap. II. above, we obtain the following interesting result. Let the lines of force be revolved round the magnet so as to generate coaxial surfaces. Then the turning moment on a conductor carrying a given current in the field of the magnet, is the same whatever be the length and position of the conductor, provided it terminate in the same two of these surfaces. The moment is equal to the difference of the parameters of these surfaces, and is therefore zero when both ends are on the same surface.

33. Moment on a conductor in the field of a single pole. By the process just employed we can show that the moment on a conductor in the field of a single magnetic pole, tending to turn it round any axis through the pole, depends only on the position of its ends. For each element can be resolved into two components, one in a plane through the element and containing the axis, the other at right angles to that plane. The force on the latter passes through the axis, and gives no moment. The former gives (supposing current and pole both unity) a moment $-\sin\theta\,d\theta$, where θ is the angle which the line drawn to it from the pole makes with the axis, and $d\theta$ is the change in θ between the ends of the element, for this is the same both for the element and its component in the plane through the axis. Hence integrating along the conductor from θ_1 to θ_2 we get $\cos\theta_2 - \cos\theta_1$ for the total moment. This is zero if $\theta_1 = \theta_2$, that is if the circuit is closed.

We obtain the same result for each of any number of magnetic poles, and hence if we have any number of magnetic poles in one line, the moment which they exert on an unclosed conductor in their field is $\Sigma dm(\cos\theta_2 - \cos\theta_1)$, the sum being taken for every element, dm, of magnetism in the magnet. Hence for a closed circuit this sum is zero round the line in which the elements lie.

It follows by equality of action and reaction that the couple which a straight linear distribution of magnetism experiences round its own line as axis, or that on a single pole round any axis, in consequence of the action of the current in a closed conductor in the corresponding field, is zero.

We can employ the result obtained above to find the equation of the lines of force of a uniformly magnetized magnet, or indeed of any straight linear distribution of magnetism. For let there be a single magnetic pole at a given point. Let the conductor be supposed made of flexible material, and to be held fixed at one end and laid along a line of force. Then let the other end be carried round the axis on which lies the magnetism, and be stretched and guided so as always to rest on the surface swept out by the line of force. No work is done against or by the action of the field, since the conductor is nowhere made to cut across lines of force. Hence for a single pole we have the equation, $\cos\theta_1 - \cos\theta_2 = 0$, that is the line of force is a straight line through the pole. In the same way we find for any assemblage of poles in a straight line the equation of the lines of force

$$\Sigma(\cos\theta_1 - \cos\theta_2) = \text{const.} \qquad \text{.......................(41)}$$

34. Reaction of the elements of a circuit on a magnetic system. It has been shown (V. 16 above) that an element of a circuit carrying a current in a magnetic field, in which the induction at the element is $\mathbf{B}$, is acted on by a force at right angles to the element and the direction of the magnetic induction, of amount $\mathbf{B}\gamma \sin\theta\, ds$, where ds is the length of the element and θ the angle between its direction and that of the magnetic induction. We may of course suppose the induction at the element to be due to a single magnetic pole of proper strength, and properly situated in a field of uniform permeability. The reaction of the element on the magnetic system in the general case, and in this on the pole will therefore be $\mathbf{B}\gamma \sin\theta\, ds$.

If we suppose the action on the element to be a single force of magnitude $\mathbf{B}\gamma \sin\theta\, ds$ applied at the element itself, the reaction on the pole will be an equal and opposite force *at the element*, and this is equivalent to an equal force at the pole and a couple. The moment of the integral couple due to the whole circuit is zero, as has just been seen. We may therefore use the elementary forces on unit pole to calculate the magnetic action of the circuit upon it, that is the magnetic field-intensity which the circuit there produces, with certainty that no action between the pole and the circuit will be neglected, although these, or any other terms which integrated round the circuit give a zero result, are left out of account.

Thus we take as the intensity of the field produced at the pole by the element the expression $\mathbf{B}\gamma \sin\theta\, ds$. But $\mathbf{B}$ is $\mu\mathbf{H}$, where $\mathbf{H}$ is the intensity of the field produced at the element by the pole, and if a spherical surface of radius r equal to the distance between the pole and element be described round the pole as centre, we shall have $4\pi r^2\mathbf{B} = 4\pi$, or

$$\mathbf{B} = \frac{1}{r^2}, \quad \mathbf{H} = \frac{1}{\mu r^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

and $\mathbf{B}\gamma \sin\theta\, ds = \gamma \sin\theta\, ds/r^2$.* The direction of this force is at right angles to the plane through the element and the line joining its centre to the pole. To specify the direction of the field-intensity due to an element, let an observer be supposed immersed in the current in the element, so that it flows from his feet to his head, and have his face turned towards the pole, then the latter, if positive or north-seeking, will tend to move towards his right hand.

It is to be observed that this result illustrates the fact stated in V. 14 above, that the action of a circuit on a magnetic pole, and therefore on any other distribution of magnetism, is independent of the nature of the medium occupying the field.

It also gives another definition of unit current (which may be compared with that given in V. 3 above), as that current which flowing in a thin wire forming a circle of unit radius acts on a magnetic pole placed at the centre with unit force per unit length of the circumference. Since the forces due to all the elements are in the same direction, the force at the centre of a circle of radius r carrying a current of strength γ is $2\pi\gamma/r$. Unit current is therefore that current which, flowing in a circle of unit radius, produces a magnetic field-intensity at the centre of 2π units.

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* This law is generally attributed to Laplace, who wrote nothing on electromagnetism. It appears to have been given orally by Laplace in a discussion at the Académie des Sciences, in 1820, of a paper by Biot and Savart, entitled, "Note sur le magnétisme de la pile de Volta," which was never published. [See *Annales de Chimie et de Physique*, xv. 1820.]

CHAPTER VI.

MAGNETIC FIELDS DUE TO CURRENTS. MAGNETIC ACTION OF COILS.

1. Unit current. The numerical measure of a current is defined by the intensity of the magnetic field produced by it at a given point. This mode of numerically reckoning currents, which we take as the fundamental method, gives results which are consistent with those obtained by other methods which are sometimes used, for example that of electrolysis. The definition may also be stated as follows.

Unit current is that which, flowing in a plane linear circuit of unit area, can be replaced by a magnet of unit magnetic moment, placed within the circuit and at right angles to it, without altering appreciably (according to the amount of inaccuracy tolerated) the magnetic field at a point at a distance from the circuit which can be regarded as small in comparison with any dimension of the circuit or the magnet. In the c.g.s. system unit magnetic moment is the moment of a doublet composed of two opposite point-charges of magnetism, each 1 c.g.s. unit, placed at a distance of 1 cm apart. Thus when the area of the circuit is 1 sq cm, and it is replaceable as regards magnetic action by such a doublet, the current flowing is 1 c.g.s. unit.

2. Magnet equivalent to a current. Magnetic shell. The equivalence of a linear circuit and a magnetic shell has been discussed in IV. 3, 4. We there saw that the magnet equivalent at distant points to the plane circuit may be supposed replaced by a very large number of equal small magnets uniformly distributed over the area enclosed by the circuit, with their centres in and their lengths at right angles to this plane.

Also it was shown in Fig. 35 that the circuit may be converted by cross conductors into a network without any displacement of the boundary, and that round each mesh a current γ may be supposed to flow in the same direction as that flowing in the original conductor. We have shown that the action of the boundary current is the same as that of the system of mesh currents imagined. But each mesh may be taken so small that it may be regarded, with as little error as we please, as a plane circuit: and each of these small circuits is replaceable, as we have seen, by a small magnet, or by a magnetic shell of strength equal to the current. This replacing of each of the meshes by a shell would give a shell of strength γ bounded by the circuit.

Any point at which the action of the finite shell is considered need only be at a distance from any part of the equivalent shell great in comparison with the dimensions of a mesh, hence the limitation as to distance imposed in the preliminary theorem does not apply. It is only necessary to take into account the finite thickness of the wire, and therefore consider magnetic action at points at a distance of several diameters of the wire from the boundary.

It is also clear that, provided the boundary of the shell, that is, the circuit, be undisturbed, the meshes may be supposed to have any positions we please, in other words the shell is defined by its boundary alone. [See however, also, II. 21.]

Since a circuit carrying a current is equivalent to a magnetic shell of strength equal to the current, all the theorems in Chapter IV. regarding the energy of a magnetic distribution, hold for fields of currents, and the whole mathematical theory can be transferred to the magnetic action of currents, or, as we call it, *electromagnetics*. For example, we have seen that the energy of a magnetic shell is equal numerically to the product of the strength of the shell and the integral of magnetic induction enclosed by its boundary, and we shall now derive from this theorem important results. We have therefore only to substitute in the equations of Chapter IV. Section III., γ for ϕ. The sketch of the subject of the equivalence of linear currents and magnetic shells just given should be supplemented by reference to Ampère's *Memoir* and to Maxwell's treatise or other works, *e.g.* the author's *Magnetism and Electricity*, Vol. I.

3. Method of vector potential. In II. 18, we have set forth briefly the method of expressing the magnetic induction by means of the auxiliary function called vector potential. All the results obtained for the vector potential of a system of magnetized bodies hold also for the magnetic fields produced by current-carrying circuits. There has been given (*loc. cit.*) a specification of the vector potential which we must now restate so as to make it more suitable for the discussion of fields due to linear conductors. This has practically been done in II. 23, in an important case; we add here some further particulars of specification as modified for the present purpose.

The direction of the vector **A** as specified for the magnetic element, of moment $\mathbf{I}\,\delta v$, of the magnetized body, corresponds precisely to that of the magnetic force at a point P, due to an element of length ds of a current-carrying linear circuit replacing $\mathbf{I}\,\delta v$, so that the direction of the current is the same as that of magnetization. The brief discussion in 6 below may help to make this point and its consequences clear to the reader.

4. Electrokinetic energy of currents. The field is the seat of magnetic energy, which depends on the state of the field at the instant considered, and is not necessarily equal to the energy which has up to that time been thrown into the field from a battery or other source. That there is energy depending on the state of the field, as distinguished from the total amount which has been furnished by the source, is clear from the dissipation of energy in hysteresis, a subject to which we shall return later. We here regard magnetic energy as kinetic, or, as we call it, electrokinetic energy, and hence in a system not subject to dissipative forces we must choose the sign of the energy so that the mutual forces of the parts of the system will tend to cause the amount of energy to *increase*. Thus a circuit brought into a field must tend, in virtue of the forces exerted upon it by the field, to move so as to increase its electrokinetic energy, that is we must choose the sign of the surface integral of magnetic induction enclosed by the circuit so that in any actual case the mutual forces may tend to its increase. Thus if T_{cf} denote the mutual energy of the circuit and field, and $d\psi$ any small change of position or configuration of the circuit, and Ψ the force producing it, due to the mutual action of the circuit and field, the work done by this force is $\Psi\, d\psi$. Thus

$$\Psi\, d\psi = dT_{cf},$$

or

$$\Psi = \frac{dT_{cf}}{d\psi} = \gamma \frac{dN}{d\psi}, \qquad \text{.............................(1)}$$

if N denote the magnetic induction through the circuit and γ be maintained constant in the change.

It is to be observed that we have no means of telling what are the actions between elements of currents, or between elements of currents and the different parts of the magnetic distribution. For example, if we consider as fundamental the action of an element of a current-circuit on a pole, the action of the pole on the element of circuit must be taken as equal and opposite to that, and as existing *at the pole*. Thus there arise a force *on the element* and a couple. It may be shown that the resultant couple is zero for the whole circuit. This will be referred to again below.

It is in fact to be remembered that the division of the circuit into elements is artificial, though it can be depended on to give the forces here discussed. For this there is sufficient experimental evidence. The couples are here ignored.

The force Ψ which thus tends to increase T_{cf} by increasing the magnetic induction through the circuit is called the electromagnetic force on the circuit. If free to move as a rigid whole, the circuit will change its position in obedience to this force so as to increase N, and if flexible, so as to be capable of changing its form, will tend to increase its area so as to include a larger total induction.

The resultant of electromagnetic force on each element of a circuit can only be in the direction at right angles at once to the magnetic force and to the element, because the element, if free to move in that direction, would increase the magnetic induction through the circuit at the greatest rate. Thus there is no electromagnetic force in the direction of the magnetic force on an element, since a displacement in that direction would not alter the electrokinetic energy of the circuit.

In general, however, the elements of the circuit are inclined to the direction of the magnetic induction. Let the angle between the direction of the current in an element of the circuit and the positive direction of the induction be θ, and let the element be displaced through a distance $d\psi$ in a direction at once normal to itself and to the direction of the magnetic induction. The element may be supposed moved out along guiding wires placed in this direction at its extremities. Let ds be its length. The change in N is the product of the induction **B** at the element into the component of the length of the element in a plane at right angles to **B** into the displacement; that is,

$$dN = B \sin\theta \, ds \, . \, d\psi .$$

Hence

$$d T_{cj} = \gamma B \sin\theta \, ds \, d\psi$$

and the force on the element $d\psi$ is

$$d\Psi = \gamma B \sin\theta \, ds. \qquad \text{.................................(2)}$$

5. Currents distributed in space of three dimensions. If the direction cosines of ds be l, m, n, we have, using the components a, b, c of **B**, the equation

$$\sin\theta = \frac{\{(mc-nb)^2 + (na-lc)^2 + (lb-ma)^2\}^{\frac{1}{2}}}{B}.$$

and therefore for (2) the alternative form

$$d\Psi = \gamma \{(mc-nb)^2 + (na-lc)^2 + (lb-ma)^2\}^{\frac{1}{2}} ds. \qquad \text{...........(3)}$$

Substituting in this for the values $(l, m, n)\gamma/\sigma$, the components u, v, w of the current in the direction of the axes, taken per unit of the area σ of the cross-section of the conductor, we find

$$d\Psi = \{(vc-wb)^2 + (wa-uc)^2 + (ub-va)^2\}^{\frac{1}{2}} ds \, . \, \sigma. \qquad \text{............(4)}$$

From this, supposing ds in the direction of y, so that $l = n = 0$, and **B** in the plane of y, z, so that $a = 0$, we get

$$d\Psi_1 = (vc - wb)\sigma \, . \, ds, \qquad \text{..............................(5)}$$

that is $vc - wb$ is the electromagnetic force per unit of volume on the conductor in the direction of x. Denoting the components per unit volume in the directions of x, y and z, by X, Y, Z, we get

$$dX = (vc - wb)\sigma \, dx, \quad dY = (wa - uc)\sigma \, dy, \quad dZ = (ub - va)\sigma \, dz. \qquad \text{.....(6)}$$

6. Specification of vector-potential. We now recall the specification of vector potential in order to adapt it to the case of fields due to currents. Consider an element, volume δv, of the magnetized substance, at which the intensity of magnetization is **I**. The magnetic moment of the element is $\mathbf{I}\,\delta v$. Then the vector-potential due to this element is

$\mathbf{I}\,\delta v \sin\phi/r^2$, at a point distant r from the element on a line making an angle ϕ, say, with the positive direction of magnetization. The direction of this element of $\mathbf{A}$ is at right angles to the plane of the angle ϕ, and in accordance with the direction chosen as that of integration round a circuit or path, appears to an eye regarding the path of integration of $\mathbf{A}$ in the direction opposed to that of $\mathbf{I}$, to be directed round the curve in the counter-clock direction.

The direction of $\mathbf{A}$, thus specified for the element of magnetic moment $\mathbf{I}\,\delta v$, corresponds precisely to that of the direction of the magnetic force at P, due to an element of a current-carrying circuit replacing $\mathbf{I}\,dv$, so that the direction of the current is the same as that of magnetization.

We verify this specification as follows. Denote the direction cosines of I by l, m, n, the coordinates of $I\,dv$ by x, y, z, and the coordinates of the point considered by ξ, η, ζ; then, putting I for the scalar value of $\mathbf{I}$, we have

$$I\,\delta v\frac{\sin\theta}{r^2}=\frac{I\,\delta v}{r^3}\left[\{m(\zeta-z)-n(\eta-y)\}^2+\ldots\right]^{\frac{1}{2}}, \quad \ldots\ldots\ldots\ldots(7)$$

and therefore

$$dF=\frac{I\,\delta v}{r^3}\{m(\zeta-z)-n(\eta-y)\},\quad dG=\frac{I\,\delta v}{r^3}\{n(\xi-x)-l(\zeta-z)\},$$

$$dH=\frac{I\,\delta v}{r^3}\{l(\eta-y)-m(\xi-x)\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

The values of F, G, H are to be obtained by putting A, B, C for Il, Im, In and integrating throughout the whole distribution of magnetism. This process leads by the theory of magnetism to the equations

$$a=\frac{\partial H}{\partial\eta}-\frac{\partial G}{\partial\zeta}=\mu_0\alpha+4\pi A',\quad b=\frac{\partial F}{\partial\zeta}-\frac{\partial H}{\partial\xi}=\mu_0\beta+4\pi B',$$

$$c=\frac{\partial G}{\partial\xi}-\frac{\partial F}{\partial\eta}=\mu_0\gamma+4\pi C', \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

where A', B', C' are the components of magnetization at the point ξ, η, ζ, and μ_0 is the inductive capacity of the medium, with respect to which the components of magnetization are taken. The terms $4\pi(A', B', C')$ are zero if the point considered does not fall within the distribution of magnetism.

7. Vector potential for a magnetic shell. Mutual inductance of two shells. Now let the distribution of magnetism producing the field be a uniform magnetic shell produced by a current of strength γ flowing in a linear circuit, which forms by Ampère's law the edge of the shell, and consider a second circuit carrying a current γ' in the field of the first. The magnetic induction through the second circuit is

$$\int(l'a+m'b+n'c)\,dS',$$

where now l', m', n' denote the direction cosines of the normal drawn outward from the positive side of the element dS' of the surface of that shell, and we have

$$\int\left(F\frac{dx'}{ds'}+G\frac{dy'}{ds'}+H\frac{dz'}{ds'}\right)ds'=\int(l'a+m'b+n'c)\,dS', \quad\text{.........(10)}$$

where dx', dy', dz' are the projections on the axes of an element ds' of the second circuit. But by (8) above, we have, since $I\,\delta v=\gamma\,dS$ in the present case, and $r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$,

$$F=\gamma\int\left(m\frac{\partial}{\partial z}\frac{1}{r}-n\frac{\partial}{\partial y}\frac{1}{r}\right)dS, \quad\text{......................(11)}$$

with similar equations for G and H. The integration is taken over the surface of the first shell, and reduces obviously for the F-term to

$$\gamma\int\frac{1}{r}\frac{dx}{ds}ds,$$

taken round the first circuit. Thus we get finally the equation

$$\gamma\int\frac{1}{r}\left(\frac{dx}{ds}\frac{dx'}{ds'}+\frac{dy}{ds}\frac{dy'}{ds'}+\frac{dz}{ds}\frac{dz'}{ds'}\right)ds\,ds'=\int(l'a+m'b+n'c)\,dS',$$

or, as we may write it,

$$\gamma\int\frac{\cos\epsilon}{r}\,ds\,ds'=\int(l'a+m'b+n'c)\,dS', \quad\text{.................(12)}$$

where ϵ is the angle between the two elements, ds of the first circuit and ds' of the second, and r is the distance between the elements.

If we multiply both sides by γ', we have

$$\gamma\gamma'\int\frac{\cos\epsilon}{r}\,ds\,ds'=\gamma'\int(l'a+m'b+n'c)\,dS'.$$

We infer from the left-hand side that

$$\gamma\int(la'+mb'+nc')\,dS=\gamma'\int(l'a+m'b+n'c)\,dS', \quad\text{............(13)}$$

that is, that the magnetic induction through the first circuit due to the current in the second is equal to the magnetic induction through the second circuit due to the current in the first. If both currents are unity we get for the magnetic induction M through either the equation

$$M=\int\frac{\cos\epsilon}{r}\,ds\,ds'. \quad\text{.........................(14)}$$

M is called the mutual inductance of the two circuits. The calculation of M is of great importance, and the equation just found gives in certain cases a ready means of performing it.

The magnetic induction N' through the second circuit due to current γ in the first is, as we have seen,

$$\gamma\iint\frac{1}{r}\left(\frac{dx}{ds}\frac{dx'}{ds'}+\frac{dy}{ds}\frac{dy'}{ds'}+\frac{dz}{ds}\frac{dz'}{ds'}\right)ds\,ds',$$

or, if we write γ_x, γ_y, γ_z for the components of γ parallel to the axes, it is

$$\iint\left(\frac{\gamma_x}{r}\frac{dx'}{ds'}+\frac{\gamma_y}{r}\frac{dy'}{ds'}+\frac{\gamma_z}{r}\frac{dz'}{ds'}\right)ds\,ds'.$$

Comparing with (10) above, we see that we may write for a linear current

$$F=\int\frac{\gamma_x}{r}ds,\quad G=\int\frac{\gamma_y}{r}ds,\quad H=\int\frac{\gamma_z}{r}ds. \quad\ldots\ldots\ldots\ldots(15)$$

It is to be observed that this does not give a unique determination of F, G, H, since any terms of proper dimensions which introduced into the line integral would give a zero result for a complete circuit may be added. Still, for complete circuits, this specification of F, G, H is sufficient. The components γ_x, γ_y, γ_z vary from point to point, but they give for every point of the circuit the relation

$$\gamma=\gamma_x\frac{dx}{ds}+\gamma_y\frac{dy}{ds}+\gamma_z\frac{dz}{ds}. \quad\ldots\ldots\ldots\ldots\ldots(16)$$

8. Second specification of magnetic induction through a circuit. We now consider another method by which the magnetic induction through a circuit, A, can be found, by first calculating the magnetic induction at a specified point due to the current in the field producing circuit, B, and then finding by integration the total induction through A. When the current producing the field is unity *the magnetic induction through A is called the inductance.* It follows from the mutuality of the energy of either circuit in presence of the other that the inductance through A due to unit current in B is equal to the inductance through B due to unit current in A. Hence we speak of the "mutual inductance" of the two circuits. See also (13) above.

We have seen in 4 above that the electromagnetic force exerted on an element of a circuit in which there is a current γ, at a place where the magnetic induction is $\mathbf{B}$, is $\mathbf{B}\boldsymbol{\gamma}\sin\theta\,ds$, that is, it is the vector product of $\mathbf{B}$ and $\boldsymbol{\gamma}$, being at right angles to the plane of these two directed quantities. Now we may suppose the induction produced by a single unit pole properly placed. The reaction on the field exerted *at the element,* is $\mathbf{B}\boldsymbol{\gamma}\sin\theta\,ds$ in the opposite direction, and therefore the action on the pole is a parallel force $\mathbf{B}\boldsymbol{\gamma}\sin\theta\,.\,ds$ together with a couple of moment $\mathbf{B}\boldsymbol{\gamma}r\sin\theta\,.\,ds$, where r is the distance of the element from the pole. The total couple due to the whole closed circuit is zero, as we have seen, and so the action on the pole may be calculated by finding the resultant of all the forces $\mathbf{B}\boldsymbol{\gamma}\sin\theta\,ds$ supposed acting at the pole.

9. Magnetic field-intensity due to an element of a current-carrying circuit. Thus we obtain as the intensity of the field produced at the pole the value $B\gamma\sin\theta\,ds$. But if $\mathbf{H}$ be the force produced at the element by the pole, we know that $\mathbf{B}=\mu\mathbf{H}$; and if r be the distance of the pole from the element and the medium be isotropic, we have

$4\pi r^2 B = 4\pi$, or $B = 1/r^2$, $H = 1/\mu r^2$, where B and H are the scalar values of the directed quantities $\mathbf{B}$, $\mathbf{H}$, so that

$$B\gamma \sin\theta\, ds = \gamma \frac{\sin\theta\, ds}{r^2}. \qquad \text{.......................(17)}$$

The expression on the right-hand side is the magnetic field intensity produced by the element ds at the point where the pole is supposed to be situated, and the resultant intensity is to be obtained by proper summation of the forces produced by the several elements. The direction of this elementary force is at right angles to the plane through the element and the point considered.

The direction may be specified as follows. Let the right-hand be held open with the palm down, and the thumb pointing downward. Let the element be represented by the thumb, and the current be supposed to flow in the direction in which the thumb points, and the point considered be at the extremity of the forefinger, then the field-intensity would tend to move a north pole there towards the next finger.

The rule here stated for the specification of the amount and direction of the field-intensity due to an element of a current-carrying circuit is generally attributed to Laplace, but sometimes to Ampère. The specification like that of vector potential is not unique, and for a similar reason. It is however applicable to a complete circuit or to a complete turn of a circuit.

10. Mutual inductance between two coaxial circular conductors. We shall now calculate as an illustration the mutual inductance M between two coaxial circular conductors, a quantity of great importance in electrical practice. It will be seen that the first method explained above gives M by one operation, while the second proceeds by calculation of the field-intensity at any point, and then finds M by a second integration. The second method is necessary for galvanometry, for the first integral gives a component of the field which it is necessary to know at the positions of the poles of the suspended needle.

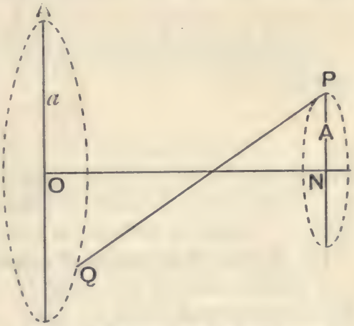

FIG. 51.—Foreshortened circles represented by narrow ellipses.

Let the circles be situated as in the figure with their planes parallel and coaxial, at distance b apart and have the radii $OA = a$, $NP = A$. Take an element of the left-hand circle at Q subtending the angle $d\theta$ at O, and denote the supplement of the angle AOQ by θ. By (14) the mutual inductance between an element at P subtending an angle $d\phi$ at N and the element at Q is $Aa\cos\theta\, d\theta\, d\phi / PQ$, where PQ is given by

$$PQ^2 = r^2 = A^2 + a^2 + 2Aa\cos\theta + b^2.$$

Clearly, for the mutual inductance of the two circles, we have

$$M = -2\pi Aa\int_0^{2\pi}\cos\theta\frac{d\theta}{r} = -2\pi Aa\int_0^{2\pi}\frac{\cos\theta\, d\theta}{(A^2+a^2+b^2+2Aa\cos\theta)^{\frac{1}{2}}}. \quad (18)$$

For θ write 2ω, and put

$$(A+a)^2+b^2=r_2{}^2, \quad (A-a)^2+b^2=r_3{}^2,$$

$$\gamma^2=4Aa/\{(A+a)^2+b^2\}=(r_2{}^2-r_3{}^2)/r_2{}^2, \quad \gamma'^2=1-\gamma^2.$$

Then using the notation $\Delta^2\omega=1-\gamma^2\sin^2\omega$, we find that

$$M=2\pi r_2\int_0^{\frac{1}{2}\pi}\gamma^2\frac{\sin^2\omega-\cos^2\omega}{\Delta\omega}d\omega=2\pi r_2\{G-H-(H-\gamma'^2G)\}, \quad \ldots(19)$$

since $\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\gamma^2\sin^2\omega\, d\omega}{\Delta\omega}=G-H, \quad \int_0^{\frac{1}{2}\pi}\frac{\gamma^2\cos^2\omega\, d\omega}{\Delta\omega}=H-\gamma'^2G,$

where $G=F(\gamma)$, $H=E(\gamma)$, the complete elliptic integrals of the second and first kinds to modulus γ.* The reader may verify by writing down the appropriate analysis that the radial gravitational or electric force at a point in the circumference of one circle due to a uniform disk of gravitating matter, or a disk uniformly charged with electricity, filling the other circle, is, to a constant, equal to the magnetic induction through either circle due to a current in the other. The equation may also be written

$$M=-4\pi\sqrt{Aa}\left\{\frac{2}{\gamma}H+\left(\gamma-\frac{2}{\gamma}\right)G\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(20)$$

the form in which it is usually given.

By Landen's transformation, in which we put $\gamma=2\sqrt{\gamma_1}/(1+\gamma_1)$, this becomes

$$M=8\pi\frac{\sqrt{Aa}}{\sqrt{\gamma_1}}(G_1-H_1), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(21)$$

where G_1, H_1 now denote the complete elliptic integrals to modulus γ_1. This form is more advantageous when the modulus γ is nearly unity. We shall give some account of methods of calculation later.

11. Mutual inductance of circle and helix. Let us now suppose that one of the circles, say that of radius a, is replaced by a closely and regularly wound helix of wire, carrying a current of unit strength in each turn. As we shall show presently, the helix may be considered as a uniform solenoidal current round the cylinder on which it is wound, of such strength that the current per unit length of the cylinder is n, where n is the number of turns per unit of the axial length. The current for an axial step dz is thus $n\,dz$. Thus for a band dz at distance z from the circle of radius A the inductance is

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* The use of γ here for the modulus of an elliptic integral will not be confused with its use to denote a current. The notation adopted leaves E and F free for other purposes. [See Greenhill, "The Elliptic Integral in Electromagnetic Theory," *Trans. Amer. Math. Soc.* 8, p. 447, 1907.]

$$M\,dz = -2\pi Aan\,dz\int_0^{2\pi}\frac{\cos\theta\,d\theta}{(R^2+z^2)^{\frac{1}{2}}}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

where $R^2 = A^2+a^2+2Aa\cos\theta$. For limits z_0, z_1 of z the whole inductance, I say, is given by

$$I=\int_{z_0}^{z_1} M\,dz = -2\pi Aan\int_0^{2\pi}\int_{z_0}^{z_1}\frac{\cos\theta\,dz\,d\theta}{(R^2+z^2)^{\frac{1}{2}}}. \quad\ldots\ldots\ \ldots..(23)$$

Integration with respect to z gives for the integral of inductance

$$I = -4\pi nAa\int_0^{\pi}\cos\theta\,d\theta\{\log(z_1+\sqrt{R^2+z_1^2})-\log(z_0+\sqrt{R^2+z_0^2})\}.$$

Integrating now by parts and noticing that the integrated terms vanish, we get

$$I = -4\pi nA^2a^2\int_0^{\pi}\sin^2\theta\,d\theta\left\{\frac{1}{(\sqrt{R^2+z_1^2}+z_1)\sqrt{R^2+z_1^2}}\right.$$
$$\left.-\frac{1}{(\sqrt{R^2+z_0^2}+z_0)\sqrt{R^2+z_0^2}}\right\}.$$

When the numerator and denominator of the first fraction is multiplied by $\sqrt{R^2+z_1^2}-z_1$, and those of the second by $\sqrt{R^2+z_0^2}-z_0$, this equation for the integral of inductance becomes

$$I = -4\pi nA^2a^2\int_0^{\pi}\sin^2\theta\,d\theta\left(\frac{z_0}{R^2\sqrt{R^2+z_0^2}}-\frac{z_1}{R^2\sqrt{R^2+z_1^2}}\right);$$

and, substituting 2ω for θ, we find

$$I = -32\pi n\frac{A^2a^2}{(A+a)^2}\left\{\frac{z_0}{\{(A+a)^2+z_0^2\}^{\frac{1}{2}}}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\omega\cos^2\omega\,d\omega}{(1-\beta^2\sin^2\omega)(1-\gamma_0^2\sin^2\omega)^{\frac{1}{2}}}\right.$$
$$\left.-\frac{z_1}{\{(A+a)^2+z_1^2\}^{\frac{1}{2}}}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\omega\cos^2\omega\,d\omega}{(1-\beta^2\sin^2\omega)(1-\gamma_1^2\sin^2\omega)^{\frac{1}{2}}}\right\},$$

where

$$\beta^2=4Aa/(A+a)^2,\quad \gamma_1^2=4Aa/\{(A+a)+z_1^2\},\quad \gamma_0^2=4Aa/\{(A+a)^2+z_0^2\}.$$

Now it is easy to show that

$$\frac{\beta^2\sin^2\omega\cos^2\omega}{1-\beta^2\sin^2\omega}=\beta^2\frac{\sin^2\omega-\sin^4\omega}{1-\beta^2\sin^2\omega}$$
$$=-\left(1-\frac{1}{\beta^2}\right)+\sin^2\omega+\left(1-\frac{1}{\beta^2}\right)\frac{1}{1-\beta^2\sin^2\omega},$$

so that we get, finally,

$$I = 2\pi n(A+a)\beta\left\{z_0\gamma_0\left[\frac{1}{{\gamma_0}^2}(G_0 - H_0) + \frac{1-\beta^2}{\beta^2}(G_0 - \Pi_0)\right]\right. \cdot$$
$$\left. - z_1\gamma_1\left[\frac{1}{{\gamma_1}^2}(G_1 - H_1) + \frac{1-\beta^2}{\beta^2}(G_1 - \Pi_1)\right]\right\}, \quad \text{........(24)}$$

where G_0, H_0, Π_0, G_1, H_1, Π_1 are the complete elliptic integrals of the second, first, and third kinds to the respective moduli γ_0, γ_1.

This result agrees with one obtained by a somewhat different process in a paper by Dr. A. Russell "On the Magnetic Field of Circular Currents," *Phil. Mag.*, April, 1907.

If the circle lie in a plane bisecting the helix at right angles, we have $z_1 = -z_0 = z$, and obtain

$$I = 4\pi n(A+a)\beta z\gamma\left[\frac{1}{\gamma^2}(G-H) + \frac{1-\beta^2}{\beta^2}(G-\Pi)\right]. \quad \text{.......(25)}$$

If instead of $2\pi n \times 2z$ we write Θp, where p is the pitch of the helix, and Θ the whole angle turned through by a radius the outer extremity of which traverses the whole length of wire, we have

$$I = \Theta p\beta\gamma(A+a)\left[\frac{1}{\gamma^2}(G-H) + \frac{1-\beta^2}{\beta^2}(G-\Pi)\right]. \quad \text{.........(26)}$$

This equation also serves for the case in which the circle coincides with one end of the helix, provided Θ be the angle of the windings of wire.

It is important to observe that if the circle and coil be made of the same radius, an arrangement which is possible if the circle (the revolving disk of the Lorenz apparatus) is placed outside the coil, the factor $(1-\beta^2)/\beta^2 = 0$ and the formulae do not involve the elliptic integral of the third kind, and the numerical calculations are much simplified. We get then by (24)

$$I = 4\pi nA\left\{\frac{z_0}{\gamma_0}(G_0 - H_0) - \frac{z_1}{\gamma_1}(G_1 - H_1)\right\}, \quad \text{............(27)}$$

and the calculations can be carried out very expeditiously and accurately by means of Legendre's tables of the complete integrals G and H (F and E). The distances z_0, z_1 are those of the near and far ends of the coil from the circle. This remark is due to Greenhill.* The formulae have a direct application to "Current Weighers" [see below 17, *et seq.*], and this remark is of importance in that connection.

12. Mutual inductance of helix and cylindrical current sheet. The investigation may be extended to give the mutual inductance of a helix and a cylindrical current sheet. The mutual inductance of the helix and a coaxial circle is found, and then the calculation is extended by integration to the current sheet. This has been done by Dr. Russell (*loc. cit. supra*) for a current sheet and a helix coaxial and concentric with it, with the following result. Let N_1 be the number of turns of the helix, $2z$, its axial length, and p its pitch, and N_2, $2z_2$ the number of turns and axial length of the current sheet, which may be a closely wound coil of fine wire. Also let a be the radius of the helix, A that of the cylindrical sheet, and

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* "Electromagnetic Integrals," *Phil. Trans.*, Dec. 15th, 1919.

$$\gamma_1^2 = 4Aa/\{(A+a)^2+(z_1+z_2)^2\}, \quad \gamma_2^2 = 4Aa/\{(A-a)^2+(z_1-z_2)^2\}.$$

Then if we write $R_1^2=(A+a)^2+(z_1+z_2)^2$, $R_2^2=(A+a)^2+(z_1-z_2)^2$, the result is

$$\begin{aligned} I = 4\pi Aa\frac{N_1N_2}{z_1z_2}\bigg[& R_1\left(1-\frac{\gamma_1^2}{\beta^2}\right)\left\{\frac{1-\beta^2}{\beta^2}(G_1-\Pi_1)+\frac{1}{\gamma_1^2}(G_1-H_1)\right\} \\ & + R_1\gamma_1^2\left\{\frac{2-\gamma_1^2}{3\gamma_1^2}H_1-\frac{2-2\gamma_1^2}{3\gamma_1^4}G_1\right\} \\ & - R_2\left(1-\frac{\gamma_2^2}{\beta^2}\right)\left\{\frac{1-\beta^2}{\beta^2}(G_2-\Pi_2)+\frac{1}{\gamma_2^2}(G_2-H_2)\right\} \\ & - R_2\gamma_2^2\left\{\frac{2-\gamma_2^2}{3\gamma_2^2}H_2-\frac{2-2\gamma_2^2}{3\gamma_2^4}G_2\right\}\bigg]. \quad \ldots\ldots\ldots\ldots\ldots(28) \end{aligned}$$

The current sheet may be replaced by a helix if a correction be applied. This may be determined by observing the change of I produced by turning one of the cylinders about its axis into successive positions, so as to obtain the mean value of I. Two positions distant 180°, or, better, four at intervals of 90°, suffice to give the correction.

13. Computation of mutual inductances. The value of I may be computed by means of Legendre's tables of elliptic integrals; but Dr. Russell has shown that the computation may be carried out very readily by means of the power series which follow:

$$\begin{aligned} I = \pi^2A^2\frac{N_1N_2}{z_1z_2}\bigg\{ & R_1\left(1-\tfrac{1}{2}q_2\gamma_1^2-\frac{1.1}{2.4}q_3\gamma_1^4-\frac{1.1.3}{2.4.6}q_4\gamma_1^6-\ldots\right) \\ & - R_2\left(1-\tfrac{1}{2}q_2\gamma_2^2-\frac{1.1}{2.4}q_3\gamma_2^4-\frac{1.1.3}{2.4.6}q_4\gamma_2^6-\ldots\right)\bigg\}, \quad \ldots(29) \end{aligned}$$

where q_n is given by the recurrence formula [in which $a>A$]

$$q_n=\frac{(A+a)^2}{4Aa}q_{n-1}-\frac{1}{n}\frac{1.3.5.\ldots.2n-3}{2.4.6.\ldots.2n-2}\frac{a}{A}. \quad \ldots\ldots\ldots(30)$$

When $z_1=z_2$ $(=z)$, the convergence of the second line of (29) may be slow. We can then substitute for it

$$\frac{8}{3}\frac{\pi Aa}{(A+a)}\frac{N_1N_2}{z^2}\{(A^2+a^2)(G-H)-2AaG\},$$

in which the modulus of G and H is $2\sqrt{Aa}/(A+a)$.

With regard to the calculation of the numerical values of elliptic integrals which are not given in tables one or two remarks may be made here. Legendre has given [*Traité des Fonctions Elliptiques* t. I.] the following relation:

$$F-\Pi=\frac{\beta}{\gamma'^2\sin\alpha\cos\alpha}[E.F(\gamma',\alpha)+F.E(\gamma',\alpha)-F.F(\gamma',\alpha)-\tfrac{1}{2}\pi], \quad (31)$$

where E, F are complete elliptic integrals of the second and first kind, and Π is a complete elliptic integral of the third kind. $E(\gamma', a)$, $F(\gamma', a)$ are incomplete elliptic integrals of amplitude a, and

$$\sin a = (1-\beta^2)^{\frac{1}{2}}/\gamma'.$$

The modulus γ' is $\sqrt{1-\gamma^2}$. This enables the tables of incomplete integrals given in t. II. of the *Fonctions Elliptiques* to be used for the calculation of Π.

Unless however exceedingly high accuracy is required, elliptic integrals of the third kind, whether complete or incomplete, can be calculated by the ordinary process of computing a sufficient number of ordinates of the curve given by successive values of the integral, and then deducing the area. [See Gray's *Gyrostatics*, p. 259, *et seq.*]

With regard to the numerical values of the elliptic integrals, they can be obtained with all needful accuracy from the great tables of complete and incomplete integrals given by Legendre in his *Fonctions Elliptiques*, t. II., but if, as often happens, these tables are not available, the values may be calculated in various ways. For example M may be found from the power series given for F and E (here G and H) in treatises on Elliptic Integrals. Information as to such computations is given with examples in Gray's *Gyrostatics*, chap. xii., which the reader may consult. Tables of elliptic integrals will be found in Appendix IV.

14. Computation of mutual inductance of two coaxial circles by q-series. The following method of calculating

$$M = 8\pi\frac{\sqrt{Aa}}{\sqrt{\gamma_1}}(G_1 - H_1) \quad \text{..........................(32)}$$

[see (21) above] is due to H. Nagaoka [*Phil. Mag.* 6, 1903]. The formulae were obtained by the theory of ϑ functions, into which we have not space to enter; but we shall indicate how the principal series can be found by direct substitution. Denoting, as above, for brevity, the complete elliptic integral $F(\gamma)$ to modulus γ by G, and the corresponding integral $F(\gamma')$ to the complementary modulus γ' $(=\sqrt{1-\gamma^2})$ by G', we define an auxiliary function q, used first by Jacobi, by the equation

$$q = e^{-\pi G'/G}. \quad \text{..................................(33)}$$

In the present connection G'/G is wholly real, so that q is less than unity, and the series formed with it below are all convergent. Generally it can be arranged that they are very highly convergent.

The value of q can be calculated from that of the modulus γ' as follows. It will be noticed that if we put $\gamma = \sin a$, we have $\gamma' = \cos a$. Then, as can easily be proved [see also (40)],

$$\gamma'^{\frac{1}{2}} = \frac{1-2q+2q^4-2q^9+\dots}{1+2q+2q^4+2q^9+\dots}, \quad \text{.....................(34)}$$

which gives

$$q = \tfrac{1}{4}\tan\tfrac{1}{2}\alpha + \tfrac{1}{16}\tan^6\tfrac{1}{2}\alpha + \tfrac{17}{512}\tan^{10}\tfrac{1}{2}\alpha + \tfrac{45}{2048}\tan^{14}\tfrac{1}{2}\alpha + \dots, \quad \dots\dots(35)$$

by which q can be calculated. We have then, as will be proved below,

$$M = 16\pi^2\sqrt{Aa}\, q^{\frac{3}{2}}(1 + 3q^4 - 4q^6 + 9q^8 - 12q^{16} + \dots). \quad \dots\dots\dots\dots(36)$$

An alternative equation for M was given by Nagaoka [*Tokyo Math. Phys. Soc.*, 6, p. 10] in 1911, as follows:

$$M = 4\pi\sqrt{Aa}\left\{4\pi q_1^{\frac{3}{4}}\frac{1 - 4q_1^3 + 9q_1^8 - \dots}{1 - 3q_1^2 + 5q_1^6 - \dots}\right\}, \quad \dots\dots\dots\dots\dots(37)$$

where the general term of the numerator is $(-1)^{n-1}n^2q_1^{n^2-1}$ and that of the denominator $(-1)^m(2m+1)q_1^{\frac{1}{4}\{(2m+1)^2-1\}}$, and q_1 is calculated from $\gamma_1'^{\frac{1}{2}}$, as q is by (34). In either case the equation

$$q = \tfrac{1}{2}l + 2(\tfrac{1}{2}l)^5 + 15(\tfrac{1}{2}l)^9 + \dots \quad \dots\dots\dots\dots\dots\dots\dots\dots(38)$$

may be used. The value of l is $(1 - \gamma'^{\frac{1}{2}})/(1 + \gamma'^{\frac{1}{2}})$ or $(1 - \gamma_1'^{\frac{1}{2}})/(1 + \gamma_1'^{\frac{1}{2}})$, according as q or q_1 is to be computed.

The modulus γ_1 being that derived from γ by Landen's transformation $[\gamma = 2\sqrt{\gamma_1}/(1 + \gamma_1)]$, that is $\gamma_1 = (1 - \gamma')/(1 + \gamma')$, we have $G_1'/G_1 = 2G'/G$, so that $q_1 = q^2$.

Using then the transformed integrals G_1, H_1 of (32) above, we obtain [Gray's *Gyrostatics* (*G.G.*), xii. 16 (14) and 11 (3)] the equation

$$G_1 - H_1 = \frac{4\pi q}{\gamma'^{\frac{1}{2}}(2G_1/\pi)^{\frac{3}{2}}}(1 - 4q^3 + 9q^8 - 16q^{15} + \dots). \quad \dots\dots\dots(39)$$

But by the values of $\gamma'^{\frac{1}{2}}$ and $2G_1/\pi$ [*G.G.*, xii. 10 (3)] and the equation

$$\gamma_1^{\frac{1}{2}} = 2q_1^{\frac{1}{4}}\frac{1 + q_1^2 + q_1^6 + q_1^9 + \dots}{1 + 2q_1 + 2q_1^4 + 2q_1^9 + \dots}, \quad \dots\dots\dots\dots\dots\dots(40)$$

this reduces to

$$\frac{1}{\sqrt{\gamma_1}}(G_1 - H_1) = 2\pi q_1^{\frac{3}{4}}\frac{1 - 4q_1^3 + 9q_1^8 - 16q_1^{15} + \dots}{1 + q_1^2 + q_1^6 + q_1^9 + \dots}$$

$$\times\frac{1}{(1 + 2q_1 + 2q_1^4 + 2q_1^9 + \dots)(1 - 2q_1 + 2q_1^4 - 2q_1^9 +)\dots},$$

which, when the denominators are multiplied out, gives

$$\frac{2}{\sqrt{\gamma_1}}(G_1 - H_1) = 4\pi q_1^{\frac{3}{4}}\frac{1 - 4q_1^3 + 9q_1^8 - \dots}{1 - 3q_1^2 + 5q_1^6 - \dots}.$$

Thus we get finally

$$M = 16\pi^2\sqrt{Aa}\,q_1^{\frac{3}{4}}\frac{1 - 4q_1^3 + 9q_1^8 - \dots}{1 - 3q_1^2 + 5q_1^6 - \dots}. \quad \dots\dots\dots\dots\dots\dots(41)$$

Expanding we may write equation (41) in the form

$$M = 16\pi^2\sqrt{Aa}\,q_1^{\frac{3}{4}}(1 + 3q_1^2 - 4q_1^3 + 9q_1^4 - 12q_1^5 + \dots). \quad \dots\dots(42)$$

As we have seen $q_1 = q^2$, and so we have, with q as defined by (34) or (35),

$$M = 16\pi^2\sqrt{Aa}\,q^{\frac{3}{2}}(1 + 3q^4 - 4q^6 + 9q^8 - 12q^{10} + \ldots),$$

which is (36).

When the circles are a considerable distance apart the value of M may be taken as given by

$$M = 16\pi^2\sqrt{Aa}\,q_1^{\frac{3}{4}} \text{ or } 16\pi^2\sqrt{Aa}\,q^{\frac{3}{2}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(42')$$

When the circles are not far apart it is advisable to use the value of q derived, by (37) above, from $l = (1 - \gamma^{\frac{1}{2}})/(1 + \gamma^{\frac{1}{2}})$, which we denote here by q'. We then obtain, by a process set forth in Nagaoka's paper (*loc. cit. supra*),

$$M = 4\pi\sqrt{Aa}\,\frac{1}{2(1 - 2q')^2}\left\{(1 + 8q' - 8q'^2 + \epsilon)\log\frac{1}{q'} - 4\right\}, \quad \ldots\ldots(43)$$

where $\epsilon = 32q'^3 - 40q'^4 + 48q'^5 - \ldots$. Tables have been constructed by Nagaoka for the calculation of M by these formulae. They are given in full in the *Bulletin of the Bureau of Standards at Washington* (*B.B.S.W.*), 8, No. 1, 1912.

15. Relations between magnetic field-intensities and inductances. Some remarkable relations exist between field-intensities of certain distributions and mutual inductances. [See papers by the author in the *Phil. Mag.*, May and August, 1919.] For example the mutual inductance of two concentric coplanar circles is proportional to the electric (or gravitational) field-intensity produced by a uniformly charged disk, the edge of which coincides with one circle, at a point on the circumference of the other. Of course this field-intensity is in the plane of the circles and radially directed, while the inductance is the surface integral of the magnetic field-intensity for a point within one circle due to a unit current in the other, and this field-intensity is axially directed at each element of surface [see 10 above].

Another proposition follows from this for a circle and a coaxial cylindrical current sheet. Take an axis of x along the axis of figure and an axis of y along a radius of the circle meeting the circle in a point P. Now imagine a uniform cylindrical volume distribution of electricity or matter of density ρ, to have its surface coincident with the cylindrical current sheet. The y-component of force at P due to the volume distribution is, to a constant, identical in numerical value with the total magnetic induction through the circle due to the current sheet. Some other results are given in the papers referred to above.

The calculation of M may be carried out for the two coplanar coaxial circles by finding for a point P on, say the outer circle, the electric field-intensity due to a uniform charge of surface density unity, on the plane surface enclosed by the inner circle. If the circles are not coplanar, but are coaxial, the same result holds; the radial field-intensity at P due to the disk-distribution is, to a constant, numerically the total magnetic induction, through the inner circle due to a current in the outer, or through the outer circle due to a current in the inner. Then by integration along the axis, the result for the current sheet can be obtained. To carry out the calculation would involve the repetition of a considerable part of the analysis already given above.

16. Equivalence of a helical current and a current sheet. The result stated in (24) agrees with an expression obtained by Viriamu Jones (*Proc. R.S.*, 1897, p. 198) for the mutual inductance of a helix of wire, regularly laid on in a screw-thread cut on a circular cylinder, and a coaxial circle midway between the ends of the helix. This arrangement constituted the principal part of an improved apparatus for carrying out with great accuracy a determination of the ohm by Lorenz's method. An ingenious method of avoiding the direct calculation of the elliptic integrals of the third kind was employed by Jones in his final calculations, and was given in the first edition of this book, but it will not be reproduced here, as the computations have all been carried out with the utmost accuracy at the Bureau of Standards at Washington, and elsewhere. Further, we shall have to return to the subject in connection with a description of a new machine installed at the National Physical Laboratory by the Drapers' Company of London as a Memorial after Professor Jones's death in 1900.

The mathematical analysis for the helix disclosed its equivalence to a current sheet. The proof of this equivalence which follows was suggested by the author to Professor Jones, about the end of 1897. It is practically identical with one given by Lord Rayleigh in 1899 (*B.A. Rep.*, 1899).

Let it be supposed that there is an exact whole number of turns round the cylinder, and that the return wire is taken back to the starting point parallel to the axis. Consider a small step ds along the conductor. By the law of Laplace, this may be regarded as producing an element of induction through the circle. This induction is dependent on the position of the element on that circle of the cylinder which passes through its centre and is at right angles to the axis. As only the component in this circle produces the element of magnetic induction, it is clear that this component may, without alteration of the induction it produces, be equally distributed round the zone of the cylinder across which ds lies obliquely. But this converts the element ds into a band of a current sheet, together with a component along the axis which produces no effect on the induction through the circle.

It is important to notice that whether the wire be thin or not, and whatever the angle of the helix, this equivalence holds, and that the sheet will have a certain total thickness, equal to the diameter of the wire, and varying in current density from zero at the inner surface through a maximum to zero again at the outer surface, in a manner which it is easy to express quantitatively if it is necessary to do so. If the insulating covering (should any be used) be uniformly laid on, and the wire be under constant tension while the coil is being wound, the inner surface will be determinate with all needful accuracy. But in later coils used for various purposes the need for a covering has been avoided by laying the wire on a cylinder of non-conducting material, *e.g.* marble, found to contain no magnetic matter.

In the discussion of determinations of the ohm given in a later chapter all necessary particulars of the arrangement and the precautions taken to avoid errors of measurement will be given.

17. Theory of a current weigher. The mutual energy of two current sheets may be regarded either as potential energy or as kinetic energy according to the point of view adopted.

The quantity $-M$, when M is defined by the equation

$$M=\iint\frac{\cos\epsilon}{r}\,ds\,ds,$$

in which the integrals are taken round the two circuits, is to be regarded as potential energy per unit of the product of the currents, or $+M$ may be regarded as the same measure of the kinetic energy. For if we take the particular case in which the circuits are in parallel planes it is necessary, in order to separate the currents still further, to apply external force to pull them apart, that is of course in the normal case in which they flow the same way round. In other words, when they are left to themselves, they approach one another, and M is increased. Thus if z be any coordinate specifying the configuration, the force between the currents towards increasing the variable z is dM/dz.

Let M_0 denote the value of M for any epoch and M_1 that for any subsequent epoch. Then, in the bringing about of the change, work M_1-M_0 has been done by the circuits on themselves. Thus, if the symbol z represent the configuration, we have

$$M_1-M_0=\int\frac{dM}{dz}\,dz. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(44)$$

Now let the force (as in a current-weighing apparatus) be applied from without, and the change be M_1-M_0: we can easily find an expression for this quantity in certain practical cases, which is very convenient. To fix the ideas, and take the most important case which occurs, consider two solenoids in any relative positions. Each solenoid may be regarded as consisting of two end disks covered with fictitious magnetic matter of opposite polarities. It may or may not be of circular cross-section, and the sections may have different forms in the two cases. What is essential for the present purpose is that the coils should both be cylindrical. Call then these disks a, b for one solenoid and c, d for the other. Let M_{ab} be the magnetic induction through the disk a due to the disk b, or, as we say, through a due to b. Then the total induction M between the solenoids is given by

$$M=M_{ac}+M_{bc}+M_{ad}+M_{bd}. \quad\ldots\ldots\ldots\ldots\ldots\ldots(45)$$

This may be written

$$M = M_{ab\cdot c} + M_{ab\cdot d} = M_{cd\cdot a} + M_{cd\cdot b}, \quad \text{................(45')}$$

where $M_{ab\cdot c}$ is the induction through the end c of the solenoid c, d, due to the end disks of the other, and $M_{ab\cdot d}$ is the same quantity for the end d of c, d. The equivalent form of M given in (45) is the induction through the ends a and b of a, b due to the solenoid c, d.

Now let us suppose that the solenoid a, b is held fixed while the other is displaced a small distance dz parallel to its axis in the direction from c to d. Then the change in M is $(M_{ab\cdot d} - M_{ab\cdot c})\,dz$, and the force F in the direction dz applied from without is given by

$$F = -(M_{ab\cdot d} - M_{ab\cdot c}). \quad \text{........................(46)}$$

For by the nature of a solenoid the change is that which would be effected by removing a length dz from the end c of the solenoid c, d to the end d.

Similarly the force required to move the solenoid a, b parallel to its axis is

$$F = M_{cd\cdot b} - M_{cd\cdot a}. \quad \text{........................(47)}$$

18. Mutual inductance of two close nearly equal coaxial circles. We take now some special cases of inductance which occur in practice. Consider first the particular case of two coaxial circles of nearly equal radius and only a small distance apart. First let the circles be in the same plane, the radius of the inner be A and that of the outer a. If a point P be taken within the inner circle the magnetic force dF there, at right angles to the plane, due to unit current in the element ds of the outer circle at E (Fig. 52) is given by

$$dF = \cos\theta \frac{ds}{r^2}, \quad \text{..........(48)}$$

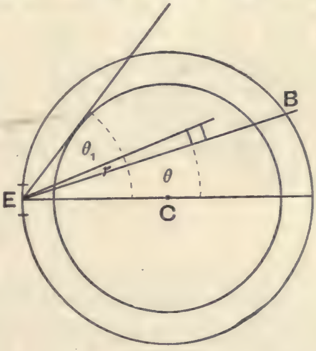

FIG. 52.

where r is the distance of P from the element E, and θ is the angle BEC. Multiplying by the area $r\,d\theta\,dr$, we get for M the equation

$$M = \iiint \cos\theta \frac{dr\,d\theta\,ds}{r}. \quad \text{........................(49)}$$

If we suppose θ to vary from 0, when EB is along EC, the upper limit is $\sin^{-1}(A/a)$. We call this θ_1. The limits of r are the two roots of the equation

$$r^2 - 2ar\cos\theta + a^2 - A^2 = 0. \quad \text{........................(50)}$$

Approximately these roots are $2a\cos\theta$ and $(a-A)/\cos\theta$; a closer approximation gives $2a\cos\theta - (a-A)/\cos\theta$ and $(a-A)/\cos\theta$. Confining ourselves in the first place to the rougher approximation, we get

$$M = 2\int ds \int_{(a-A)/\cos\theta}^{2a\cos\theta} \int_0^{\theta_1} \frac{\cos\theta}{r}\, d\theta\, dr$$

or

$$M = 4\pi a \log\left(\frac{8a}{a-A} - 2\right). \quad \text{.....................(51)}$$

If x denote the smallest distance between the circles, we have thus

$$M = 4\pi a \log\left(\frac{8a}{x} - 2\right). \quad \text{.....................(51')}$$

The more exact value of the larger root of (50) might have been used without much added difficulty, but a more accurate solution can easily be derived in another way, due to Maxwell, which we shall give below.

The order of approximation so far adopted takes that part of the induction, which does not pass through the inner coaxial circle, as equal to that which might be computed by taking the outer circle elements as straight. Of course if the conductor represented by the outer circle is infinitely thin this part is infinite, but the infinity is avoided in any actual case by taking the cross-section of the conductor as finite though very small. This term also appears in the induction through the circle of radius a, so that M for the smaller circle has the value stated in (51′).

Now let the smaller circle be moved out of the plane of the larger through a small distance y. If d now denote the shortest distance between the circles, while x denotes the former shortest distance, the additional induction which escapes passing through the circle of radius a is $4\pi b(\log d - \log x)$. Hence to the same order of approximation as before

$$M = 4\pi a \log\left(\frac{8a}{d} - 2\right). \quad \text{.....................(52)}$$

This equation is not exact; but by a process which will be found set forth in the *Electricity and Magnetism*, Maxwell obtained a convergent series by which the computation can be carried out to any needful degree of accuracy. We simply quote the result, putting as above a, A for the radii of the outer and inner circles, x for the (positive) difference $a-A$, y for the small distance apart of the planes of the circles, and $r = \sqrt{x^2+y^2}$. Then

$$M = 4\pi A\left\{\log\frac{8A}{r}\left(1 + \frac{x}{2A} + \frac{x^2+3y^2}{16A^2} - \frac{x^3+3xy^2}{32A^3} + \dots\right)\right.$$
$$\left. - \left(2 + \frac{x}{2A} - \frac{3x^2-y^2}{16A^2} + \frac{x^3-6xy^2}{48A^3} - \dots\right)\right\}. \quad \text{......(53)}$$

Of course these formulae are not to be used unless x/A and y/A are mall. If $y/A = 0{\cdot}1$ the largest term neglected in (53) is less than two ts in a million.

This approximation has been carried further by J. G. Coffin of the Bureau of Standards, Washington (see *B.B.S.W.* 8 (1912), p. 14). The new formula is, if A (as above) be the radius of the smaller circle and y be the distance between the planes of the circles,

$$M = 4\pi A\left\{\log\frac{8A}{y}\left(1+\frac{3y^2}{16A^2}-\frac{15y^4}{8\,.\,128A^4}+\frac{35y^6}{128^2A^6}-\frac{1575y^8}{2\,.\,128^3A^8}+\dots\right)\right.$$
$$\left.-\left(2+\frac{y^2}{16A^2}-\frac{31y^4}{16\,.\,128A^4}+\frac{247y^6}{6\,.\,128^2A^6}-\frac{7795y^8}{8\,.\,128^3A^8}+\dots\right)\right\}. \quad \dots(54)$$

An extension of Maxwell's formula is also given in the *Bulletin of the Bureau, loc. cit.*

Two formulae are given by J. H. Havelock (*Phil. Mag.* 15, 1908). They were derived from certain definite integrals of Bessel Functions. We give one, as extended by a comparison with the series of Coffin just quoted. If $r^2=(a-A)^2+y^2$, $\alpha=(A/a)r^2/A^2$, the formula stands

$$M = 4\pi\sqrt{Aa}\left\{\left[1+\frac{3}{16}\alpha-\frac{15}{1024}\alpha^2+\frac{35}{128^2}\alpha^3-\frac{1575}{2\,.\,128^2}\alpha^4+\dots\right]\log\frac{8\sqrt{Aa}}{r}\right.$$
$$\left.-\left(2+\frac{1}{16}\alpha-\frac{31}{2048}\alpha^2+\frac{247}{6\,.\,128^2}\alpha^3-\frac{7795}{8\,.\,128^3}\alpha^4+\dots\right)\right\}. \quad \dots(55)$$

This expression is said to give very accurate results for values of y almost as great as the smaller radius A. A small number of terms suffices for most practical purposes.

When the circles are coaxial and very close the following power series in terms of γ' [the complementary modulus of the elliptic integral in (18) above] is rapidly convergent. It is due to Weinstein.

$$M = 4\pi\sqrt{Aa}\left\{\left(1+\frac{3}{4}\gamma'^2+\frac{33}{64}\gamma'^4+\frac{107}{256}\gamma'^6+\frac{5913}{16384}\gamma'^8+\dots\right)\left(\log\frac{4}{\gamma'}-1\right)\right.$$
$$\left.-\left(1+\frac{15}{128}\gamma'^4+\frac{185}{1536}\gamma'^6+\frac{7465}{65536}\gamma'^8+\dots\right)\right\} \quad \dots\dots(56)$$

A selection of formulae for inductances in more general cases will be given later [see Chapter XIII. below].

19. Total induction is stream function of magnetic potential in case of axial symmetry. An interesting relation exists between the induction I through a circle in a magnetic field, and the magnetic potential Ω of the field. Let the position of the circle be defined by the distance x of the centre from a fixed point on the axis of the circle, and let its radius increase from A to $A+dA$ (or, as we shall write it in order to refer to ordinary coordinates, x, y, from y to $y+dy$), while its centre, and the circle as a whole, moves along the axis a distance dx. If Ω be the magnetic potential at any point of the circle, the radial magnetic force at the point is $-\partial\Omega/\partial y$, while the axial component is $-\partial\Omega/\partial x$. Clearly an addition is made to I in consequence of the enlargement of the radius, and a diminution takes place in consequence of the escape from the circle of the lines which formerly passed through the cylindrical strip of length dx, and radius y. Thus we get

$$\frac{\partial I}{\partial x} = \int_0^{2\pi} y \frac{\partial \Omega}{\partial y} d\theta, \quad \frac{\partial I}{\partial y} = -\int_0^{2\pi} y \frac{\partial \Omega}{\partial x} d\theta,$$

which gives

$$\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} = -\int \frac{\partial \Omega}{\partial x} d\theta$$

or

$$\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} - \frac{1}{y}\frac{\partial I}{\partial y} = 0. \quad \text{.........................(57)}$$

This holds in the general case whether there is symmetry of the magnetic field about the axis of the circle or not.

As to the differential equation fulfilled by Ω we get, by considering an element of volume of dimensions dx, dy, $y\,d\theta$,

$$\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} + \frac{1}{y}\frac{\partial \Omega}{\partial y} + \frac{1}{y^2}\frac{\partial^2 \Omega}{\partial \theta^2} = 0; \quad \text{..................(58)}$$

so that if there is symmetry of the field about the axis of the circle, such that $\partial^2\Omega/\partial\theta^2 = 0$, we have

$$\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} + \frac{1}{y}\frac{\partial \Omega}{\partial y} = 0. \quad \text{.......................(58')}$$

It follows that in this case I is the *stream function* for the potential Ω. At a great distance from the axis, Ω and I fulfil the usual relations of conjugate functions.

If we consider any unclosed surface in the symmetrical field, and draw through the points of its edge lines of magnetic force, we see that any other surface with its edge on these lines is a surface for which I has the same value, and moreover I has the same value for any surface the edge of which coincides with the circle.

If we put $\Omega = I \cos\theta/y$ it will be found that the differential equation (58) for Ω is satisfied, in virtue of that, (57), satisfied by I.

The present discussion is to some extent a digression from our subject and we shall not pursue it further. The reader may however refer to Greenhill's paper, *loc. cit. supra*.

20. Calculation by zonal harmonic series. In many cases of circular currents and currents in coils, for example when elliptic integrals appear, or when the axes of the circles do not coincide, we have to depend on power series for the numerical calculation of inductances. A valuable method is that of expression of the potential of a coil or the electrokinetic energy of a system in convergent series of zonal harmonics. It is well known that the mutual electrokinetic energy of the currents in two circular conductors can be thus expressed. But this series when used in the ordinary way to find the energy of the currents in two cylindrical coils (and hence also the induction coefficients of the coils), by expansion of each term of the series and *subsequent* integration yields expressions which are inconvenient for practical applications, as the work of numerical calculation of their values in actual cases is long and tedious. In a paper in the *Philosophical Magazine* for January 1892,* it is shown that it is possible very simply to integrate each term without expansion. The form of the result is remarkable, and enables a pair of coils to be constructed in such a way that the zonal harmonic expression reduces to a very brief and manageable formula, from which the energy of the currents and the mutual action of the coils can be very readily obtained. This formula has been tested by its use in many accurate numerical determinations of constants by the United States Bureau of Standard at Washington. See the *Bulletin* of the Bureau [*B.B.S.W.*], vols. 3 and 6, *passim*. The result, as we shall see, is interpretable also in power series applicable to important practical cases.

21. Integration of zonal harmonic series for two circles with intersecting axes. The integration of the zonal harmonic expression for the general case of two circles with intersecting axes, so as to find the mutual energy of two single-layer coils, can, as will be seen presently, be carried out with great simplicity by a process of successive differentiation [see 24, 25 *et seq.* below]. A convergent series [see (76) below] is obtained, of which the even terms all vanish when one at least of the coils is placed with its centre at the intersection of the axes. The third term also vanishes if the smaller of the two coils is so placed, and has its length and diameter in the ratio of $\sqrt{3}$ to 2; and the fifth term also disappears when the larger coil fulfils the same conditions. Further, if both coils are thus proportioned and placed, the even terms, so to speak, doubly vanish, so that any inaccuracy in the placing of the coils can only very slightly affect the result.

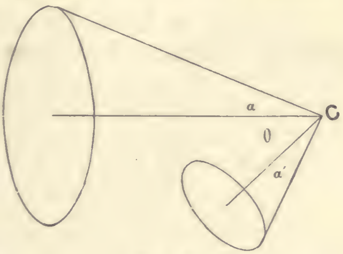

FIG. 53.

Only the seventh, ninth, eleventh, etc., terms of the series in (76) are then left after the first. If one coil has half the radius of the other, the error, made by taking only the first term in calculating the inductance, etc., of the pair of coils is only about 1 in 26,000, and if the ratio of the radii is as great as $\frac{2}{3}$ only 1 in 4500.

The zonal harmonic expression for the mutual electrokinetic energy of two circles, carrying unit currents, is given by the equation

$$T = 4\pi^2 r'^2 \sin^2\alpha \sin^2\alpha' \sum \frac{1}{i(i+1)} {}_{\alpha}Z_i' \cdot {}_{\alpha'}Z_i' \cdot {}_{\theta}Z_i \left(\frac{r'}{r}\right)^i \quad (r' < r), \quad \ldots(59)$$

where, as shown in the diagram, α, α' are the angles which the radii of the circles subtend at the intersection C of the axes, which is taken as the origin of the spherical harmonics; ${}_{a}Z_i'$ is the differential coefficient with respect to $\cos\alpha$ of the zonal harmonic of the i^{th} order for the angle α; ${}_{a'}Z_i$ the corresponding function for α'; ${}_{\theta}Z_i$ the zonal harmonics of the i^{th} order in terms of the angle θ between the axes of the circles; and r, r' ($r' < r$) are the distances of the circular arcs from the origin. The derivation of (59) will be found in the Appendix in Spherical Harmonics: assuming it here we go on to its applications.

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* "On the Calculation of the Induction-Coefficients of Coils and the Construction of Standards of Inductance, and on Absolute Electrodynamometers." By A. Gray.

Putting then a, a' (see Fig. 53) for the radii of the larger and smaller circles respectively, and x, x' for the distances of their planes from the origin, we have

$$\sin\alpha = \frac{a}{r}, \quad \sin\alpha' = \frac{a'}{r'}.$$

Substituting in the zonal harmonic expressions, as given in the Appendix, their values in terms of a, x, a', x', we obtain

$$\begin{aligned} T = \pi^2\gamma\gamma'\frac{a^2a'^2}{r^3}\{&1\,.\,2\cos\theta + 2\,.\,3\frac{x}{r^2}x'(\cos^2\theta - \tfrac{1}{2}\sin^2\theta) \\ &+ 3\,.\,4\frac{x^2 - \frac{1}{4}a^2}{r^4}(x'^2 - \tfrac{1}{2}a'^2)(\cos^3\theta - \tfrac{3}{2}\sin^2\theta\cos\theta) \\ &+ 4\,.\,5\frac{x(x^2 - \frac{3}{4}a^2)}{r^6}x'(x'^2 - \tfrac{3}{4}a'^2)(\cos^4\theta - 3\cos^2\theta\sin^2\theta + \tfrac{3}{8}\sin^4\theta) \\ &+ 5\,.\,6\frac{x^4 - \frac{3}{2}x^2a^2 + \frac{1}{8}a^4}{r^8}(x'^4 - \tfrac{3}{2}x'^2a'^2 + a'^4) \\ &\times\left(\cos^5\theta - 5\cos^3\theta\sin^2\theta + \frac{3\,.\,5}{8}\sin^4\theta\right) + \ldots\}. \quad \ldots\ldots(60) \end{aligned}$$

The couple Θ due to the mutual action of the two circular circuits tending to increase θ is $\partial T/\partial\theta$. Hence

$$\begin{aligned} \Theta = -\pi^2\gamma\gamma' a^2a'^2\sin\theta\{&1\,.\,2\frac{1}{r^3} + 2\,.\,3\frac{x}{r^2}x'.\,3\cos\theta \\ &+ 3\,.\,4\frac{x^2 - \frac{1}{4}a^2}{r^4}(x'^2 - \tfrac{1}{4}a'^2)\,.\,2\,.\,3(\cos^2\theta - \tfrac{1}{4}\sin^2\theta) \\ &+ 4\,.\,5\frac{x(x^2 - \frac{3}{4}a^2)}{r^4}x'(x'^2 - \tfrac{3}{4}a'^2)\,.\,2\,.\,5\cos\theta(\cos^2\theta - \tfrac{3}{4}\sin^2\theta) + \ldots\}. \quad (61) \end{aligned}$$

The attraction between the circuits when they are coaxial, that is when $\theta = 0$, may be found by putting $\theta = 0$ in the value of T given in (60), and calculating $\partial T/\partial x'$. We have

$$\begin{aligned} T = \pi^2\gamma\gamma'\frac{a^2a'^2}{r^3}\{&1\,.\,2 + 2\,.\,3\frac{x}{r^2}x' + 3\,.\,4\frac{x^2 - \frac{1}{4}a^2}{r^4}(x'^2 - \tfrac{1}{4}a'^2) \\ &+ 4\,.\,5\frac{x(x^2 - \frac{3}{4}a^2)}{r^6}x'(x'^2 - \tfrac{3}{4}a'^2) + \ldots\}. \quad (62) \end{aligned}$$

$$\begin{aligned} \frac{\partial T}{\partial x'} = \pi^2\gamma\gamma'\frac{a^2a'^2}{r^4}\{&1\,.\,2\,.\,3\frac{x}{r} + 2\,.\,3\,.\,4\frac{x^2 - \frac{1}{4}a^2}{r^3}x' \\ &+ 3\,.\,4\,.\,5\frac{x(x^2 - \frac{3}{4}a^2)}{r^5}(x'^2 - \tfrac{1}{4}a'^2) + \ldots\}. \quad (63) \end{aligned}$$

22. Discussion of two single-layer coils. We conclude the present chapter with an investigation of the mutual energy of two cylindrical coils, each consisting of a single layer of fine wire carrying currents γ, γ' and so placed that their axes intersect at an angle θ as shown in Fig. 54. The discussion of some more general cases will be given in the next chapter.

Single-layer coils are capable of being constructed so that their constants can be determined with very great accuracy, and the expression of the electrokinetic energy of the arrangement enables the inductances to be found exactly for a number of important cases. Let x_1, x_1', x_2, x_2' be the distances of the nearer and farther ends of the coils from the intersection of their axes, x, x' those of two circular elements of lengths dx, dx'. If now n, n' be the numbers of turns per unit length and γ, γ' the currents, in the two coils, the currents in the elements are $n\gamma\, dx$, $n'\gamma'\, dx'$. Writing down then by (1) the expression for the energy of the two elements, and integrating from $x = x_1$ to $x = x_2$ in the one case, and from $x' = x_1'$ to $x' = x_2'$ in the other, we get for the mutual electrokinetic energy of the two coils of lengths $x_2 - x_1$, $x_2' - x_1'$ the expression

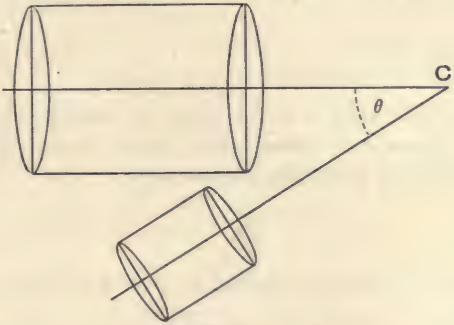

FIG. 54.

$$T = 4\pi^2 nn'\gamma\gamma' a^2 a'^2 \sum \frac{1}{i(i+1)} \cdot {}_\theta Z_i \int_{x_1}^{x_2} \frac{{}_a Z_i'}{r^{i+2}}\, dx \int_{x_1'}^{x_2'} r'^{i-1} \cdot {}_{a'}Z_i'\, dx. \quad \ldots (64)$$

The quantities ${}_a Z_i'$, ${}_{a'}Z_i'$ can be found by differentiation with respect to $\cos a$, $\cos a'$ of the well-known expressions for ${}_a Z_i$, ${}_{a'}Z_i'$, and the integrals then got by direct integration; but the following theorems for zonal harmonics of even and odd orders respectively, yield at once the integrals required. [See the paper cited on p. 201, footnote.]

(a) The solid angle subtended by one of the circles, say that of radius a and axial distance x, at a point distant ρ from the origin is given if $\rho < r$ by the equation

$$\omega = 2\pi \left\{ 1 - \cos a + \sin^2 a \sum \frac{1}{i}\, {}_a Z_i' \cdot {}_\theta Z_i \left(\frac{\rho}{r}\right)^i \right\}, \quad \ldots\ldots .(65)$$

where θ is the angle between the axis of the circle and the line from the origin C to the point in question.

23. Particular case of axes of coils at right angles. Method of integration. Now let the angle θ be 90°, and write y for ρ in this case, since it is the distance of the point considered from the axis. The integrals to be found do not depend on the value of θ, a. Then all the harmonics ${}_\theta Z_i$ of odd order vanish for $\theta = 90^\circ$, and the general expression for the harmonic of even order $2i$ is

$$(-1)^i \frac{1 \,.\, 3 \ldots (2i-1)}{2 \,.\, 4 \ldots 2i}.$$

Hence $$\omega = 2\pi\left[1 - \frac{x}{r} - a^2 \left\{\frac{1}{2^2}\frac{y^2}{r^4} \,.\, {}_\alpha Z_2' - \frac{1 \,.\, 3}{2 \,.\, 4^2}\frac{y^4}{r^6} \,.\, {}_\alpha Z_4' + \ldots\right\}\right]. \quad \ldots(66)$$

But this is of the form

$$\omega = 2\pi(A_0 + A_1 y^2 + A_2 y^4 + \ldots), \quad \ldots\ldots(67)$$

where $2\pi A_0$ is the value of ω for $y=0$, so that $A_0 = 1 - x/r$. Now ω must satisfy Laplace's equation, which, since there is symmetry round the axis of the circle, is for the present case

$$\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} + \frac{1}{y}\frac{\partial\omega}{\partial y} = 0. \quad \ldots\ldots(68)$$

Differentiating (67) and substituting in (68), we find

$$\begin{aligned} &\frac{\partial^2 A_0}{\partial x^2} + \frac{\partial^2 A_1}{\partial x^2} y^2 + \frac{\partial^2 A_2}{\partial x^2} y^4 + \ldots \\ &+ 2A_1 + 3 \,.\, 4A_2 y^2 + 5 \,.\, 6A_3 y^4 + \ldots \\ &+ 2A_1 + \quad 4A_2 y^2 + \quad 6A_3 y^4 + \ldots = 0. \end{aligned}$$

The coefficients of the different powers of y in this series equated separately to zero give

$$A_1 = -\frac{1}{2^2}\frac{\partial^2 A_0}{\partial x^2}, \quad A_2 = \frac{1}{2^2 \,.\, 4^2}\frac{\partial^4 A_0}{\partial x^4}, \quad A_3 = -\frac{1}{2^2 \,.\, 4^2 \,.\, 6^2}\frac{\partial^6 A_0}{\partial x^6}, \ldots$$

so that $$\omega = 2\pi\left(A_0 - \frac{y^2}{2^2}\frac{\partial^2 A_0}{\partial x^2} + \frac{y^4}{2^2 \,.\, 4^2}\frac{\partial^4 A_0}{\partial x^4} - \ldots\right). \quad \ldots\ldots(69)$$

Comparing this with (66) we see that

$$a^2 \frac{{}_\alpha Z_2'}{r^4} = \frac{\partial^2 A_0}{\partial x^2}, \quad 3\,!\, a^2 \frac{{}_\alpha Z_4'}{r^6} = \frac{\partial^4 A_0}{\partial x^4}, \quad 5\,!\, a^2 \frac{{}_\alpha Z_6'}{r^8} = \frac{\partial^6 A_0}{\partial x^6}, \ldots.$$

Thus, neglecting constants of integration,

$$\left.\begin{aligned} a^2 \int \frac{{}_\alpha Z_2'}{r^4}\,dx &= \frac{\partial A_0}{\partial x}, \\ 3\,!\, a^2 \int \frac{Z_4'}{r^6}\,dx &= \frac{\partial^3 A_0}{\partial x^3}, \end{aligned}\right\} \quad \ldots\ldots(70)$$

and we are able to calculate the integrals of even order required for (64) by successive differentiation.

To find the integrals of odd order, let us assume that

$$A\int\frac{{}_{a}Z'_{2i+1}}{r^{2i+3}}\,dx=\frac{\partial^{2i}A_0}{\partial x^{2i}}, \quad \text{......\ ....}(71)$$

where A is a constant to be determined. Differentiating we obtain from this equation and (70) the relation

$$A\,\frac{{}_{a}Z'_{2i+1}}{r^{2i+3}}=\frac{\partial^{2i+1}A_0}{\partial x^{2i+1}}=(2i+1)\,!\,a^2\int\frac{{}_{a}Z_{2i+2}}{r^{2i+4}}\,dx,$$

and therefore also

$$A\{(1-\mu^2)\,{}_{a}Z''_{2i+1}-(2i+3)\mu\,{}_{a}Z'_{2i+1}\}=(2i+1)\,!\,a^2\,.\,{}_{a}Z'_{2i+2}, \quad \text{......}(72)$$

where $\mu=\cos\alpha$.

The assumption made in (71) will be justified if the relation expressed in (72) holds for a constant value of A. Now if Z_i denote a zonal harmonic of any order i, we have, by the fundamental relations of zonal harmonics,

$$\mu Z_i-Z_{i-1}=-\frac{1}{i}(1-\mu^2)Z_i', \quad Z_i-\mu Z_{i-1}=-\frac{1}{i}(1-\mu^2)Z'_{i-1}.$$

By elimination of first Z_{i-1}, and then of Z_i, we find

$$Z_i=\frac{1}{i}(\mu Z_i'-Z'_{i-1}), \quad Z_{i-1}=\frac{1}{i}(Z_i'-\mu Z'_{i-1}). \quad \text{...........}(73)$$

Differentiating these with respect to μ and eliminating Z_i'', we get

$$(1-\mu^2)Z''_{i-1}-\mu(i+1)Z'_{i-1}=-(i-1)Z_i',$$

which, with $2i+2$ written for i, agrees with (72) if we put

$$A=-(2i-1)\,!\,a^2.$$

Thus the assumption is justified.

Hence neglecting as before constants of integration, we obtain

$$\left.\begin{aligned} a^2\int\frac{{}_{a}Z_1}{r^3}\,dx&=-A_0,\\ 2\,!\,a^2\int\frac{{}_{a}Z_3}{r^5}\,dx&=-\frac{\partial^2A_0}{\partial x^2},\\ 4\,!\,a^2\int\frac{{}_{a}Z_5}{r^7}\,dx&=-\frac{\partial^4A_0}{\partial x^4},\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ &\cdot\ \cdot\ \cdot\ \cdot\end{aligned}\right\} \quad \text{........................}(74)$$

so that (70) and (74) give by the same process all the required integrals. Taken together they give the theorem, of great importance, in this connection at least,

$$\int\frac{Z_i'}{r^{i+2}}\,dx=(-1)^i\frac{1}{(i-1)\,!\,a^2}\,\frac{\partial^{i-1}A_0}{\partial x^{i-1}}, \quad \text{..................}(75)$$

where i is any positive integer. A similar theorem holds of course, *mutatis mutandis*, that is with a' substituted for a, x' for x, and ρ for r_1, for the harmonics in a', and can be used as indicated above for the calculation of the second integrals in (74).

24. Formulae of calculation. The first seven derived functions of $A_0(=1-x/r)$ are as follows:

$$\frac{\partial A_0}{\partial x} = -\frac{a^2}{r^3}, \quad \frac{\partial^2 A_0}{\partial x^2} = \frac{3a^2x}{r^5}, \quad \frac{\partial^3 A_0}{\partial x^3} = -3\frac{a^2}{r^7}(4x^2-a^2),$$

$$\frac{\partial^4 A_0}{\partial x^4} = 3 \,.\, 5\frac{a^2x}{r^9}(4x^2-3a^2),$$

$$\frac{\partial^5 A_0}{\partial x^5} = -3^2 \,.\, 5\frac{a^2}{r^{11}}(8x^4-12x^2a^2+a^4),$$

$$\frac{\partial^6 A_0}{\partial x^6} = 3^2 \,.\, 5\frac{a^2x}{r^{13}}(56x^4-140x^2a^2+35a^4),$$

$$\frac{\partial^7 A_0}{\partial x^7} = -3^2 \,.\, 5\frac{a^2}{r^{15}}(448x^6-1680x^4a^2+840x^2a^4-35a^6).$$

Substituting these values in (70), (74) or (75), we obtain, for $\gamma=\gamma'=1$,

$$T = \pi^2 nn'a^2a'^2\{K_1k_1 \,.\, {}_\theta Z_1 + K_2k_2 \,.\, {}_\theta Z_2 + \ldots\}, \quad \ldots\ldots\ldots\ldots(76)$$

where

$$K_1 = \frac{2}{a^2}\left(\frac{x_2}{r_2}-\frac{x_1}{r_1}\right), \quad K_2 = -\left(\frac{1}{{r_2}^3}-\frac{1}{{r_1}^3}\right), \quad K_3 = -\frac{1}{2}\left(\frac{x_2}{{r_2}^5}-\frac{x_1}{{r_1}^5}\right),$$

$$K_4 = -\frac{1}{8}\left\{\frac{1}{{r_2}^7}(4{x_2}^2-a^2)-\frac{1}{{r_1}^7}(4{x_1}^2-a^2)\right\},$$

$$K_5 = -\frac{1}{8}\left\{\frac{x_2}{{r_2}^9}(4{x_2}^2-3a^2)-\frac{x_1}{{r_1}^9}(4{x_1}^2-3a^2)\right\},$$

$$K_6 = -\frac{1}{8}\left\{\frac{1}{{r_2}^{11}}(4{x_2}^2-6{x_2}^2a^2+\tfrac{1}{2}a^4)-\frac{1}{{r_1}^{11}}(4{x_1}^4-6{x_1}^2a^2+\tfrac{1}{2}a^4)\right\},$$

$$K_7 = -\frac{1}{8}\left\{\frac{x_2}{{r_2}^{13}}(4{x_2}^4-10{x_2}^2a^2+\tfrac{5}{2}a^4)-\frac{x_1}{{r_1}^{13}}(4{x_1}^4-10{x_1}^2a^2+\tfrac{5}{2}a^4)\right\},$$

. . . .

$$k_1 = x_2'-x_1', \quad k_2 = x_2'^2-x_1'^2, \quad k_3 = 2x_2'^3-\tfrac{3}{2}x_2'a'^2-2x_1'^3+\tfrac{3}{2}x_1'a'^2,$$

$$k_4 = 2x_2'^4-3x_2'^2a'^2-2x_1'^4+3x_1'^2a'^2,$$

$$k_5 = 2x_2'^5-5x_2'^3a'^2+\tfrac{5}{4}x_2'a'^4-2x_1'^5+5x_1'^3a'^2-\tfrac{5}{4}x_1'a'^4,$$

$$k_6 = 2x_2'^6-\tfrac{15}{2}x_2'^4a'^2+\tfrac{15}{4}x_2'^2a'^4-2x_1'^6+\tfrac{15}{2}x_1'^4a'^2-\tfrac{15}{4}x_1'^2a'^4,$$

$$k_7 = 2x_2'^7-\tfrac{21}{2}x_2'^5a'^2+\tfrac{35}{4}x_2'^3a'^4-\tfrac{35}{32}x_2'a'^6-2x_1'^7+\tfrac{21}{2}x_1'^5a'^2-\tfrac{35}{4}x_1'^3a'^4+\tfrac{35}{32}x_1'a'^6.$$

. . . .

If one of the coils, say that of radius a, have its centre at the origin $x_2=-x_1$, the terms of even order all vanish, since K_2, K_4, ...

vanish. If, besides, the other coil have its centre at the origin, $x_2' = -x_1'$, and the even terms doubly vanish.

Further, if besides being so placed, the second coil be constructed so that its length $2x_2' = \sqrt{3}a'$, the third term of the series in (76) will vanish; and similarly the fifth term will disappear if the first coil fulfil the relation $2x_2 = \sqrt{3}a$. Thus under these conditions all the terms in (76) between the first and the seventh disappear.

25. Application of results to construction of absolute galvanometers and electro-dynamometers. The first term will give T to a sufficient degree of approximation for all practical purposes if $a' \not> \frac{2}{3}a$, as then the coefficient of ${}_\theta Z_7$ does not amount to more than 1/1770 of that of ${}_\theta Z_1$, and the terms of higher order are relatively unimportant. If $a' \not> \frac{1}{2}a$, the seventh coefficient in (76) is at most only about 1/10000 of the first. With the former ratio of radii, the error made in taking only the first term of the series amounts to a quantity quite inappreciable in experimental measurements made with standard coils, and an important application of the result can be made in the construction of electro-dynamometers.

If the coils be concentric and at right angles, and N, N' be the whole number of turns, the couple on the suspended coil may be written

$$\Theta = \frac{2\pi^2 a^2 N N' \gamma\gamma'}{(a^2+x^2)^{\frac{1}{2}}}\left\{1 + \cdot 0001851\left(\frac{a'}{a}\right)^6 + \cdot 0000307\left(\frac{a'}{a}\right)^8 + \ldots\right\}, \ldots (76')$$

which shows better that the value of the couple depends, for any reasonable ratio a'/a, almost entirely on the first term of the series in brackets. If, for example, $a'/a = \frac{1}{2}$, the correction terms following the first amount to only about 3 parts in a million.

A single-layer coil made as here of considerable length seems very suitable also for use as an absolute galvanometer. It has sufficient uniformity of field to render the very exact placing of the needle at the centre quite unessential, and it can be made sufficiently sensitive, so that it possesses most of the advantages of the Helmholtz double-coil arrangement, without the uncertainty which exists in the latter as to the distribution of the different turns of wire in the two multiple-layer bobbins, or requiring the correction terms which the bobbins involve on account of their finite cross-section.

We may find the couple acting on the needle of such a galvanometer as follows, provided the needle be suspended with its axis intersecting that of the coil. The suspended coil in the above discussion may be taken as a solenoidal magnet of magnetic moment $\pi a'^2 n' \gamma'$ per unit of length, and therefore of total magnetic moment $M = \pi a'^2 n' \gamma' (x_2' - x_1')$. Hence by (76), we have, since $\Theta = \partial T / \partial \theta$,

$$\Theta = -\pi n \gamma a^2 M \sin\theta \frac{1}{x_2' - x_1'} \{K_1 k_1 \cdot {}_\theta Z_1' + K_2 k_2 \cdot {}_\theta Z_2' + \ldots\}, \ldots (77)$$

from which by means of the values of K_1, K_2, ... k_1, k_2, ... given above the value of Θ in the general case can be calculated.

If the coil and solenoidal magnet be concentric all the even terms vanish as before, and by making the length of the coil $\sqrt{3}$ times its radius we can cause the fifth term to disappear. The couple therefore to the seventh term inclusive is given by

$$\Theta = -2\pi n\gamma a^2 \boldsymbol{M} \sin\phi \left\{ \frac{a^2}{2}\frac{x_2}{r_2} - \frac{1}{4}\frac{x^2}{{r_2}^5}(4x_2'^2 - 3a'^2)\,{}_\theta Z_3' \right.$$
$$- \frac{1}{8}\frac{x_2}{{r_2}^{13}}\left(4{x_2}^4 - 10{x_2}^2a^2 + \frac{5}{2}a^4\right)\left(2x_2'^6 - \frac{21}{2}a^2x_2'^4\right.$$
$$\left.\left. + \frac{35}{4}x_2'^2a^4 - \frac{35}{32}a'^6\right){}_\theta Z_7' \right\}. \quad \ldots\ldots(78)$$

With an actual magnet it is impossible to set up any definite relation between x_2' and a_2'; but by using a thin uniform needle it is possible to make $2a'$, which is a quantity of the order of magnitude of the thickness, small compared with x_2', and therefore practically zero. Then by making x_2', which for a thin needle of uniform thickness is approximately its half-length, small in comparison with r_2, the second and third terms in (78) may be made quite negligible. For example, if a needle 1 cm long be used in a coil of 20 cm radius, and therefore of axial length 34·64 cm, and the value of θ be approximately 90°, the second term in (78) is only about 1/(6500) of the first.

26. Coefficient of mutual induction of two coils: standards of inductance. We may notice here (though the subject of induction coefficients belongs to the next section of this chapter), that in (74) $T/\gamma\gamma'$ is the coefficient of mutual induction of the two coils. Thus if two coils of considerably different radii, but each having its length $\sqrt{3}$ times its radius, be arranged concentrically, their mutual induction coefficient is given for any angle between their axes with accuracy by the first term of (74). In this way standards of mutual inductance could be easily made with very considerable exactness.

By supposing the coils equal in every respect and coincident, we can calculate the self-induction coefficient of each, by taking the value of $T/\gamma\gamma'$ given by (76). In this case however the first term does not give an exact result, and it is necessary to take in at least one more term of the series.

For two coaxial and concentric solenoids of half lengths, x and x', and radii a and a', the mutual inductance, calculated by (76) and the values of the *K*s and *k*s which follow that equation, comes out

$$M = \frac{8\pi^2}{r}nn'xx'a'^2\{1 - \frac{a^2}{8r^4}(4x'^2 - 3a'^2) - \frac{a^2}{64r^8}(4x^2 - 3a^2)(8x'^4 - 20x'^2a'^2 + 5a'^4)$$
$$- \frac{a^2}{1024r^{12}}(8x^4 - 20x^2a^2 + 5a^4)(64x'^6 - 336x'^4a'^2 + 280x'^2a'^4 - 35a^6)\} - \ldots \quad (79)$$

Here r is the length of the diagonal of half the outer coil, that is, $r^2=x^2+a^2$. [See Fig. 54.]*

With a certain amount of accuracy the single-layer coils discussed above might be replaced by coils consisting of several layers, the ends of the channel in each case being frustums of a cone having its vertex at the common centre and semi-vertical angle equal to $\tan^{-1}(2/\sqrt{3})$. This makes each layer (unless a whole number of turns cannot be made in each case to fulfil the relation) have its length equal to $\sqrt{3}$ times its radius. The mutual energy and the action of one coil on the other can then be calculated by considering separately each pair of single-layer coils which can be formed by taking one layer in each coil. In such an arrangement, however, as in all multiple-layer coils, the distribution of the wires would be to a certain extent irregular.

In the calculations of coil constants given in later chapters no allowance is made for insulation space between the turns of wire. This matter, however, will be found shortly dealt with in XIII. 51, below.

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* The zonal harmonic expression of the electrokinetic energy of two coils was integrated in a very general form and all the formulae and factors, here used, worked out in the paper by the author, *loc. cit. supra.* Equivalent expressions for some of these results were given later by Searle and Airey in the *Electrician*, 56, 1905.

It was observed after Chapters VI. and VII. were in type that in the discussions in VI. 23, and VII. 1 ... 3, which were written at different times and with different applications in view, the symbol A_0 of VI. 23 has the same meaning, $1-x/r$, as the $-\partial A_0/\partial x$ of VII. 3, 17. The discussions are thus independent, and one is to some extent a check on the other. As each chapter is complete in itself, it has not been thought necessary to rewrite, and reprint, any of the subject matter of either.

CHAPTER VII.

CALCULATION OF CONSTANTS OF COILS AND COEFFICIENTS OF INDUCTION. MAGNETIC ACTION OF CIRCUITS AND COILS.

1. Solid angle subtended by circle. Potential due to circular current. It has been proved above that the solid angle subtended by a circle at any point is given by the equation

$$\omega = 2\pi\left\{1 - \cos\alpha + \sin^2\alpha \sum \frac{1}{i} \cdot {}_{\alpha}Z_i' \cdot {}_{\theta}Z_i \cdot \left(\frac{r'}{r}\right)^i\right\}, \ldots\ldots\ldots\ldots(1)$$

where (Fig. 55) θ is the angle between CP and the axis, α the angle between CA and the axis, ${}_{\theta}Z_i$ the zonal spherical harmonic of order i, ${}_{\alpha}Z_i'$ the differential coefficient of ${}_{\alpha}Z_i$ with respect to $\cos\alpha$, r' the distance CP, and r the distance CA. If as indicated in the figure we take $\theta = 90°$, then all the zonal harmonics of odd order vanish, and the general expression of the zonal harmonic of even order $2i$ is

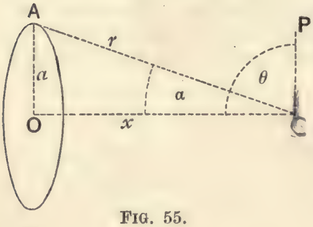

FIG. 55.

$$(-1)^i \frac{1 \cdot 3 \ldots (2i-1)}{2 \cdot 4 \ldots 2i}.$$

Now the solid angle subtended at any point by a closed curve is equal to the potential which a unit current flowing in the curve would produce at that point. Hence if a current γ flow in the circle, and Ω be the potential which the current produces, we have, writing y for CP in the particular case in which it is at right angles to the axis, and x for OC,

$$\Omega = 2\pi\gamma\left\{1 - \frac{x}{r} - \frac{3}{2^2}\frac{y^2}{r^5}a^2x - \frac{3 \cdot 5}{2^2 \cdot 4^2}\frac{y^4}{r^9}a^2x(3a^2 - 4x^2) - \frac{3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2}\frac{y^6}{r^{13}}a^2x(35a^4 - 140a^2x^2 + 56x^4) - \ldots\right\}, \ldots.(2)$$

a series which is convergent if $y < r$.

This equation can be found as follows without the use of zonal harmonics, and a comparison of the processes gives results which will be of great service in some more complex applications of zonal harmonics later in the present chapter.

2. Potential due to circular surface distribution of magnetism. Consider the potential produced at any point P (Fig. 55), at a distance y from the axis of the circle, by a circular plane distribution of magnetism. Let the dimensions be as in Fig. 55, and denote by σ the surface density of the magnetic distribution, and by dV the potential at C produced by a narrow concentric ring of the magnetism of radius p and breadth dp. Then

$$dV = 2\pi\sigma \frac{p\,dp}{\sqrt{p^2+x^2}}.$$

Hence integrating from $p=0$ to $p=a$, we find

$$V = 2\pi\sigma(\sqrt{a^2+x^2}-a)$$

for the potential at C due to the whole distribution.

Now assume for the potential at P,

$$V = 2\pi\sigma(A_0 + A_1 y^2 + A_2 y^4 + \ldots), \qquad\text{.......................(3)}$$

where A_0, A_1, ... are functions of x. No odd powers of y can enter, since the potential is not altered by reversing the sign of y; and since, when $y=0$, the value of V reduces to that for C we have [see Note, p. 209]

$$A_0 = \sqrt{a^2+x^2} - x. \qquad\text{....................(4)}$$

3. Potential due to disk, and potential due to circular magnetic shell, found by solving Laplace's equation. At all points external to the distribution V must satisfy Laplace's equation, which, for the case of symmetry round the axis of x, takes the form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{1}{y}\frac{\partial V}{\partial y} = 0. \qquad\text{...........................(5)}$$

Differentiating (3) and substituting in (4) we find

$$\begin{aligned}\frac{\partial^2 A_0}{\partial x^2} + \frac{\partial^2 A_1}{\partial x^2} y^2 + \frac{\partial^2 A_2}{\partial x^2} y^4 + \ldots& \\ + 2A_1 + 3\,.\,4A_2 y^2 + 5\,.\,6A_3 y^4 + \ldots& \\ + 2A_1 + \quad 4A_2 y^2 + \quad 6A_3 y^4 + \ldots &= 0.\end{aligned}$$

The coefficients of the different powers of y in this series equated separately to zero give

$$A_1 = -\frac{1}{2^2}\frac{\partial^2 A_0}{\partial x^2}, \quad A_2 = \frac{1}{2^2\,.\,4^2}\frac{\partial^4 A_0}{\partial x^4}, \quad A_3 = -\frac{1}{2^2\,.\,4^2\,.\,6^2}\frac{\partial^6 A_0}{\partial x^6}, \ldots.$$

Hence finally

$$V = 2\pi\sigma\left(A_0 - \frac{y^2}{2^2}\frac{\partial^2 A_0}{\partial x^2} + \frac{y^4}{2^2\,.\,4^2}\frac{\partial^4 A_0}{\partial x^4} - \ldots\right), \qquad\text{.............(6)}$$

where $A_0 = \sqrt{a^2+x^2} - x$. [See footnote at end of Chap. VI.]

From (6), of course, by differentiation with respect to x and y respectively, the axial and radial component forces at the point x, y, can be obtained for the given distribution.

If now another circular plane distribution, of equal density but opposite sign, be supposed placed coaxial with and at a distance $-dx$ from the former, its potential at the point (x, y) will be the same as that produced at the point $(x+dx, y)$ by the former distribution, except that the sign will be changed. Thus it is $-(V+\partial V/\partial x \,.\, dx)$. The potential at the point (x, y), due to the two plane distributions together, is thus $-\partial V/\partial x \,.\, dx$. Calling this Ω we have

$$\Omega = 2\pi\sigma\, dx\left(-\frac{\partial A_0}{\partial x} + \frac{y^2}{2^2}\frac{\partial^3 A_0}{\partial x^3} - \frac{y^4}{2^2 \,.\, 4^2}\frac{\partial^5 A_0}{\partial x^5} + \dots\right). \quad \dots\dots\dots\dots(7)$$

This is the potential at x, y of a magnetic shell of strength $\sigma\, dx$. If the shell be replaced by a current of strength γ flowing in a circle coinciding with the edge of the shell, we have $\sigma\, dx = \gamma$. Performing then the differentiations of A_0 (writing for brevity r for $\sqrt{a^2+x^2}$, and replacing $\sigma\, dx$ by γ), we find again (2).

By comparison of (1) and (7), remembering that θ' is now 90°, and $r' = y$, we see that

$$\left.\begin{aligned} 1 - \frac{x}{r} &= -\frac{\partial A_0}{\partial x}, \\ (2i-1)!\,\frac{a^2}{r^{2i+2}}\,{}_{a}Z'_{2i} &= -\frac{\partial^{2i+1} A_0}{\partial x^{2i+1}}, \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots(8)$$

a general result which enables $\int\{{}_{a}Z'_{2i}/r^{2i+2}\}dx$ to be calculated for any value of i by successive differentiation of A_0. By the process of VI. 23, it can be proved that a similar theorem is true for zonal harmonics of odd order. The $-\partial A_0/\partial x$ of (8) is the A_0 of VI. 23.

4. Magnetic forces due to circular magnetic shell. The axial and radial component forces F, R, are $-\partial\Omega/\partial x$, $-\partial\Omega/\partial y$ respectively. These could be obtained directly from (2), but it is easier to differentiate (7) with respect to x and y, and insert the differential coefficients of A_0 in the result. Thus

$$\begin{aligned} F = -\frac{\partial\Omega}{\partial x} = 2\pi\gamma\frac{a^2}{r^3}\Big\{1 &+ \frac{3}{2^2}\frac{y^2}{r^4}(a^2 - 4x^2) + \frac{3^2 \,.\, 5}{2^2 \,.\, 4^2}\frac{y^4}{r^8}(a^4 - 12a^2x^2 + 8x^4) \\ &+ \frac{3^2 \,.\, 5}{2^2 \,.\, 4^2 \,.\, 6^2}\frac{y^6}{r^{12}}(35a^6 - 840a^4x^2 + 1680a^2x^4 - 448x^6) + \dots\Big\}, \quad \dots\dots(9) \end{aligned}$$

$$\begin{aligned} R = -\frac{\partial\Omega}{\partial y} = 3\pi\gamma\frac{a^2xy}{r^5}\Big\{1 &+ \frac{5}{2 \,.\, 4}\frac{y^2}{r^4}(3a^2 - 4x^2) \\ &+ \frac{3 \,.\, 5}{2 \,.\, 4^2 \,.\, 6}\frac{y^4}{r^8}(35a^4 - 140a^2x^2 + 56x^4) + \dots\Big\}. \quad \dots\dots\dots\dots\dots\dots(10) \end{aligned}$$

The field due to a circular conductor is shown in section by Fig. 56, which is taken from Maxwell's *Electricity and Magnetism*, vol. ii., with an extension to show the field symmetrically about the centre of the circular conductor.

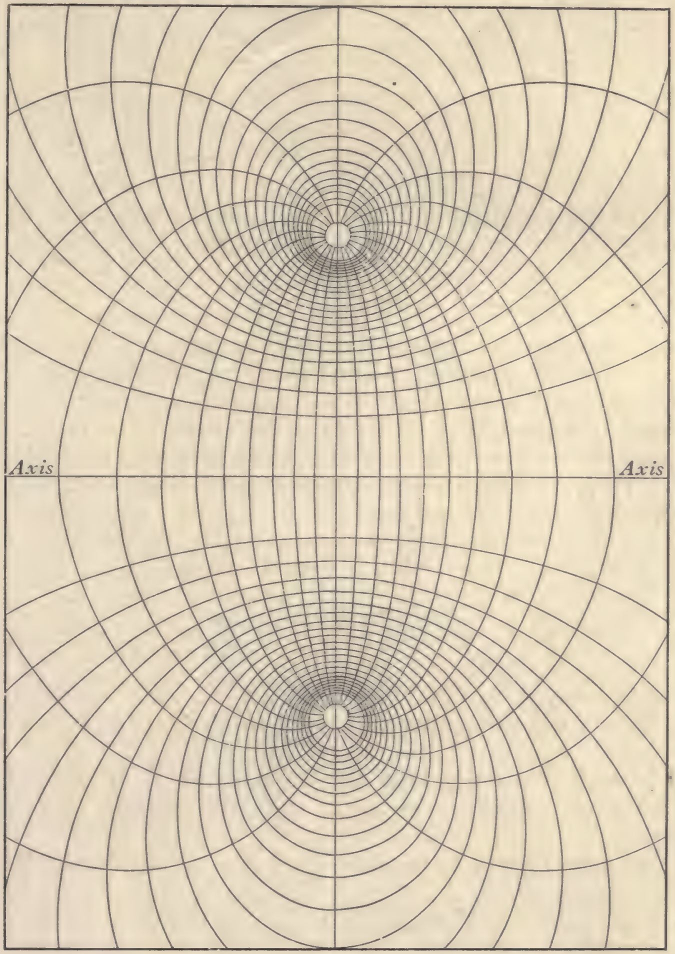

FIG. 56.

5. Couple on magnetic needle produced by circular current. From these results we could calculate the couple on a thin uniformly magnetized needle A, B (Fig. 57) placed with its centre on the axis, and deflected into any given position; but the following method is preferable. Let $2l$ be the length of the needle, and θ the angle which its axis makes with the plane of the circuit. The coordinates of its ends are $x+l\sin\theta$, $l\cos\theta$ for A, and $x-l\sin\theta$, $-l\cos\theta$ for B. Now if

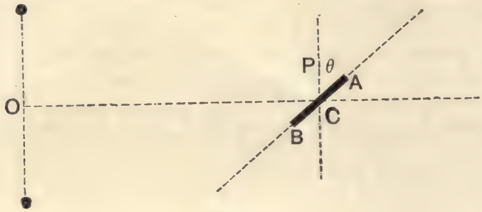

FIG. 57.

Ω_1, Ω_2 be the potentials at A and B respectively, we have by Taylor's theorem,

$$\left.\begin{matrix}\Omega_1\\ \Omega_2\end{matrix}\right\} = \Omega \pm l\left(\sin\theta\frac{\partial\Omega}{\partial x} + \cos\theta\frac{\partial\Omega}{\partial y}\right)$$
$$+\frac{l^2}{1\,.\,2}\left(\sin^2\theta\frac{\partial^2\Omega}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2\Omega}{\partial x\,\partial y} + \cos^2\theta\frac{\partial^2\Omega}{\partial y^2}\right) \pm \dots \quad (11)$$

Thus if the strength of each pole of the needle be m, the energy of the needle in the given position is $m(\Omega_2-\Omega_1)$, supposing the positive end at B. By (11) we have, writing $\boldsymbol{M}$, the magnetic moment of the needle, for $2ml$, and D for the operator

$$\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y},$$

$$m(\Omega_2-\Omega_1) = -\boldsymbol{M}\left(D + \frac{l^2}{3!}D^3 + \frac{l^4}{5!}D^5 + \dots\right)\Omega, \quad\dots\dots\dots\dots(12)$$

where Ω is given by (7). The expansion is a little troublesome but not difficult, and the reader may verify that the result stated in (13) is correct.

If instead of a single turn of wire there be N turns which may be taken as coincident, we must write $N\gamma$ instead of γ in this equation.

The couple Θ acting on the needle is thus given numerically by

$$\Theta = m\frac{\partial(\Omega_2-\Omega_1)}{\partial\theta} = 2\pi N\gamma\boldsymbol{M}\cos\theta\frac{a^2}{r^3}\left\{1 + \frac{3}{2^2}\frac{l^2}{r^4}(a^2-4x^2)(1-5\sin^2\theta)\right.$$
$$\left.+\frac{3^2\,.\,5}{2^2\,.\,4^2}\frac{l^4}{r^8}(a^4-12a^2x^2+8x^4)(1-14\sin^2\theta+21\sin^4\theta)+\dots\right\}, \quad\dots(13)$$

a formula of great importance in galvanometry.

If the needle be not uniformly magnetized the value of l is not definite. It is easy to see however that $\boldsymbol{M}$, the magnetic moment of the magnet, should be used in the first term. In the other terms

l^2, l^4, etc., should be replaced by quantities depending on the distribution of magnetism on the needle. This however it is in general impossible to determine for a small needle.

If l be very small the expression on the right of (13) reduces to the first term approximately; and if also $x=0$, that is, if the centre of the needle is at the centre of the circle, we have

$$\Theta = 2\pi N\gamma M \cos\theta/a. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13')$$

6. Modification of formulae to allow for dimensions of coil-section. The principal term in the expression on the right of (13) is the first $2\pi N\gamma M\cos\theta a^2/r^3$, which, by (9) and (10), is the value of the couple when l is so small that the component R of magnetic force is negligible, and the value which F has at the centre of the needle is taken as the force at each pole. Now we have for the couple in that case

$$\Theta = 2\pi N\gamma M\cos\theta a^2/r^3 = FM\cos\theta,$$

so that $F = 2\pi N\gamma a^2/r^3$. We have to find what takes the place of $2\pi N\gamma a^2/r^3$, or F, in (13) when the coil cannot be treated as a simple circular conductor. For the other terms, unless the dimensions of the bobbin are larger than usual, the coil may be taken as a single circular conductor coinciding with the mean circle of the bobbin, and carrying the whole current. The case of a long bobbin we shall consider specially.

Let the breadth in the direction of the axis of the cross-section of the coil by a plane through the axis be $2b$, and the radial depth of the section $2d$. Let BC (Fig. 58) be a radius drawn from the centre C of the coil in that plane which cuts the coil into two equal and similar coils, and taking $DE\ (=h)$, $CD\ (=k)$ at right angles to one another, we have $dh\,dk$ for the area of the element E of the cross-section of the coil by a plane passing through the axis and through BC. Also $PE^2 = (x-h)^2 + k^2$. Let, further, n be the number of turns crossing unit of area of cross-section, and γ the current in each. The current crossing the element E is $n\gamma\,dh\,dk$, for we here suppose the wire so fine that we may suppose that everywhere the current crossing any area of cross-section is proportional to that area. [When the layers of wire form each a helix we here neglect the axial component of flow. How this may be compensated will be explained later.] Hence by the law (p. 178 above), which we may assume, as to the magnetic action of the elements of a circuit, the force exerted on a unit magnetic pole at P, by an element, of area $dh\,dk$ and length ds, at right angles to the plane of the paper is $n\gamma\,ds\,dh\,dk/\{(x-h)^2+k^2\}^{\frac{3}{2}}$. Hence if dF be the component in this

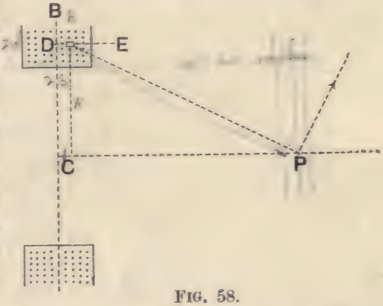

FIG. 58.

direction due to the whole ring, of which the element E is the cross-section,

$$dF = 2\pi n\gamma \frac{k^2 dh\, dk}{\{(x-h)^2 + k^2\}^{\frac{3}{2}}}.$$

The whole magnetic force parallel to the axis is therefore

$$F = 2\pi n\gamma \int_{-b}^{+b}\int_{a-d}^{a+d} \frac{k^2 dh dk}{\{(x-h)^2 + k^2\}^{\frac{3}{2}}},$$

or after integration,

$$F = 2\pi n\gamma \left\{ (x+b)\log\frac{a+d+\sqrt{(x+b)^2+(a+d)^2}}{a-d+\sqrt{(x+b)^2+(a-d)^2}} \right.$$

$$\left. -(x-b)\log\frac{a+d+\sqrt{(x-b)^2+(a+d)^2}}{a-d+\sqrt{(x-b)^2+(a-d)^2}} \right\}, \quad \text{.........(14)}$$

which reduces, when $x = 0$, to

$$F = \pi N\gamma \frac{1}{d}\log\frac{a+d+\sqrt{(a+d)^2+b^2}}{a-d+\sqrt{(a-d)^2+b^2}}, \quad \text{..................(15)}$$

and, when b and d are small enough, to

$$F = 2\pi N\gamma/a, \quad \text{............................(16)}$$

where N is the total number of turns in the coil.

The value of F in (14) is to be used, when required, instead of $2\pi N\gamma a^2/r^3$, so far as the first term of the series for F is concerned; the remainder of the series is to be retained without alteration as sufficiently accurate for practical purposes.

7. Removal of second term in series for F. Gaugain's galvanometer. The second term of the series in (13), involving the product

$$(a^2 - 4x^2)(1 - 5\sin^2\theta),$$

may be made to vanish by arranging so that one or both of the factors may vanish. The value of the second factor is 1 when $\theta = 0$, and diminishes as θ (whether positive or negative) increases in numerical value, until, when $\theta = \pm\sin^{-1}(1/\sqrt{5}) = \pm 26^\circ\, 34'$, it is zero. Thereafter it becomes negative, and approaches -4 as θ approaches 90°. At 45° its value is $-3/2$.

The first factor may be made to vanish by placing the needle so that $x = a/2$. This was done by Gaugain in his galvanometer, which consisted of a vertical coil with a needle so suspended that its centre was as nearly as possible on the axis of the coil, at a distance equal to half its radius. The uncertainty as to the proper distance, caused by the dimensions of the cross-section of the coil itself, was got over by winding the wire on a conical surface of semi-vertical angle $\tan^{-1}2$, so that the distance of the needle, suspended with its centre as nearly as possible at the vertex, might be in the proper position relative to each spire.

With proper arrangements this winding of the coil, though more difficult than that of an ordinary bobbin, might be carried out with sufficient exactness; but any inaccuracy in the placing of the needle is serious. For by (13) the value of $\partial\Theta/\partial x$ is $-2\pi N\gamma M\cos\theta\,.\,3a^2/r^4\,.\,\partial r/\partial x$, and therefore Θ requires correction for an error dx in placing the needle, by multiplication by the factor $1+1/\Theta\,.\,\partial\Theta/\partial x\,.\,dx$, or $1-3x\,dx/r^2$, or since $x=a/2$, by the factor $1-6dx/5a$. Thus if dx is sensible, this factor, depending as it does on $1/a$, seriously affects the value of Θ.

8. Helmholtz's arrangement of galvanometer coils. Gaugain's galvanometer has been improved upon by von Helmholtz, in whose arrangement two equal parallel coils are placed with their medial planes at a distance apart equal to their mean radius. The needle is suspended with its centre as nearly as may be on the axis, at a point about which the arrangement of coils is symmetrical; and the coils are so joined that the current flows in the same direction round both. This makes $a^2-4x^2=0$, very approximately, in Θ, and further obviates the uncertainty just referred to. For any displacement of the needle towards the coil is attended by a diminution of the couple due to the other coil, and a very nearly equal increase of the couple due to that which is approached.

The field due to the arrangement is shown in Fig. 59, which is taken from Maxwell's *Electricity and Magnetism*, vol. ii., and may be contrasted with that for a simple coil shown in Fig. 56. It will be seen from the diagram of lines of force, and the same thing is obvious from (10) (since the values of x for the two coils are equal and opposite), that R is zero at every point in the plane midway between the coils, and passing therefore (approximately) through the centre of the needle, and also very nearly zero at points even at some distance on either side of this plane. Thus over quite a considerable space surrounding the centre of the needle, the field due to the coils is practically uniform and parallel to the axis, and the couple practically independent of θ, and unaffected by any error in centring the needle, such as would have a serious effect on the couple in the case of a single coil.

It is clear that the energy of the needle in the field of the double coil is twice that given in (12). For the energy of the positive pole, supposed nearer to the coil from which it is repelled, is $m\Omega_2$ in the field of that coil, and $-m\Omega_1$ in the field of the other coil. The energy of the other pole has evidently the same value, so that the whole energy is $2m(\Omega_2-\Omega_1)$. The couple is thus 2Θ, where Θ is given by (13), subject to the condition that $a^2=4x^2$. It may be written therefore to terms of the fourth order inclusive,

$$\Theta=4\pi N\gamma M\cos\theta\,\frac{a^2}{r^3}\left\{1-\frac{3}{2}\,\frac{45}{64}\,\frac{l^4a^4}{r^8}\,(1-14\sin^2\theta+21\sin^4\theta)\right\}, \tag{17}$$

where, since $a^2=4x^2$, $r^2=5x^2$ and N is the number of turns in *each* coil.

Values of θ which satisfy the equation $1-14\sin^2\theta+21\sin^4\theta=0$, render the factor of the second term in brackets zero. These values are 16° 34′ and 49° 55′. The factor in brackets has two maximum numerical values, viz. 8 for $\theta=\pm 90°$ and $-4/3$ for $\theta=\pm 35° \, 16'$.

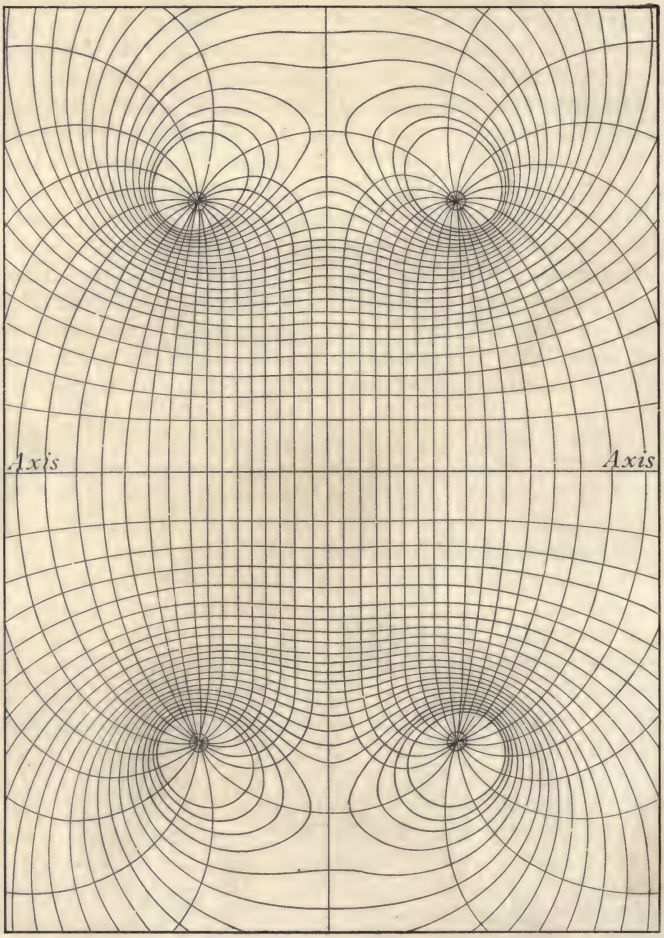

FIG. 59.

9. Effect of finite cross-section of coil. To take into account the distribution of the wire over the finite cross-section of the bobbin, we may take the coil just considered as an elementary ring of the real coil, and, regarding the distance x and radius a of this ring as subject to variation, find, from each term in the expression of any effect produced on the needle by the central ring, the corresponding term of the effect produced by any other parallel ring of the coil. From this we can find an expression for the average value of the term for the whole coil.

Thus let P_0 denote any term of the expression for the action, whatever its nature, on the needle produced by the central circular filament. If then P be the corresponding term for a filament, the coordinates of which reckoned from the centre of the cross-section coil are h, k, and the area of cross-section of which is $dh\,dk$, $\bar{P}$ the average term for the action of the whole coil, and $2b$, $2d$ be the axial breadth and radial depth of the coil, we have by definition

$$4bd\,.\,\bar{P}=\int_{-b}^{b}\int_{-d}^{d}P\,dh\,dk. \qquad\qquad (18)$$

But, since the value of P for this term is obtained by substituting in the expression $x-h$ for x and $a+k$ for a, by Taylor's theorem,

$$P=P_0-h\frac{\partial P_0}{\partial x}+k\frac{\partial P_0}{\partial a}+\frac{h^2}{1\,.\,2}\frac{\partial^2 P_0}{\partial x^2}+\frac{k^2}{1\,.\,2}\frac{\partial^2 P_0}{\partial a^2}+\dots .$$

Multiplying this value of P by $dh\,dk$, and integrating as indicated in (18) between the limits $-b$, $+b$ for h, and $-d$, $+d$ for k, we find

$$4bd\,.\,P=4bd\,.\,P_0+\frac{4b^3d}{6}\frac{\partial^2 P_0}{\partial x^2}+\frac{4bd^3}{6}\frac{\partial^2 P_0}{\partial a^2}$$
$$+\frac{4b^5d}{5!}\frac{\partial^4 P_0}{\partial x^4}+\frac{8b^3d^3}{4!3}\frac{\partial^4 P_0}{\partial x^2 da^2}+\frac{4bd^5}{5!}\frac{\partial^4 P_0}{\partial a^4}, \qquad\qquad (19)$$

since the terms of odd order vanish in the integration.

We apply this result to the correction of the values of F and Θ given in (9) and (13) by treating the terms separately as follows. It will suffice to take $F/2\pi\gamma$, as the results obtained will apply at once to Θ also.

A first approximation to F' for the whole coil is obtained by writing $4abn\gamma$ (or $N\gamma$ if N is the whole number of turns) for γ, since this is the whole current flowing across each section. To correct for the distribution of the turns, we take first the factor a^2/r^3, and call it P_0. Differentiating, we find

$$\partial^2 P_0/\partial x^2=3a^2(4x^2-a^2)/r^7, \qquad \partial^2 P_0/da^2=(2x^4-11x^2a^2+2a^4)/r^7,$$

so that taking the first three terms of (19),

$$P=\frac{a^2}{r^3}+\frac{b^2}{6}\frac{3a^2}{r^7}(4x^2-a^2)+\frac{d^2}{6}\frac{1}{r^7}(2x^4-11x^2a^2+2a^4), \qquad\qquad (20)$$

and this takes the place of a^2/r^3 in (9) and (13).

10. Application of corrections to Helmholtz double coil. If the coil is a Helmholtz arrangement, in which $4x^2=a^2$, the second term disappears, and we have after reduction,

$$\bar{P}=\frac{8}{5\boxed{\sqrt{5a}}}\left(1-\frac{1}{15}\frac{d^2}{a^2}\right),$$

→ $\sqrt{5}\,a$

and the first term of F takes the corrected form

$$\frac{32\pi N\gamma}{5\sqrt{5}a}\left(1-\frac{1}{15}\frac{d^2}{a^2}\right),$$

where N is the number of turns in each of the two coils.

The second term of F may be corrected in the same way by taking $a^2(a^2-4x^2)/r^7$ for P_0. We have $\partial^2 P_0/\partial x^2 = -3\,.\,5a^2(8x^4-12x^2a^2+a^4)/r^{11}$ and $\partial^2 P_0/\partial a^2 = -(8x^6-136x^4a^2+159x^2a^4-12a^6)/r^{11}$, so that to three terms

$$\bar{P}=\frac{a^2}{r^7}(a^2-4x^2)-\frac{b^2}{6}\frac{3\,.\,5}{r^{11}}a^2(8x^4-12x^2a^2+a^4)$$
$$-\frac{d^2}{6}\frac{1}{r^{11}}(8x^6-136x^4a^2+159x^2a^4-12a^6), \text{......(21)}$$

which takes the place of $a^2(a^2-4x^2)/r^7$ wherever the latter occurs.

Again, if the coil is a Helmholtz arrangement, this value of $\bar{P}$ is simplified. Its first term disappears altogether on account of the relation $4x^2=a^2$, which also reduces the remaining two terms so that

$$\bar{P}=\frac{5}{6\,.\,2^3}\frac{a^6}{r^{11}}(36b^2-31d^2), \text{ ....(22)}$$

where $r^2=5/4\,.\,a^2$.

Hence, taking in only second powers of b and d, and the first three terms of (9), we have for the Helmholtz arrangement

$$F=\frac{32\pi N\gamma}{5\sqrt{5}a}\left\{\left(1-\frac{1}{15}\frac{d^2}{a^2}\right)+\frac{2^5}{5^3}\frac{y^2}{a^4}(36b^2-31d^2)-\frac{2\,.\,3^3}{5^3}\frac{y^4}{a^4}\right\}. \text{ ...(23)}$$

The value of $\bar{P}$ in (22), and therefore also the second term of F for any arrangement, can be made to vanish by constructing the coil so that $b^2=31/36\,.\,d^2$. If this is done for a Helmholtz galvanometer, the value is, for that instrument, given to a very high degree of approximation by

$$\Theta=\frac{32\pi N\gamma M\cos\theta}{5\sqrt{5}a}\left\{\left(1-\frac{1}{15}\frac{d^2}{a^2}\right)\right.$$
$$\left.-\frac{2\,.\,3^3}{5^3}\frac{l^4}{a^4}(1-14\sin^2\theta+21\sin^4\theta)\right\}. \text{ ...(24)}$$

If the half-length l of the needle is small in comparison with a, as it ought always to be, the value of Θ for the Helmholtz arrangement may, within the limits of errors of observation, be taken as given by the formula obtained by omitting the term involving l^4/a^4 on the right in (24).

11. Galvanometer with four coaxial coils. If four coaxial coils be arranged so that the current flows through them all in the same direction, the values of F at the same point due to the separate coils will have the same sign. Consider then the component magnetic force at a point O symmetrically situated with reference to the coils, which are arranged in pairs, those of each pair having equal radii, and being at equal distances along the axis on opposite sides of the point at which F is taken. Let a, α, be the radii of the coils, x, ξ, the distances of their planes from O, N, N', the number of turns in each, and $r^2 = x^2 + a^2$, $\rho^2 = \xi^2 + \alpha^2$. Then to three terms,

$$F = 4\pi\gamma\left[N\frac{a^2}{r^3} + N'\frac{\alpha^2}{\rho^3} + \frac{3}{2^2}y^2\left\{\frac{N}{r^7}a^2(a^2 - 4x^2) + \frac{N'}{\rho^7}\alpha^2(\alpha^2 - 4\xi^2)\right\}\right.$$
$$\left. + \frac{3^2 . 5}{2^2 . 4^2}y^4\left\{\frac{N}{r^{11}}a^2(a^4 - 12a^2x^2 + 8x^4) + \frac{N'}{\rho^{11}}\alpha^2(\alpha^4 - 12\alpha^2\xi^2 + 8\xi^4)\right\}\right]. \quad (25)$$

Now we can impose the condition that $r = \rho$, that is, that the coils should lie on a sphere having its centre at O, and so choose a, α, x, ξ, that the coefficients of y^2, y^4, may vanish identically. We thus have fulfilled by these four quantities the equations

$$Na^2(a^2 - 4x^2) + N'\alpha^2(\alpha^2 - 4\xi^2) = 0,$$
$$Na^2(a^4 - 12a^2x^2 + 8x^4) + N'\alpha^2(\alpha^2 - 12\alpha^2\xi^2 + 8\xi^4) = 0.$$

We may write

$$a^2 - 4x^2 = 5a^2 - 4r^2 \quad \text{and} \quad a^4 - 12a^2x^2 + 8x^4 = 21a^4 - 28a^2r^2 + 8r^4,$$

so that calling ϕ, ϕ', the angles which the radii of the coils subtend at O, and putting m for N/N', we may write the equations in the form

$$\left.\begin{aligned} &m\sin^2\phi\,(4 - 5\sin^2\phi) + \sin^2\phi'(4 - 5\sin^2\phi') = 0, \\ &m\sin^2\phi\,(21\sin^4\phi - 28\sin^2\phi + 8) \\ &\qquad + \sin^2\phi'(21\sin^4\phi' - 28\sin^2\phi' + 8) = 0. \end{aligned}\right\} \quad \ldots\ldots\ldots(26)$$

12. Galvanometer with three coaxial coils. Conditions for uniformity of field. Since $\sin\phi$, $\sin\phi'$ can never exceed 1, these equations necessitate the fulfilment of certain conditions by m, $\sin\phi$, $\sin\phi'$, and, subject to these, any number of arrangements can be found to carry out the object stated. If however $\sin\phi = 1$, so that one pair of circles coincide in the equatorial plane through O, we have from (26),

$$21\sin^4\phi' - 33\sin^2\phi' + 12 = 0,$$

which is satisfied by $\sin^2\phi' = 4/7$, or by $\sin^2\phi' = 1$.

The second solution, in which all the coils are round the equator of the sphere, is not relevant, inasmuch as it would make $m = -1$, which may be interpreted to mean that the number of turns on each coil should be the same, and that the currents should flow in opposite directions, that is, that there should be no current at all on the whole, and therefore no magnetic effect.

The solution $\sin^2\phi' = 4/7$ gives $m = 32/49$, that is, the circles surrounding the centre should each contain 32 turns for every 49 turns contained in each of the others, and the latter should be placed on the two sides of the great circle of the sphere bisecting the axis, at a distance in each case of $\sqrt{3/7}$ of the radius, and have a corresponding radius of $2/\sqrt{7}$ of that of the sphere.

13. Long coil of single layer of fine wire. We now consider a long right cylindrical solenoid. Such a solenoid can be very approximately constructed by winding a close single layer of fine wire, so that the mean radius of the single layer may be taken with sufficient accuracy as the radius of the wire, and the wire may be regarded as everywhere at right angles to the axis. Such a single-layer coil is very convenient for accurate work, since there can be no uncertainty as to the winding.

The value of F for a single turn of such a coil is given by (9), which, taking here for convenience the origin of coordinates at the centre of the coil, and x, ξ, as the axial distances of the point P considered, and the turn in question, we may write

$$F = 2\pi\gamma \frac{a^2}{r^3}\left[1 + \frac{3}{2^2}\frac{y^2}{r^4}\{a^2 - (x-\xi)^2\} + \frac{3^2 . 5}{2^2 . 4^2}\frac{y^4}{r^8}\{a^4 - 12a^2(x-\xi)^2 + 8(x-\xi)^4\} + \dots\right], \quad (27)$$

where $r^2 = a^2 + (x-\xi)^2$.

Similarly the value of the radial component, R, may be written down. The value of the couple Θ exerted on the needle could easily be found also; but it will be given later [see equation (54) below].

If then n be the number of turns per unit of length, we have to replace γ by $n\gamma\, d\xi$. Hence if $2l$ be the axial length of the coil, we have for the total force

$$F = 2\pi n\gamma \int_{-l}^{+l} d\xi \frac{a^2}{r^3}\left[1 + \frac{3}{2^2}\frac{y^2}{r^4}\{a^2 - (x-\xi)^2\} + \dots\right]. \quad \text{........(27')}$$

But clearly the expansion in (27) is $-\partial\Omega/\partial x$, if Ω be given by (2) with x replaced by $x - \xi$. But $-\partial\Omega/\partial x$ is $+\partial\Omega/\partial\xi$, so that (2) gives at once the integral $+\Omega$ for (27′). Hence taking the integral between the limits $-l$ and $+l$ for ξ, and writing $r_1 = \sqrt{a^2 + (x-l)^2}$, $r_2 = \sqrt{a^2 + (x+l)^2}$, we find

$$F = 2\pi n\gamma \left[\frac{x+l}{r_1} - \frac{x-l}{r_2} + \frac{y^2}{2^2} 3a^2 \left\{\frac{x+l}{r_1^5} - \frac{x-l}{r_2^5}\right\} + \frac{y^4}{2^2 . 4^2} 3 . 5a \left\{\frac{x+l}{r_1^9}\left(3a^2 - 4(x+l)^2\right) - \frac{x-l}{r_2^9}\left(3a^2 - 4(x-l)^2\right)\right\} + \dots\right], \quad \text{...............(28)}$$

which holds for all points whether inside or outside the solenoid.

If $y = 0$ this gives

$$F = 2\pi n\gamma\left(\frac{x+l}{r_1} - \frac{x-l}{r_2}\right), \quad \text{.......................(28')}$$

that is if the point at which F is taken be on the axis, and ψ_1, ψ_2 be the angles which r_1, r_2 make with the axis, as shown in Fig. 60,

$$F = 2\pi n\gamma(\cos\psi_2 - \cos\psi_1). \quad \text{......................(28'')}$$

[To suit Fig. 60 the symbol ψ is here used instead of a, employed in 1...3, above.]

If the coil be very long r_1, r_2, approximate for an internal point P, not near the ends, more and more nearly to $x+l$, $l-x$, so that all terms vanish in (28) except the first two. For such points $x-l$ is negative, and approximately $(x+l)/r_1 - (x-l)/r_2 = 2$. Thus the field within a long coil is uniform except near the ends, and its intensity is given by

$$F = 4\pi n\gamma. \qquad \text{..............................(29)}$$

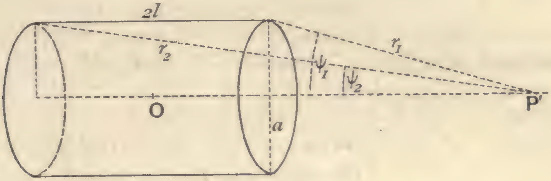

FIG. 60.

To take into account different layers if there are more than one, the best course in any practical case is (since only a limited number of layers would be employed) to calculate F, by (27) above, for each, and add the results together.

14. Direct calculation for potential and force at centre of a long coil. The result expressed in (28″) can of course be obtained at once by direct calculation. The potential due to a circular current of strength $n\gamma\, dx$, at a point P' (Fig. 60) on the axis at numerical distance x from the plane of the circle is $n\gamma\omega\, dx$, where ω is the solid angle subtended at the point by the circle. But if ψ be the angle subtended by the radius of the circle $\omega = 2\pi(1-\cos\psi)$. Thus if Ω be the potential of magnetic induction due to the whole solenoid,

$$\Omega = 2\pi n\gamma \int_{\psi=\psi_1}^{\psi=\psi_2} (1-\cos\psi)\, dx.$$

Ω is also the mutual energy of the solenoid and a unit pole placed at P'. Reckoning then x as the distance of any turn from P', the force in the direction of x on the solenoid (that is from the pole towards the solenoid) is $-\partial\Omega/\partial x$, and this is the force F on the pole at P' in the opposite direction. Thus

$$F = -2\pi n\gamma \int_{\psi=\psi_1}^{\psi=\psi_2} \frac{\partial}{\partial x}(1-\cos\psi)\, dx = 2\pi n\gamma(\cos\psi_2 - \cos\psi_1).$$

Or Ω and F may be found thus. The potential produced by a circular disk of positive magnetism of surface density $n\gamma$ and radius a, at a point on the axis distant x from the disk, is $2\pi n\gamma(\sqrt{a^2+x^2}-x)$. The repulsion due to the disk on unit pole at P' is therefore

$$2\pi n\gamma(1 - x/\sqrt{a^2+x^2}) = 2\pi n\gamma(1-\cos\psi).$$

Hence for two equal positive and negative coaxial disks subtending angles ψ_1, ψ_2 respectively at P', and at distances x_1, x_2, the potential and force are

$$2\pi n\gamma\{\sqrt{a^2+x_2{}^2}-x_2-(\sqrt{a^2+x_1{}^2}-x_1)\} \quad \text{and} \quad 2\pi n\gamma(\cos\psi_2-\cos\psi_1).$$

The magnetic potential and force at a point at distance y from the axis can also be found as follows. It has been shown in II. 22, that the energy of a magnetic shell in a magnetic field is equal to the total induction through the shell multiplied by the strength of the shell. Hence in order to find the force on a pole placed in the field of the solenoid we have to calculate the magnetic induction at the point.

15. A solenoid regarded as a lamellar magnet. We may regard the solenoid as a lamellar distribution of magnetism, the direction of magnetization of which is everywhere parallel to the axis. Hence by (69) and (70) of Chap. II. above, if Ω be the potential of magnetic induction in the interior of the solenoid,

$$\Omega = V - 4\pi\phi,$$

where $\phi(=\int\partial\phi/\partial x\,.\,dx)$ is the sum of the strengths of the shells traversed by a point imagined to move parallel to the axis from an adopted zero, to the point where the potential is to be found. But if we suppose x to increase from the negative towards the positive end of the solenoid, we have $\partial\phi/\partial x = n\gamma$, and hence, reckoning from the zero of x,

$$4\pi\phi = 4\pi n\gamma x.$$

V in the present case is simply the potential due to the ends of the solenoid, which may be regarded as two uniform parallel circular disks of magnetism having densities σ, $-\sigma$, respectively. Hence if V_1 be the potential due to the positive disk, V_2 that due to the negative, $V = V_1 - V_2$, and

$$\Omega = V_1 - V_2 - 4\pi n\gamma x. \quad \text{........ ....(30)}$$

For an external point $\Omega = V_1 - V_2$ simply. The values of V_1, V_2 can be found from (6) above and the value of F then found by differentiation of (30). The result, as the reader may verify, agrees with (28).

16. Long coil of several layers. The method described above (p. 219) may also, if desired, be employed to take into account the radial depth of the coil. Supposing the number of layers per unit of depth to be n', the number in unit area of cross-section is nn'. Thus, if $2d$ be the depth of the coil, the number of turns in unit of length is $2nn'd$, and this must replace n in (27′). Taking then as P_0 any term of the expression for the effect of the mean coaxial current sheet, the average value, $\bar{P}$, of the term, for all the coaxial sheets into which the coil may be supposed divided, is given by the equation

$$\bar{P} = P_0 + \frac{d^2}{6}\frac{\partial^2 P_0}{\partial a^2} + \frac{d^4}{5!}\frac{\partial^4 P_0}{\partial a^4}.$$

Taking first from (28),

$$P_0 = \frac{x+l}{r_1} - \frac{x-l}{r_2},$$

we have

$$\frac{\partial^2 P_0}{\partial a^2} = (x+l)\,\frac{2a^2-(x+l)^2}{r_1{}^5} - (x-l)\,\frac{2a^2-(x-l)^2}{r_2{}^5}.$$

Therefore to the second power of d,

$$P = \frac{x+l}{r_1} - \frac{x-l}{r_2} + \frac{d^2}{6}\left((x+l)\,\frac{2a^2-(x+l)^2}{r_1{}^5} - (x-l)\,\frac{2a^2-(x-l)^2}{r_2{}^5}\right).$$

Next taking the second term of (28),

$$P_0 = \frac{a^2(x+l)}{r_1{}^5} - \frac{a^2(x-l)}{r_2{}^5},$$

and therefore, as far as terms involving $1/r^9$,

$$P = \frac{a^2(x+l)}{r_1{}^5} - \frac{a^2(x-l)}{r_2{}^5} + \frac{d^2}{6}\left\{\frac{x+l}{r_1{}^9}\{12a^4 - 21a^2(x+l)^2 + 2(x+l)^4\}\right.$$
$$\left. - \frac{x-l}{r_2{}^9}\{12a^4 - 21a^2(x-l)^2 + 2(x-l)^4\}\right\}.$$

Hence to terms in d^2 and y^2, (27) becomes (x_1 being put for $2nn'd$, the number of terms per unit length)

$$F = 2\pi n_1\gamma\left[\frac{x+l}{r_1} - \frac{x-l}{r_2} + \frac{d^2}{6}\left((x+l)\,\frac{2a^2-(x+l)^2}{r_1{}^5}\right.\right.$$
$$\left. - (x-l)\,\frac{2a^2-(x-l)^2}{r_2{}^5}\right)$$
$$+ \frac{3}{2^2}\,y^2\left\{\frac{a^2(x+l)}{r_1{}^5} - \frac{a^2(x-l)}{r_2{}^5} + \frac{d^2}{6}\,\frac{x+l}{r_1{}^9}\{12a^4 - 21a^2(x+l)^2 + 2(x+l)^4\}\right.$$
$$\left.\left. - \frac{d^2}{6}\,\frac{x-l}{r_2{}^9}\{12a^4 - 21a^2(x-l)^2 + 2(x-l)^4\}\right\}\right]. \quad \ldots(31)$$

17. Potential, etc., of circular current. Expansions available for near or distant points. Equations equivalent to (2), (9), (10), (12) may be obtained by first expanding A_0 in ascending powers of x or a according as $x <$ or $> a$. These equations are convenient only when the point considered is near to or far from the plane of the circular current, as only then are the series sufficiently convergent. We have

$$\left.\begin{aligned} A_0 &= \sqrt{a^2+x^2} - x \\ &= a\left\{1 + \frac{1}{2}\frac{x^2}{a^2} - \frac{1.1}{2.4}\frac{x^4}{a^4} + \frac{1.1.3}{2.4.6}\frac{x^6}{a^6} - \ldots\right\} - x \quad (x<a) \\ &= x\left\{\frac{1}{2}\frac{a^2}{x^2} - \frac{1.1}{2.4}\frac{a^4}{x^4} + \frac{1.1.3}{2.4.6}\frac{a^6}{x^6} - \ldots\right\} \quad (x>a). \end{aligned}\right\} \quad \ldots(32)$$

Calculating $\partial A_0/\partial x$, $\partial^3 A_0/\partial x^3$, etc., from these and substituting in (6), we find if $x < a$,

$$\begin{aligned}\Omega = 2\pi\gamma\{1 - \frac{x}{a} + \frac{1}{2}\frac{x^3}{a^3} - \frac{1.3}{2.4}\frac{x^5}{a^5} + \dots \\ + \frac{1}{2^2}\frac{y^2}{a^3}\left(-1.3x + \frac{1.3.5}{2}\frac{x^3}{a^2} - \frac{1.3.5.7}{2.4}\frac{x^5}{a^4} + \dots\right) \\ + \frac{1}{2^2.4^2}\frac{y^4}{a^5}\left(-1.3^2.5x + \frac{1.3.5^2.7}{2}\frac{x^3}{a^2} - \frac{1.3.5.7^2.9}{2.4}\frac{x^5}{a^4} + \dots\right) \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \}.\end{aligned} \tag{33}$$

Or if $x > a$,

$$\left.\begin{aligned}\Omega = 2\pi\gamma\Big\{\frac{1}{2}\frac{a^2}{x^2} - \frac{1.3}{2.4}\frac{a^4}{x^4} + \frac{1.3.5}{2.4.6}\frac{a^6}{x^6} - \dots \\ + \frac{1}{2^2}\frac{y^2}{x^2}\left(-1.3\frac{a^2}{x^2} + \frac{1.3.5}{2}\frac{a^4}{x^4} - \frac{1.3.5.7}{2.4}\frac{a^6}{x^6} - \dots\right) \\ + \frac{1}{2^2.4^2}\frac{y^4}{x^4}\left(3.5.4\frac{a^2}{x^2} - \frac{1.3.5.7}{2}6\frac{a^4}{x^4} + \frac{1.3.5.7.9}{2.4}8\frac{a^6}{x^6} - \dots\right) \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \tag{33'}$$

Hence, since $F = -\partial\Omega/\partial x$,

$$\left.\begin{aligned}F = 2\pi\gamma\frac{1}{a}\Big\{1 - \frac{1.3}{2}\frac{x^2}{a^2} + \frac{1.3.5}{2.4}\frac{x^4}{a^4} - \dots \\ + \frac{1}{2^2}\frac{y^2}{a^2}\left(1.3 - \frac{1.3^2.5}{2}\frac{x^2}{a^2} + \frac{1.3.5^2.7}{2.4}\frac{x^4}{a^4} - \dots\right) \\ + \frac{1}{2^2.4^2}\frac{y^4}{a^4}\left(1.3^2.5 - \frac{1.3^2.5^2.7}{2}\frac{x^2}{a^2} + \frac{1.3.5^2.7^2.9}{2.4}\frac{x^4}{a^4} - \dots\right) \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \tag{34}$$

if $x < a$; or if $x > a$,

$$\left.\begin{aligned}F = 2\pi\gamma\frac{a^2}{x^3}\Big\{1 - \frac{1.3}{2}\frac{a^2}{x^2} + \frac{1.3.5}{2.4}\frac{a^4}{x^4} - \dots \\ + \frac{1}{2^2}\frac{y^2}{x^2}\left(-3.4 + \frac{1.3.5}{2}6\frac{a^2}{x^2} - \frac{1.3.5.7}{2.4}8\frac{a^4}{x^4} + \dots\right) \\ + \frac{1}{2^2.4^2}\frac{y^4}{x^4}\left(3.5.4.6 - \frac{1.3.5.7}{2}6.8\frac{a^2}{x^2} + \frac{1.3.5.7.9}{2.4}8.10\frac{a^4}{x^4} - \dots\right) \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \tag{34'}$$

18. Expansions for couple on small magnet with centre on axis. If, instead of a single turn of wire, the circle consist of N turns, each carrying a current λ, the above expressions must of course be multiplied by N.

Finally, multiplying these values of F by $M \cos\theta$, the second term in each by $(1 - 5\sin^2\theta)$, the third terms by $1 - 14\sin^2\theta + 21\sin^4\theta$, and changing y into $l\cos\theta$, we get the values of Θ for the respective cases $x < a$, $x > a$. [See 5 (13) above.]

It is to be carefully observed that in all these expressions it is necessary for convergence that $y < a$ when $x < a$, and $y < x$ when $a < x$.

It will be seen that the formulae just obtained are simply those previously found for the different cases, with the finite expressions which constitute the different terms in the latter replaced by infinite series. In the majority of practical cases it is much more convenient to calculate numerically the values of the finite expressions. The series are in fact only useful for points very near the plane of the circle, or very far from it. In the former case equations (33), (34) are applicable, in the latter (33′), (34′).

When the coil has a finite cross-section the last found expressions may be readily corrected by direct integration; or the process explained in 9 above may be used. We cannot here afford space for the corrected expressions, which would seldom be needed; but the reader will have no difficulty in writing them down for himself.

19. Mutual action of two circular conductors. It has been shown (in 21 above) that the mutual potential energy of two circular magnetic shells is given by the equation

$$\mathsf{E} = -4\pi^2 \Phi\Phi \sin^2\psi \sin^2\psi' \rho \sum \frac{1}{i(i+1)} \cdot {}_{\psi}Z_i' \cdot {}_{\psi'}Z_i' \cdot {}_{\phi}Z_i \left(\frac{\rho}{r}\right)^i \quad (r > \rho) \qquad (35)$$

if ψ, ψ' (Fig. 61) denote the angles which the radii of the shells subtend at the intersection of their axes, r, ρ the distances of the circular arcs from the origin, ϕ the angle between the axes of the shells (denoted by θ in Fig. 53 above), and ${}_{\phi}Z_i$ the zonal surface harmonic of the i^{th} order taken for the angle ϕ, and similarly for the others as explained at p. 201. This value of E with its sign changed, and γ, γ' written for Φ, Φ', is the mutual electrokinetic energy T of two circular currents, and is at once available for the calculation of their mutual action. The result enables the mutual action of two coils to be found, and is therefore the foundation of the theory of absolute electrodynamometers and current balances, which measure currents in absolute units by the forces exerted on a movable coil by a fixed coil, through both of which the current to be measured is flowing, or in which the currents flowing have a certain known ratio.

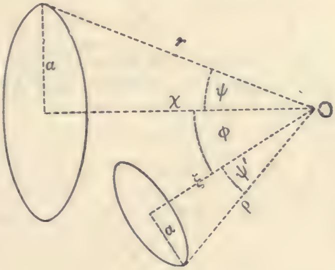

FIG. 61.

20. Electrokinetic energy of two circular currents. Extension to two coils of finite cross-section. Putting then (Fig. 61) a, α, for the radii of the larger and smaller circles respectively, and x, ξ, for the distances of their planes from the origin, we have $\sin\psi = a/r$ and $\sin\psi' = \alpha/\rho$, and substituting in the zonal harmonic expressions, as given in the Appendix on Spherical Harmonics, their values in terms of a, x, α, ξ, we have

$$
\begin{aligned}
T = \pi^2\gamma\gamma'\frac{a^2\alpha^2}{r^3}\{&1\,.\,2\cos\phi + 2\,.\,3\frac{x}{r^2}\xi(\cos^2\phi - \tfrac{1}{2}\sin^2\phi)\\
&+3\,.\,4\frac{x^2-\frac{1}{4}a^2}{r^4}(\xi^2-\tfrac{1}{4}\alpha^2)(\cos^3\phi - \tfrac{3}{2}\sin^2\phi\cos\phi)\\
&+4\,.\,5\frac{x(x^2-\frac{3}{4}a^2)}{r^6}\xi(\xi^2-\tfrac{3}{4}\alpha^2)(\cos^4\phi - 3\cos^2\phi\sin^2\phi + \tfrac{3}{8}\sin^4\phi)\\
&+5\,.\,6\frac{x^4-\frac{3}{2}x^2a^2+\frac{1}{8}a^4}{r^8}(\xi^4-\tfrac{3}{2}\xi^2\alpha^2+\alpha^4)(\cos^5\phi - 5\cos^3\phi\sin^2\phi\\
&\qquad + \frac{3\,.\,5}{8}\sin^4\phi\cos\phi) + \ldots\}. \qquad (36)
\end{aligned}
$$

As explained in 21 above, the couple Θ due to the mutual action of the two circuits tending to increase ϕ is $\partial T/\partial\phi$. Hence for this couple we have

$$
\begin{aligned}
\Theta = -\pi^2\gamma\gamma'\sin\phi\{&1\,.\,2\frac{a^2}{r^3}\alpha^2 + 2\,.\,3\frac{x}{r^2}\xi\,.\,3\cos\phi\\
&+3\,.\,4\frac{x^2-\frac{1}{4}a^2}{r^4}(\xi-\tfrac{1}{4}a^2)\,.\,2\,.\,3(\cos^2\phi - \tfrac{1}{4}\sin^2\phi)\\
&+4\,.\,5\frac{x(x^2-\frac{3}{4}a^2)}{r^4}\xi(\xi^2-\tfrac{3}{4}\alpha^2)\,.\,2\,.\,5\cos\phi(\cos^2\phi - \tfrac{3}{4}\sin^2\phi)\\
&+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \}. \qquad (37)
\end{aligned}
$$

The attraction between the circuits when they are coaxial, that is when $\phi = 0$, may be found by putting $\phi = 0$ in (35), and calculating $\partial T/\partial\xi$. We have

$$
\begin{aligned}
T = \pi^2\gamma\gamma'\frac{a^2\alpha^2}{r^3}\{&1\,.\,2 + 2\,.\,3\frac{x}{r^2}\xi + 3\,.\,4\frac{x^2-\frac{1}{4}a^2}{r^4}(\xi^2-\tfrac{1}{4}\alpha^2)\\
&+4\,.\,5\,.\,\frac{x(x^2-\frac{3}{4}a^2)}{r_6}\xi(\xi^2-\tfrac{3}{4}\alpha^2)+\ldots\}. \qquad (36')
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial T}{\partial\xi} = \pi^2\gamma\gamma'\frac{a^2\alpha^2}{r^4}\{&1\,.\,2\,.\,3\frac{x}{r} + 2\,.\,3\,.\,4\frac{x^2-\frac{1}{4}a^2}{r^2}\xi\\
&+3\,.\,4\,.\,5\frac{x(x^2-\frac{3}{4}a^2)}{r^6}(\xi^2-\tfrac{1}{4}\alpha^2)+\ldots\}. \qquad (38)
\end{aligned}
$$

We may now proceed from two simple circles to two mutually influencing cylindrical coils of finite axial breadth and radial depth of cross-section. This may be done either by the method, of correction of the successive terms, exemplified above for a coil and a magnet or by direct integration with respect to x, a, and x', a' in the two cases. Proceeding by the former method, and dealing with the terms of (36) separately, putting for the axial breadth and radial depth $2b$, $2d$ in the case of the larger coil, and $2b'$, $2d'$ in the case of the smaller (both being supposed of rectangular cross-section), while x; a, x', a' are retained for the mean filaments in the two cases, and putting N, N' for the *total* numbers of turns in the two coils, larger and smaller respectively,

$$T = NN'\gamma\gamma'\{G_1 g_1 \cdot {}_\theta Z_1 + G_2 g_2 \cdot {}_\theta Z_2 + G_3 g_3 \cdot {}_\theta Z_3 + \dots\}, \quad \dots\dots\dots(39)$$

where

$$G_1 = 2\pi\frac{a^2}{r^3}\{1 + \frac{3b^2}{6r^4}(4x^2 - a^2) + \frac{d^2}{6a^2r^4}(2x^4 - 11x^2a^2 + 2a^4) + \dots\},$$

$$G_2 = 3\pi\frac{a^2x}{r^5} + \frac{5b^2}{6r^4}(4x^2 - 3a^2) + \frac{5d^2}{6a^2r^4}(2x^4 - 21x^2a^2 + 12a^4 + \dots),$$

$$G_3 = \frac{\pi}{r^7}\{a^2(4x^2 - a^2) + \frac{3 \cdot 5a^2b^2}{6r^4}(8x^4 - 12x^2a^2 + a^4)$$

$$+ \frac{d^2}{6r^4}(8x^6 - 136x^4a^2 + 159x^2a^4 - 12a^6) + \dots\}$$

. . . .

$$g_1 = \pi(a'^2 + \tfrac{1}{3}b'^2 + \dots),$$

$$g_2 = 2\pi x'(a'^2 + \tfrac{1}{3}b'^2 + \dots),$$

$$g_3 = \pi\{\tfrac{1}{4}3a'^2(4x'^2 - a'^2) + \tfrac{1}{2}d'^2(2x'^2 - 3a'^2) + b'^2a'^2 + \dots\}$$

. . . .

21. Couple on suspended double coil of electrodynamometer. Hence we obtain from (39),

$$\Theta = -NN'\gamma\gamma'\sin\theta\{G_1 g_1 \cdot {}_\theta Z_1' + G_2 g_2 \cdot {}_\theta Z_2' + G_3 g_3 \cdot {}_\theta Z_3' + \dots\}, \quad \dots(40)$$

which is the corrected form of (37). Similarly we could write down from (38) the corrected value of the attraction between the two coils.

22. Turning couple of coil on magnetic needle galvanometers. Equation (37) is applicable to the determination of the couple due to the action of a coil on a uniformly magnetized thin magnet the centre of which is at the origin. We have only to suppose another coil equal in all respects to the smaller placed coaxial with the latter on the other side of the origin at a mean distance x' from that point, and further suppose a current of the same strength to flow in like directions round both. The couple acting on the second coil will be got from that on the first by merely supposing the angle θ to be increased by 180°, and the current in the coil to be reversed. But changing θ into $\theta + 180°$ changes the signs of ${}_\theta Z_2'$, ${}_\theta Z_3'$, etc., and taking into account the change of sign of the current and of $\sin\theta$, we have for the couple, Θ' say, on the second coil,

$$\Theta' = -NN'\gamma\gamma'\sin\theta\{G_1 g_1 \cdot {}_\theta Z_1' - G_2 g_2 \cdot {}_\theta Z_2' + G_3 g_3 \cdot {}_\theta Z_3' - \text{etc.}\}.$$

Hence for the total couple we get

$$\Theta' + \Theta' = -2NN'\gamma\gamma' \sin\phi\{G_1 g_1 \,.\, {}_\theta Z_1 + G_3 g_3 \,.\, {}_\theta Z_3' + \text{etc.}\}. \quad \ldots\ldots(41)$$

But the double coil here supposed to exist is equivalent to a needle with its centre at the origin, and of moment $M = 2\pi a^2 N'\gamma'$. Also if we make the section of each coil very small, and the radius a' very small, but preserve $2\pi a'^2 N'\gamma'$ a finite quantity, we may regard the pair of coils as equivalent to a uniformly magnetized magnet of moment $2\pi a'^2 N'\gamma'$, and of length $2x'$, and put, in the values of g_1, g_2, etc., $b' = 0$, $d' = 0$, $Ma'^2 = 0$, etc. In this way we shall obtain from (41) a formula equivalent to that given in VI. (78) when the latter is corrected for the finite cross-section of the large coil. [N' is the total number of turns in the coil.]

23. Electrodynamometer with double-coil arrangement. If instead of two single coils, one fixed and the other movable, the Helmholtz double arrangement is adopted for both the fixed and movable parts of the dynamometer, so that the centres of both are made coincident with the origin,* the expressions for their mutual action are much simplified.

Let A, B (Fig. 62) denote the large coils, A', B' the small coils. Then the mutual energy of A and A', and the couple on A' due to the action of A, are equal in numerical amount and similar in sign to those of B and B'. These are given by (39) and (40). Hence for these two pairs of coils the energy is

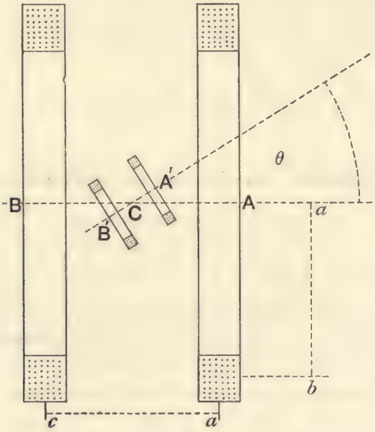

FIG. 62.

$$2T = 2NN'\gamma\gamma'\{G_1 g_1 \,.\, {}_\theta Z_1 + G_2 g_2 \,.\, {}_\theta Z_2 + G_3 g_3 \,.\, {}_\theta Z_3 + \ldots\}, \quad \ldots\ldots\ldots(42)$$

---

* This was the arrangement adopted for the Absolute Electrodynamometer made by Mr. Latimer Clark for the British Association Committee on Electrical Standards. See Chapter XII. below.

where θ is the angle ACA' indicated in Fig. 62, and $a^2 = 4x^2$, $a'^2 = 4x'^2$. Now the mutual energy of the coils B', A, is that which the value of T would become for A' and A if θ were increased by 180° and the current in A' were then reversed. The mutual electrokinetic energy of B and A' has evidently the same value. But $\cos(\theta + 180°) = -\cos\theta$, so that the zonal harmonics of odd order change sign. Hence taking into account the change of sign of current, we have for the electrokinetic energy of the other two pairs of coils A, B' and A', B the value

$$2T_1 = 2NN'\gamma\gamma'\{G_1 g_1 \,.\, {}_\theta Z_1 - G_2 g_2 \,.\, {}_\theta Z_2 + G_3 g_3 \,.\, {}_\theta Z_3 - \text{etc.}\}, \quad \ldots\ldots(43)$$

where all the quantities have the same values as before. Hence for the total energy of the arrangement, we have

$$2(T + T_1) = 4NN'\gamma\gamma'\{G_1 g_1 \,.\, {}_\theta Z_1 + G_3 g_3 \,.\, {}_\theta Z_3 + G_5 g_5 \,.\, {}_\theta Z_5 + \text{etc.}\}, \quad \ldots(44)$$

and the turning couple on the pair of small coils is

$$\Theta = -4NN'\gamma\gamma' \sin\phi\{G_1 g_1 \,.\, {}_\theta Z_1' + G_3 g_3 \,.\, {}_\theta Z_3' + G_5 g_5 \,.\, {}_\theta Z_5'\}, \quad \ldots\ldots(45)$$

The values of G_1, G_3, G_5, ... , g_1, g_2, g_3, ... are given in 20 above, and it is to be noticed that in these $4x - a^2 = 0$, $4x'^2 - a'^2 = 0$, for the double-coil galvanometer constructed according to Helmholtz's specification, so that, to a considerable degree of approximation, G_3 and g_3 vanish, and the couple reduces to

$$\Theta = -4NN'\gamma\gamma' G_1 g_1 \sin\theta \,.\, {}_\theta Z_1', \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46)$$

or, neglecting the correction terms in b^2, d^2, etc., to

$$\Theta = -64NN'\gamma\gamma' \frac{a'^2}{5\sqrt{5}a} \sin\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46')$$

Considering the movable-coil system as equivalent to a needle of moment $2N'\gamma'\pi a'^2$, we see that this agrees with (24) above.

In the same manner as in 22 we could deduce the action of a Helmholtz double coil on a magnetic needle, with its centre at the centre of symmetry, from the theory of the double electrodynamometer just given.

CHAPTER VIII.

DYNAMICAL THEORY OF MUTUALLY INFLUENCING CIRCUITS.

Measurements in Alternating-Current Circuits.

1. Electrokinetic energy of a system of circuits. The facts and theories are dealt with at some length in the author's *Treatise on Magnetism and Electricity*, vol. i. chap. ix. It is there shown that the electrokinetic energy of any system of circuits carrying currents can be written in the form

$$T=\tfrac{1}{2}L_1\gamma_1^2+M_{12}\gamma_1\gamma_2+M_{13}\gamma_1\gamma_3+\dots+\tfrac{1}{2}L_2\gamma_2^2+M_{23}\gamma_2\gamma_3 \\ +\dots+\tfrac{1}{2}L_n\gamma_n^2, \quad\dots\dots\dots\dots\dots\dots(1)$$

where, if ds_j, ds_j' denote two different elements of the circuit, which is distinguished by the suffix j, ds_i, ds_j elements of the different circuits marked by i and j, ϵ the angle, and r the distance between the elements in each case, L_j, M_{ij}, have the values indicated by the equations

$$L_j=\mu\iint\frac{\cos\epsilon}{r}ds_jds_j',\quad M_{ij}=\mu\iint\frac{\cos\epsilon}{r}ds_ids_j. \quad\dots\dots\dots\dots(2)$$

The integrals, which are typical, are taken round the circuits, the first by keeping first ds_j fixed and integrating for every ds_j', then integrating the result for every ds_j. The second integral deals with every element of the integrand which can be made up of an element ds_i in one circuit, and an element ds_j in the other. The quantities L_j are the self-inductances, and the quantities M_{ij} the mutual inductances. L_{ij} is the total magnetic induction through circuit j produced by each unit of its own current, while M_{ij} is the total magnetic induction through the circuit j due to unit current in the circuit i, or, which is the same thing, the induction through the circuit i, due to unit current in the circuit j.

2. Components of electrokinetic momentum. Differentiation of T, as expressed in (1), gives the total inductions through the different circuits. Calling these inductions N_1, N_2, ... , N_k, ... , we obtain

$$\left.\begin{aligned} N_1 &= \frac{\partial T}{\partial \gamma_1} = L_1\gamma_1 + M_{12}\gamma_2 + \ldots + M_{1k}\gamma_k + \ldots , \\ N_2 &= \frac{\partial T}{\partial \gamma_2} = M_{21}\gamma_1 + L_2\gamma_2 + \ldots + M_{2k}\gamma_k + \ldots , \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ N_k &= \frac{\partial T}{\partial \gamma_k} = M_{k1}\gamma_1 + M_{k2}\gamma_2 + \ldots + L_k\gamma_k + \ldots , \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad , \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

with $M_{ik} = M_{ki}$. The quantities N_1, N_2, ... correspond to the generalized momenta in the dynamical analogue, and we may call them the *electrokinetic momenta*.

3. Dissipation function and electrostatic energy. We notice next that the rate of dissipation, $2F$ say, of energy in heat in conductors of resistances R_1, R_2, ... , R_k, in which flow currents γ_1, γ_2, ... , γ_k, is given by

$$F = \tfrac{1}{2}(R_1\gamma_1^2 + R_2\gamma_2^2 + \ldots + R_k\gamma_k^2). \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

If now we denote the total electric energy by E, not including in that electrokinetic energy, T, as expressed in (1), but energy of charged conductors, as potential energy (electrostatic energy), and regard, as we do when we give the name *electrokinetic energy* to the quantity T in (1), the currents γ_1, γ_2, ... , γ_k, ... , as speeds $\dot{y}_1$, $\dot{y}_2$, ... , $\dot{y}_k$, ... of coordinates y_1, y_2, ... , y_k, ... , the equation for the k^{th} circuit are

$$\frac{dN_k}{dt} + \frac{\partial \mathsf{E}}{\partial y_k} + \frac{\partial F}{\partial \gamma_k} = E_k, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where E_k is the proper impressed electromotive force. We may write (5) also in the form

$$\frac{dN_k}{dt} + \frac{\partial \mathsf{E}}{\partial y_k} = E_k - R_k y_k. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

It follows from electrostatic theory that E is a homogeneous quadratic function of y_1, y_2, ... , y_k, ... , the charges of the conductors, quantities of electricity which are usually denoted by Q_1, Q_2, ... , Q_k, ... We have in fact (with the condition $p_{kh} = p_{hk}$),

$$\mathsf{E} = \tfrac{1}{2}\{p_{11}Q_1^2 + 2p_{12}Q_1Q_2 + \ldots + p_{22}Q_2^2 + 2p_{23}Q_2Q_3 + \ldots\}. \qquad \ldots\ldots\ldots(7)$$

If we write

$$V_k = \frac{\partial \mathsf{E}}{\partial Q_k} = p_{k1}Q_1 + p_{k2}Q_2 + \ldots + p_{kk}Q_k + \ldots + p_{kn}Q_n, \qquad \ldots\ldots\ldots\ldots(8)$$

we can express E as a homogeneous quadratic function of V_1, V_2, ... , V_k, ... , V_n. For of course from the equations of the form (8) we can derive expressions for the charges Q_1, Q_2, ... , in terms of the values of the V's, the potentials of the conductors. These are equations of the form

$$Q_k = c_{k1}V_1 + c_{k2}V_2 + \ldots + c_{kh}V_h + \ldots + c_{kn}V_n. \qquad \ldots\ldots\ldots\ldots\ldots(9)$$

4. Coefficients of potential, coefficients of induction, and capacities. The quantities p_{11}, p_{12}, ... are called *coefficients of potential*, the quantities c_{11}, c_{12}, ... , *coefficients of induction*. There are, if n be the number of conductors, n coefficients of the form $p_{11}, p_{22}, \ldots, p_{kk}, \ldots$, and the same number of the form $c_{11}, c_{22}, \ldots , c_{kk}, \ldots$, in each of which both the suffixes are the same. A coefficient p_{hk} is the potential of the conductor marked by the suffix h produced by unit charge on the conductor k, when all other conductors except the conductor h are without charge, and $p_{hk}=p_{hk}$. Thus p_{hh} is the potential of the conductor h produced by unit charge on the conductor itself.

Again c_{hk} is the charge on the conductor h produced (or as it is called "induced") when the conductor k is at unit potential while all the other conductors are at potential zero, and $c_{hk}=c_{kh}$. A coefficient of the form c_{hh} is the charge on the conductor h, when it is at potential unity, and all other conductors are at potential zero. Hence c_{hh} has been called the *electrostatic capacity* of the conductor.

It can be shown that

$$p_{hh} > p_{kh}(=p_{hk}) > 0, \quad p_{kk} > p_{hk} > 0.$$

The capacities are all positive, and the coefficients of induction c_{hh} all negative. Also

$$c_{hh} \geqq -\Sigma c_{kh}.^*$$

5. Dynamical theory of mutually influencing circuits. We now take some particular cases in which the dynamical equations find application. First we consider two circuits in which the currents are γ_1, γ_2, the inductances L_1, L_2, M, the electromotive forces E_1, E_2, and the resistances R_1, R_2. For the present we suppose E to be zero. The equations of currents are at once derived from the energy equation

$$T=\tfrac{1}{2}(L_1\gamma_1^2+2M\gamma_1\gamma_2+L_2\gamma_2^2)+\tfrac{1}{2}\Sigma(m\dot{x}^2), \quad \ldots\ldots\ldots\ldots(10)$$

where x denotes a typical coordinate fixing the configuration of the system. If no change of configuration is proceeding the term $\tfrac{1}{2}\Sigma(m\dot{x}^2)$ is zero.

From this we obtain for the equations of currents

$$\frac{d}{dt}(L_1\gamma_1+M\gamma_2)+R_1\gamma_1=E_1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

$$\frac{d}{dt}(L_2\gamma_2+M\gamma_1)+R_2\gamma_2=E_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Here the components of electrokinetic momentum are

$$N_1=L_1\gamma_1+M\gamma_2, \quad N_2=L_2\gamma_2+M\gamma_1. \quad \ldots\ldots\ldots\ldots(13)$$

In the general case we have also equations of the form

$$m\ddot{x}-\frac{1}{2}\left(\gamma_1^2\frac{\partial L_1}{\partial x}+2\gamma_1\gamma_2\frac{\partial M}{\partial x}+\gamma_2^2\frac{\partial L_2}{\partial x}\right)=F_x, \quad \ldots\ldots\ldots\ldots(14)$$

---

* See Gray, *Magnetism and Electricity*, pp. 126, 127.

where m is a mass coefficient corresponding to the coordinate x, and F_x is the external force. The internal electromagnetic force is

$$\frac{1}{2}\left(\gamma_1{}^2\frac{\partial L_1}{\partial x}+2\gamma_1\gamma_2\frac{\partial M}{\partial x}+\gamma_2{}^2\frac{\partial L_2}{\partial x}\right),$$

that is, this expression is due to internal action, and measures the rate of change of momentum which the action of the circuits on one another brings about. To balance this a force of equal amount and of opposite sign would have to be applied, and this is sometimes referred to as *the* "applied electromagnetic force." If such a force were really applied of course both forces would go out of the account, so that the phrase "applied force" in this particular connection is somewhat misleading.

6. Rails and slider magneto. We may take here, as an illustration, the case treated in I. 50 of the sliding conductor connecting two equidistant rails laid in a magnetic field of uniform intensity H, so that the bar moving with speed v in a direction at right angles to its length cuts perpendicularly across the lines of force of the impressed field. The rails are connected by a wire, so that the total resistance of the circuit is R, and for simplicity the rails and sliding bar are regarded as being of negligible electrical resistance, so that R may be taken as practically all contained in the wire.

The energy changes are discussed in the section cited, and it is found that half the amount of electrical work done per unit of time, over and above that dissipated in heat in the resistance R, goes to increase the molar kinetic energy of the sliding bar, and the other half to augment the electrokinetic energy of the system. Other examples of this mode of distribution of the electrical work done present themselves in electromagnetic theory.

A difficulty arises here, and in some other cases in which the magnetic field is due to permanent magnets, as to the mutual electrokinetic energy of the circuit considered and the magnets. If the magnets, as seems required by the unity of electrical action, be due to currents in circuits of a molecular character, there ought to be such mutual energy, depending on the position of the circuit in the field. This subject does not concern us here; an attempt is made to discuss it in a paper by the author [*Phil. Mag.* March, 1914].

7. General theorem regarding mutually influencing circuits. Now returning to the case of two mutually influencing circuits, and the equations of currents (11), let the circuits be rigid in form, so that L_1, L_2 do not change, while M changes in consequence of displacement produced by the mutual action of the circuits. Let dT be the change of T which takes place in a small interval of time dt; then

$$dT=L_1\gamma_1 d\gamma_1+L_2\gamma_2 d\gamma_2+M(\gamma_2 d\gamma_1+\gamma_1 d\gamma_2)+\gamma_1\gamma_2 dM. \quad \ldots(15)$$

As it is supposed that L_1, L_2 do not vary, the only function of the coordinates which expresses the configuration of the system is M. Thus

$$\frac{dT}{dx}dx = \gamma_1\gamma_2 dM = dW. \quad \text{.......................} (16)$$

The quantity on the left is the work done by mutual electromagnetic forces. It is spent in producing molar kinetic energy in the conductors, or in moving the circuits against external forces, or in both ways.

The work done by the electromotive forces, over and above that dissipated, is

$$(E_1 - R_1\gamma_1)\gamma_1 dt + (E_2 - R_2\gamma_2)\gamma_2 dt,$$

and this by (11) and (12) has the value

$$L_1\gamma_1 d\gamma_1 + L_2\gamma_2 d\gamma_2 + M(\gamma_1 d\gamma_2 + \gamma_2 d\gamma_1) + 2\gamma_1\gamma_2 dM = dT + dW. \; (17)$$

This accounts for dT and the work $\gamma_1\gamma_2 dM$ required for the displacement dM. It is remarkable that the electromotive forces furnish, in consequence of configurational change, an amount of work $2\gamma_1\gamma_2 dM$, half of which goes to increase the electrokinetic energy, and the other half to do work against external forces, either applied by external bodies, or arising from the inertia of the circuits.

When the circuits move from rest to rest again, then both before and after the displacement, $E_1 = R_1\gamma_1$, $E_2 = R_2\gamma_2$, and so

$$dT + dW = 2\gamma_1\gamma_2 dM = 2dW. \quad \text{.......................} (18)$$

Thus the batteries furnish energy $2\gamma_1\gamma_2 dM$, of which one half is accounted for in dT, and the other in dW.

If the work done by electromagnetic forces is spent against friction, it appears from the result just obtained that the batteries furnish the energy dissipated, and exactly just as much more to increase the electrokinetic energy.

A more general theorem can be established in which $2\Sigma(\gamma_j\gamma_k dM_{jk})$ replaces $2\gamma_1\gamma_2 dM$.

8. Two circuits, a primary and a secondary. Theorem: $L_1L_2 > M^2$. We now consider two mutually influencing circuits invariable in form and position, and shall suppose that E_1 and E_2 are constants. The equations can be written, by separating the symbols and grouping into one operator all that act on one quantity, in the form

$$\left(L_1\frac{d}{dt} + R_1\right)\gamma_1 + \qquad M\frac{d}{dt}\gamma_2 - E_1 = 0, \quad \text{............} (19)$$

$$M\frac{d}{dt}\gamma_1 + \left(L_2\frac{d}{dt} + R_2\right)\gamma_2 - E_2 = 0. \quad \text{............} (20)$$

Hence we operate on the first of these by $L_2 d/dt + R_2$, and on the second by Md/dt, and subtract. The result is

$$(L_1L_2 - M^2)\frac{d^2\gamma_1}{dt^2} + (L_2R_1 + L_1R_2)\frac{d\gamma_1}{dt} + R_2(R_1\gamma_1 - E_1) = 0. \; \text{....} (21)$$

The complete solution of this equation is

$$R_1\gamma_1 - E_1 = A_1e^{\alpha t} + B_1e^{\beta t}, \quad \text{.........................(22)}$$

where A_1, B_1 are constants, and α, β the roots of the quadratic

$$(L_1L_2 - M^2)x^2 + (L_2R_1 + L_1R_2)x + R_1R_2 = 0, \quad \text{............(23)}$$

that is α, β are given by

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = -\frac{1}{2(L_1L_2 - M^2)}[L_2R_1 + L_1R_2 \pm \{(L_2R_1 + L_1R_2)^2 - 4R_1R_2(L_1L_2 - M^2)\}^{\frac{1}{2}}]. \quad \text{...(24)}$$

Similarly, by eliminating γ_2, we get

$$R_2\gamma_2 - E_2 = A_2e^{\alpha t} + B_2e^{\beta t}, \quad \text{.........................(25)}$$

where α, β have the same values as before, and A_2, B_2 are other constants.

The values of γ_1, γ_2, given by these equations for any time t, depend on the constants A_1, B_1, A_2, B_2, which must be determined to suit the given circumstances of the case.

The quantities α, β depend on the form and dimensions of the circuits. They are real, for the roots of the quadratic (23) are real if

$$(L_1R_2 + L_2R_1)^2 > \{(L_1R_2 + L_2R_1)^2 - 4(L_1L_2 - M^2)R_1R_2\},$$

which is true if $L_1L_2 > M^2$. This is obvious from the energy equation, or from the lines of induction of a unit current flowing in either circuit. These lines all pass through the circuit from which they originate, but do not all pass through the other. The other circuit may, however, consist of n turns, and hence $M < nL_1$. Again a unit current flowing in the second circuit gives a total induction through it of value L_2, and all these do not pass through the first. We shall suppose that the second circuit has the greater effective area. Thus $M < L_2$. But if L'_2 be the average inductance of a single turn of the second coil, $L_2 = n^2L'_2$, since the lines due to each turn give an induction nL'_2, and these are n turns. But clearly also $M < nL'_2$, that is $M < L_2/n$. Hence $M^2 < nL_1L_2/n$, or $M^2 < L_1L_2$.

9. March of the currents in the primary and secondary. Let both circuits be closed at the zero of reckoning of t. Then for $t=0$, $\gamma_1 = 0$, $\gamma_2 = 0$, and we get from (22) and (25),

$$-E_1 = A_1 + B_1, \quad -E_2 = A_2 + B_2. \quad \text{....................(26)}$$

Hence (22) and (25) become

$$R_1\gamma_1 = E_1(1 - e^{\beta t}) + A_1(e^{\alpha t} - e^{\beta t}), \quad \text{.....................(27)}$$

$$R_2\gamma_2 = E_2(1 - e^{\beta t}) + A_2(e^{\alpha t} - e^{\beta t}). \quad \text{.....................(28)}$$

Of course A_1, A_2 could be determined from the initial values of $\dot\gamma_1$, $\dot\gamma_2$; but as a rule it will be more convenient to determine the constants to suit the particular circumstances of the actual cases to which the equations are applied. An important case which we shall consider is that of a secondary circuit for which $E_2=0$. We shall suppose the secondary circuit to be kept closed, while the primary circuit is made or broken. Then differentiating (27) and (28), putting $t=0$, and substituting in (11), we get

$$\left.\begin{aligned} A_1 &= \frac{E}{\alpha-\beta}\left(\beta+\frac{L_2R_1}{L_1L_2-M^2}\right), \\ A_2 &= -\frac{E}{\alpha-\beta}\frac{R_2M}{L_1L_2-M^2}. \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

The march of the primary and secondary currents is shown in the figure for the case of $E=100$ volts, applied to a primary of resistance 10 ohms and self inductance 0·05 henry, between which and a secondary resistance 5 ohms and self inductance 0·4 henry, the mutual inductance is 0·02 henry.

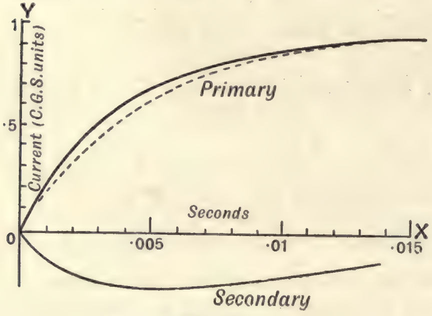

FIG. 63.

The upper curve in Fig. 63 * shows the rise of the primary current from zero to its steady value, while the lower shows the march of the secondary current, which is in the opposite direction. The dotted curve shows the rise of the primary current for the case of no secondary; and it appears that the effect of the mutual inductance is, as we should expect, to make the rise more rapid at first, and afterwards to retard it; as will be seen, the dotted line continued would cross the full curve.

The secondary current rises to its maximum in time t given by

$$t=\frac{1}{\alpha-\beta}\log\frac{\beta}{\alpha}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30)$$

as may be shown by finding $d\gamma_2/dt$ from (28), and then equating the value to zero. This value of t is least (it is, in fact, zero) when $M^2=L_1L_2$. This condition is never fulfilled, but it is most nearly fulfilled when the primary and secondary coils are equal and as nearly as possible coincident. If they were absolutely coincident, we should have $L_1=L_2=M$. In point of fact M^2 is always less than L_1L_2. The ratio $M/\sqrt{L_1L_2}$ has been called the coefficient of coupling of the circuits. The coupling of an induction coil is somewhat different. [See Appendix on the action of the Induction Coil.]

10. Total flow at "make" and at "break." Current in the secondary at break of the primary. The reader may easily verify the well-known result that the quantities of electricity which flow in the secondary at make and at break of the primary are the same in amount and opposite in sign, being in the former case $-\gamma_1 M/R_2$ and in the latter $\gamma_1 M/R_2$, where γ_1 is the primary steady current. This result has been verified experimentally, and affords evidence of the correctness of the theory from which the result has been derived. The ratio M/R_2 is of course capable of being regarded as an interval of time.

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* From *Alternate-Current Working*, by Alfred Hay, London, Biggs and Co.

It is interesting to study the march of the current in the secondary, at break of the primary. We suppose as before that the secondary is kept closed. Let the variable stage of the primary current extend over a time τ, then τ is the duration of the break. Integrating over this interval, we get from the differential equation of the secondary

$$-M\gamma_1 + L_2\gamma_2 + R_2\int_0^\tau \gamma_2 dt = 0, \quad \text{......................(31)}$$

where, in the integrated terms, γ_1 is the steady value of the primary current at the commencement of the break, and γ_2 is the value of the secondary current at the end of the time τ, while in the third term γ_2 is the value of the secondary current at the passing of the element dt of the interval of break.

We may suppose the break to be made by a very sudden rise of the resistance in the primary from R_1 to infinity. During this change the differential equation of the primary circuit no longer holds, but that of the secondary is

$$L_2\frac{d\gamma_2}{dt} + M\frac{d\gamma_1}{dt} + R_2\gamma_2 = 0. \quad \text{.....................(32)}$$

If we integrate this from the beginning of the break to any time t, at which it is yet incomplete, we have, since γ_2 is initially zero, and γ_1 has the steady value of the primary current,

$$L_2\gamma_2 + M(\gamma - \gamma_1) + R_2\int_0^t \gamma_2 dt = 0, \quad \text{................(33)}$$

where γ is the value of the primary current at time t, and γ_2 that of the secondary current at the same instant, except, of course, in the integral, where γ_2 is the value of the secondary current at the passing of the element of time dt. It is clear that γ_2 is finite, and therefore, if we suppose the break effected in an extremely short interval, we may neglect the time integral of γ_2 over that interval, and write

$$L_2\gamma_2 - M\gamma_1 = 0. \quad \text{..............................(34)}$$

Here γ_2 is the secondary current at the close of the break of the primary, and γ_1 the steady current which initially flowed in the primary. Thus

$$\gamma_2 = \frac{M}{L_2}\gamma_1. \quad \text{..............................(34')}$$

Up to the end of the break

$$L_2\gamma_2 = M(\gamma_1 - \gamma) - R_2\int_0^t \gamma_2 dt; \quad \text{...... ....(35)}$$

so that until then γ_2 continually increases. The graph of γ_2 is therefore that here shown in Fig. 64. The duration of the break is OM and the ordinate MP is approximately $M\gamma_1/L_2$.

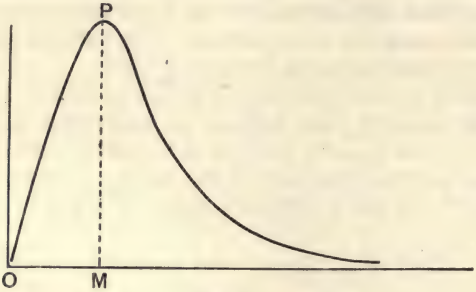

FIG. 64.

The energy of the secondary current during the dying-away stage, that is after the lapse of the time OM, is $\frac{1}{2}L_2\gamma_2^2$, and at the beginning of that stage is $\frac{1}{2}L_2M^2\gamma_1^2/L_2^2$ or $\frac{1}{2}M^2\gamma_1^2/L_2$. The rate at which the energy is dissipated in heat is $R_2\gamma_2^2$, and so we have

$$\frac{d}{dt}(\tfrac{1}{2}L_2\gamma_2^2) + R_2\gamma_2^2 = 0,$$

that is

$$L_2\frac{d\gamma_2}{dt} + R_2\gamma_2 = 0.$$

Integrating we obtain, reckoning t from the end of the break,

$$\gamma_2 = \frac{E}{R_1}\frac{M}{L_2}e^{-\frac{L_2}{R_2}t}, \quad\text{.........................}(36)$$

which shows how the current dies away in the secondary.

It is to be understood that if the primary or secondary circuit consist of coils surrounding iron cores, the march of the induced current is very different from that discussed here. The inductances are no longer constant, but functions of the current. For results in such cases, the reader may consult a paper by the late Professor T. Gray, *Phil. Trans. R.S.* 184 (1893), A. Much information will also be found in various treatises, *e.g.* Russell's *Treatise on Alternating Currents*.

11. Theory of a single circuit with self inductance. The theory of a single circuit with resistance and self inductance can be written down at once. The differential equation is

$$L\frac{d\gamma}{dt} + R\gamma = E, \quad\text{.........................}(37)$$

which gives, if the integral is taken from the instant at which $\gamma = 0$,

$$\gamma = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right), \quad\text{.........................}(38)$$

of which the graph is shown in Fig. 65. The current divides into two parts, the final steady current E/R, and $-Ee^{-tR/L}/R$. The whole quantity of electricity which may be regarded as passing in this second part is

$$q = -\frac{E}{R}\int_0^\infty e^{-\frac{R}{L}t}\,dt = -\frac{EL}{R^2}. \qquad \text{.....................(39)}$$

This is the deficiency in the quantity of electricity carried in the time of rise of the current to its steady value, caused by induction.

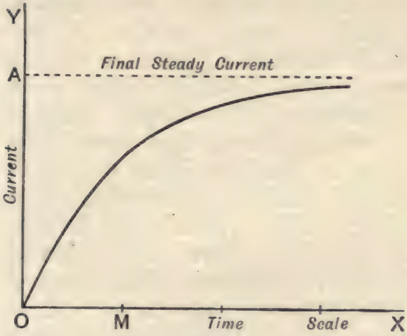

FIG. 65.

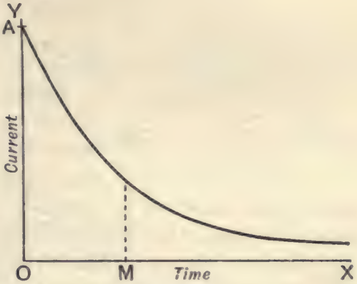

FIG. 66.

If, while the current is flowing steadily, the electromotive force is removed at a given instant, taken as $t=0$, the current after any interval t has elapsed is

$$\gamma = \frac{E}{R}e^{-\frac{R}{L}t}, \qquad \text{.................................(40)}$$

which is illustrated by Fig. 66. OM is the interval $t=L/R$, in which the current falls to $1/e$ of its initial value. This is called the time-constant of the circuit. The curves in Figs. 65 and 66 are the same, but are differently placed with respect to the axes.

12. Equations for the circuits of a network of conductors. We have in many examples of measurements considered below to deal with a set of conductors which are joined so as to form a network. The dynamical equations are at once applicable to such a system, in the same way as to a system of complete circuits, provided we use instead of resistances, inductances, and electromotive forces in circuits, the resistances, inductances (self and mutual) of the conductors, and the impressed differences of potential between their terminals.

The two fundamental principles from which the results for steady flow in a network are obtained in IV. 6 and 7 above, are here also applicable. The principle of continuity requires no modification; the statement of the second principle requires to be changed in the manner indicated below.

A difficulty exists in deciding what is the self-inductance of a conductor joining two points in a circuit or the mutual inductance of two conductors in the same circuit or in different circuits. There is no real practical difficulty except in a few special cases, for example in Hertzian vibrators of different forms: in most cases the conductors are coils, which may be regarded as each so many complete circuits given in position and dimensions by the turns of wire. The magnetic induction through each turn is quite definite and can be calculated.

This difficulty is at first sight in a manner avoided by the use of Maxwell's cycle method of dealing with a network, that is of regarding it as made up of a series of meshes or cells, as in Fig. 67, which consists of three distinct meshes *ADCA*, *ABDA*, *CDBC*. Each individual conductor is common to two meshes, except those conductors which form the outer edge of the network. Suppose a current to circulate round each mesh in the same direction, so that the actual current in each conductor is made up of the currents in two adjoining meshes. Each mesh is from this point of view a complete circuit with its own current flowing round it, and the inductances appear quite definite, being those of the distinct circuits. Each conductor, however, forms part of each of two adjacent circuits, and the determination of the inductances is not an easy matter.

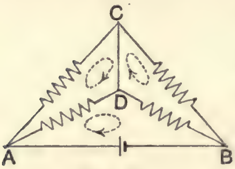

FIG. 67.

On the whole the usual method is the more convenient, and we shall adhere to it, as we avoid various perplexing questions which arise in complicated systems as to expressions for the energy. We denote by L_1, L_2, …, M_{12}, M_{23}, …, the self-inductances of the conductors 1, 2, …, and the mutual inductances of the pairs of conductors 12, 23,…. The electrokinetic energy has the value

$$T=\tfrac{1}{2}(L_1\gamma_1^2+2M_{12}\gamma_1\gamma_2+\ldots+L_2\gamma_2^2+2M_{23}\gamma_2\gamma_3+\ldots), \quad\ldots\ldots(41)$$

and the dissipation function is

$$F=\tfrac{1}{2}(R_1\gamma_1^2+R_2\gamma_2^2+\ldots). \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(42)$$

If there is electric energy E of condensers situated in the conductors and carrying charges y_1, y_2, …, the equations of the circuits are of the type

$$\frac{d}{dt}\frac{\partial T}{\partial \gamma_k}+\frac{\partial F}{\partial \gamma_k}+\frac{\partial \mathsf{E}}{\partial y_k}=E_k-V_k, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43)$$

where E_k is the *internal* electromotive force in the conductor, and V_k is the difference of potential between its terminals, taken with the negative sign, since we shall suppose E_k to act with the current, and V_k to oppose it.

Adding these equations for all the conductors forming a circuit, we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \gamma_j}+\frac{\partial T}{\partial \gamma_{j+1}}+\ldots\right)+\frac{\partial \mathsf{E}}{\partial y_j}+\frac{\partial \mathsf{E}}{\partial y_{j+1}}+\ldots+\frac{\partial F}{\partial \gamma_j}+\frac{\partial F}{\partial \gamma_{j+1}}+\ldots=E, \quad (44)$$

where E is the total electromotive force in the circuit. The sum of the differences of potential between the terminals of the conductors is of course zero for every complete circuit.

If C_j, C_{j+1}, ... be the capacities in the successive conductors of the circuit, and y_j, y_{j+1}, ... denote as above the corresponding charges, we have

$$\mathsf{E} = \tfrac{1}{2}\left(\frac{y_j^2}{C_j} + \frac{y_{j+1}^2}{C_{j+1}} + \dots\right). \quad \text{........................}(45)$$

Hence the last equation of currents may be written

$$\sum\left\{\frac{d}{dt}\left(L_j\gamma_j + M_{jk}\gamma_{jk}\right) + R_j\gamma_j + \frac{y_j}{C_j}\right\} = E, \quad \text{............}(44')$$

in which form we shall generally use it. This may be taken as the generalized form of the so-called second law of Kirchhoff for a system of linear conductors.

13. Battery with induction coil and cross-connection. As an example we take the case of a battery and coil in circuit, with a cross-connection between them as shown in Fig. 68. If γ_1, γ_2 be the currents in the coil and the cross-connection, respectively, and γ the current through the battery, r_1, r_2, r the resistances of the coil, cross-connection and battery with its connecting wires to AB, we have

$$\gamma = \gamma_1 + \gamma_2,$$

and on the supposition that the only inductance to be considered is L, that of the coil,

$$r_2\gamma_2 + r\gamma = E, \quad L\frac{d\gamma_1}{dt} + r_1\gamma_1 + r\gamma = E,$$

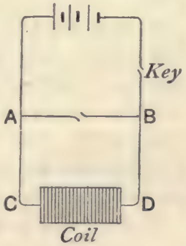

FIG. 68.

for the two circuits $EABE$, $EACDBE$. The first of these equations used in the second two gives

$$\left.\begin{aligned} r_2\gamma_2 + r(\gamma_1+\gamma_2) &= E, \\ L\frac{d\gamma_1}{dt} + (r+r_1)\gamma_1 + r\gamma_2 &= E. \end{aligned}\right\} \quad \text{........ ....}(45')$$

Eliminating γ_2 between the two last equations, we obtain for γ_1 the equation

$$L\frac{d\gamma_1}{dt} + \frac{rr_1 + r_1r_2 + r_2r}{r+r_2}\gamma_1 = \frac{r_2E}{r+r_2}. \quad \text{.....................}(46)$$

Writing Σrr_1 for $rr_1 + r_1r_2 + r_2r$, and integrating, we get

$$\gamma_1 = \frac{r_2E}{\Sigma rr_1}\left\{1 - e^{-\frac{\Sigma rr_1}{L(r+r_2)}t}\right\}; \quad \text{......................}(47)$$

and hence, for γ_2,

$$\gamma_2 = \frac{E}{r+r_2}\left\{1 - \frac{rr_2}{\Sigma rr_1}\left(1 - e^{-\frac{\Sigma rr_1}{L(r+r_2)}t}\right)\right\}. \quad \text{...............}(48)$$

The reader may now prove that the quantity, q, of electricity which flows through the coil in any interval τ, reckoned from the closing of the circuit, is given by

$$q=\int_0^\tau \gamma_1 dt=\frac{r_2 E}{\Sigma r r_1}\left[\tau-\frac{L(r+r_2)}{\Sigma r r_1}\left\{1-e^{-\frac{\Sigma r r_1}{L(r+r_2)}\tau}\right\}\right]. \quad \text{......(48')}$$

If when the interval τ has elapsed the circuit of the battery be broken, the quantity of electricity which flows through the coil after the instant of break is

$$q'=\frac{r_2 LE}{(r_1+r_2)\,\Sigma r r_1}\left(1-e^{-\frac{L(r+r_2)}{\Sigma r r_1}\tau}\right), \quad \text{..................(48'')}$$

as the reader may prove by writing down the differential equation for the current in the coil after the break, and then integrating from the instant (the end point of τ) for which (47), with $t=\tau$, gives the current.

14. Electrical oscillations. Theory. A condenser, of capacity C, is charged to a difference of potential V_0, and its plates are then connected by a coil of self-inductance L and resistance R. The condenser begins to discharge by a current in the wire. Let the difference of potential between the plates be V at any time t, and the current γ. The energy stored at the moment in the condenser is $\frac{1}{2}CV^2$, and the discharging current has electrokinetic energy $\frac{1}{2}L\gamma^2$. Hence the total electric energy is $\frac{1}{2}(CV^2+L\gamma^2)$. The total time rate of diminution of this energy must be equal to the rate at which energy is being transformed into heat in the circuit, *plus that at which energy is being radiated from the varying current system.*

If radiation is neglected we have, by what has been stated,

$$\frac{1}{2}\frac{d}{dt}(CV^2+L\gamma^2)+R\gamma^2=0, \quad \text{.....................(49)}$$

with $\gamma=-C\,dV/dt$. Thus the equation just written becomes

$$CL\frac{d^2V}{dt^2}+RC\frac{dV}{dt}+V=0, \quad \text{....................(50)}$$

of which the complete solution is

$$\left.\begin{aligned} V&=e^{-\frac{R}{2L}t}(Ae^{at}+Be^{-at}),\\ \text{where}\quad a&=\frac{1}{2L}\left(R^2-4\frac{L}{C}\right)^{\frac{1}{2}}. \end{aligned}\right\} \quad \text{........................(51)}$$

We may write the solution in the form

$$V=e^{-\frac{R}{2L}t}D\cosh(at-\zeta), \quad \text{........................(51')}$$

where A and B, or D and ζ are constants to be determined from the initial circumstances for any particular case.

If a is real this represents an ordinary discharge, that is a progressive non-oscillatory subsidence of the difference of potential between the plates, in which, theoretically, complete equalization of the potentials is only reached in an infinite time.

If a is imaginary the solution may be obtained in a realised form by writing cos for cosh, and ia for a, in the last equation. Thus we obtain

$$V = e^{-\frac{R}{2L}t} D \cos\left\{\frac{1}{2L}\left(4\frac{L}{C} - R^2\right)^{\frac{1}{2}} t - \theta\right\}, \quad \ldots\ldots\ldots\ldots\ldots(52)$$

where D and θ are constants. This represents an oscillatory discharge, with gradually diminishing range of difference of potential. The period of oscillation is given by

$$T = \frac{4\pi L}{\left(\frac{4L}{C} - R^2\right)^{\frac{1}{2}}}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(53)$$

and the logarithmic decrement of the difference of potential is $RT/4L$.

The discharging current $-(dV/dt)C/R$ is obtained from (52) or from (51), according as the discharge is oscillatory or non-oscillatory.

Thus the existence of an oscillatory discharge depends on the relation of L to R and C. If the inductance is great enough, electrical oscillations will take place, and there is no doubt that many discharges which appear to be single sparks are successions of backward and forward discharges caused by successive oscillations.

15. Dynamical analogies in electrical oscillations. The discharge of a condenser is thus similar to the motion of a deflected spring when resisted by a frictional force proportional to the speed of displacement. For we may write the equation of discharge as

$$L\frac{d^2V}{dt^2} + R\frac{dV}{dt} + \frac{1}{C}V = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(54)$$

which shows that L corresponds to the inertia of the matter moved, V to the displacement, $1/C$ to the return force of the spring per unit of displacement (that is C may be regarded as the modulus of yielding, or *permittance* as Heaviside calls it), and R to the resisting force per unit of the speed. In such a case we know that, if the inertia is small and the resisting force is large enough, the spring will simply slip back to its equilibrium position without oscillation about it, just as does a pendulum bob, of small inertia, deflected in a highly viscous fluid (*e.g.* treacle) and then left to itself. If however the spring has a certain inertia it will get into motion, and, as it nears the equilibrium position, move more and more quickly, will overshoot that position, and end by oscillating about it with diminishing range. When the inertia is such that the spring is just brought to rest without passing the equilibrium position, and the slightest addition of mass would cause it to pass that position without coming to rest, the motion is "dead beat," and the relation $R^2C = 4L$ is fulfilled. Up to this limit the addition of inertia diminishes (see below) the time of return, that is the time of "discharge."

Thus the addition of the analogous quantity, self-inductance, to the discharging conductor increases the rapidity of discharge of a condenser. For example the self-inductance of a lightning conductor may facilitate the discharge of a thunder cloud.

When the problem of the discharge of a condenser through a coil was first discussed mathematically [by Lord Kelvin, *Phil. Mag.* June, 1853] and the conditions of oscillatory discharge were set forth, the existence of radiation of energy from the system was not suspected. If it had been, perhaps a more complete theory might have been worked out, and the history of the electromagnetic theory of light been different from what it is. As a matter of fact the discovery of the electromagnetic theory was not very long delayed, as Clerk Maxwell's famous memoir was given to the Royal Society in 1864.* The possibility of oscillatory discharge had however been suggested by Helmholtz † in 1847, from certain unexplained phenomena of magnetization produced by passing Leyden jar discharges through a coil surrounding a bar of steel.

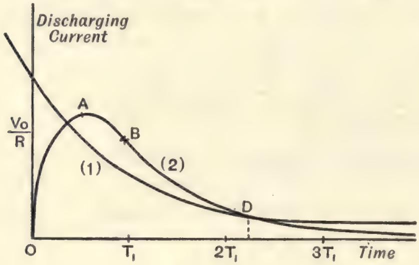

FIG. 69.

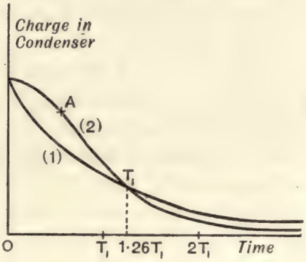

FIG. 70.

The rate of discharge and amount of charge left in the condenser are shown in Figs. 69 and 70 [from a paper by Lodge, *Electrician*, May 18, 1888] for the cases of (1) zero self-inductance, (2) just as much self-inductance as can exist without oscillatory discharge. If the coil possesses no inductance the equation is reduced to

$$V = V_0 e^{-\frac{t}{RC}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55)$$

which gives the potential at time t in terms of the initial potential V_0 and the time interval t. It is clear that t/RC is a mere number, that is to say RC is a time, and is the interval in which the potential is diminished from any value V_0 to V_0/e, and is called the time constant. V_0/e is the common ratio of the geometrical progression the terms of which are the values of V after successive intervals each equal to RC. Since $e = 2{\cdot}71828\ldots$, e^{10} is about 20000, and so in an interval 10 times RC, the potential has fallen to about 1/20000 of what it was at the beginning of the interval.

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* *Phil. Trans.* 155 (1865). † *Die Erhaltung der Kraft*, 1847.

16. Time constants in oscillatory discharge of a condenser. Writing the first of equations (51) in the form

$$V = Ae^{-\left(\frac{R}{2L}-a\right)t} + Be^{-\left(\frac{R}{2L}+a\right)t}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(56)$$

we see that in the general case there are two time constants, $1/(R/2L - a)$ and $1/(R/2L + a)$. If the roots $a - R/2L$, $-(a + R/2L)]$ of the auxiliary quadratic are real, both of these time constants are positive, since in that case $a = (R/2L)(1 - 4L/CR^2)^{\frac{1}{2}}$, and is real and less than $R/2L$.

The time of discharge depends mainly on the first of these time constants, which is the larger, since the term depending upon it remains still sensible after the other term has practically been wiped out.

Full particulars with graphs of the oscillatory and non-oscillatory discharge will be found in the author's *Treatise on Magnetism and Electricity*, i. p. 360 *et seq.*

17. Harmonic electromotive forces. Rule for solution of differential equations for forced oscillations. We now consider, as a preliminary to the discussion of the measurement of power, etc., in circuits carrying alternating currents, some examples of the action of harmonic electromotive forces. We take first a simple circuit containing self-inductance L, resistance R, a condenser of capacity C, and an electromotive force represented by $E_0 \sin nt$. The arrangement is that of the diagram with the coil p, shown in parallel with the condenser, removed. The alternating machine indicated on the left is supposed to produce the harmonic electromotive force in the circuit of the coil and condenser joined in series. Reckoning from any epoch of time for which the charge of the condenser may be taken as zero, we have as the equation of current

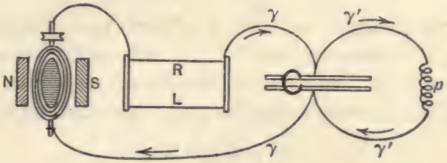

FIG. 71.

$$L\dot{\gamma} + R\gamma + \frac{1}{C}\int_0^t \gamma\, dt = E_0 \sin nt, \quad \ldots\ldots\ldots\ldots\ldots\ldots(57)$$

which is equivalent to the equation

$$L\ddot{\gamma} + R\dot{\gamma} + \frac{1}{C}\gamma = nE_0 \cos nt. \quad \ldots\ldots\ldots\ldots\ldots\ldots(57')$$

The following theorem of differential equations will be of much use in what follows. If we put D for d/dt, the equation just written takes the form

$$\left(LD^2 + RD + \frac{1}{C}\right)\gamma = nE_0 \cos nt. \quad \ldots\ldots\ldots\ldots\ldots\ldots(57'')$$

Now in order to find γ we notice that we may interpret the equation as a statement that γ is that function of t, which, by the operation $LD^2 + RD + 1/C$ performed upon it, generates $nE_0 \cos nt$. But if we perform the operation on $nE_0 \cos(nt + a)$, the result is

$$nE_0 \left\{ \left(-Ln^2 + \frac{1}{C} \right)^2 + n^2 R^2 \right\}^{\frac{1}{2}} \cos \left(nt + a + \tan^{-1} \frac{Rn}{-Ln^2 + \frac{1}{C}} \right).$$

The effect of the operation is thus to multiply the operand by the factor

$$\left\{ \left(-Ln^2 + \frac{1}{C} \right)^2 + n^2 R^2 \right\}^{\frac{1}{2}},$$

and to advance the phase of the harmonic factor by the angle

$$\tan^{-1}\{Rn/(-Ln^2 + 1/C)\}.$$

Hence if we had taken as the operand

$$\gamma = \frac{nE_0}{\left\{ \left(-Ln^2 + \frac{1}{C} \right)^2 + n^2 R^2 \right\}^{\frac{1}{2}}} \cos \left(nt - \tan^{-1} \frac{Rn}{-Ln^2 + \frac{1}{C}} \right), \;\ldots\ldots(58)$$

we should have obtained as the result of the operation $nE_0 \cos nt$. Thus (58) is a particular solution of the differential equation (57′).

The performance of the inverse operation $(AD^2 + ND + M)^{-1}$ on the function $C \cos(nt + a)$ *divides* the function operated on by

$$\{(-An^2 + M)^2 + N^2 n^2\}^{\frac{1}{2}},$$

and turns the phase *back* through the angle $\tan^{-1}\{Nn/(-An^2 + M)\}$. This gives an easily remembered rule for applying the symbolical method of treatment, which we shall find very useful in what follows.

Of course this process gives the particular solution required to express what is called the forced vibration brought about by the harmonically varying force $nE_0 \cos nt$. The complete solution requires the addition of the so-called complementary function which is the complete solution of the differential equation when the right-hand side is zero. For forced oscillation proceeding in steady regime this addition to the solution is not required.

It may be added here for reference that the simpler inverse operation $(AD + N)^{-1}$ performed on $C \cos(nt + a)$ turns the phase back through the angle $\tan^{-1}(nA/N)$ and divides the operand by the factor $(A^2 D^2 + N^2)^{\frac{1}{2}}$. Thus if we are given the differential equation

$$\left(L\frac{d}{dt} + R \right) \gamma = C \cos(nt + a), \;\ldots\ldots\;\ldots\ldots\ldots\ldots(59)$$

we see at once that the particular solution for forced oscillation is

$$\gamma = \frac{C}{(R^2 + n^2 L^2)^{\frac{1}{2}}} \cos \left(nt + a - \tan^{-1} \frac{nL}{R} \right). \;\ldots\ldots\ldots\ldots(60)$$

We now return to the problem of the alternating current in a condenser circuit, as stated above. By the method just explained we obtain the solution in (58). We omit the complementary function $Ae^{\alpha t} + Be^{\beta t}$, since, whether the roots of the auxiliary quadratic be real or complex, that part of the solution will ultimately, if the flow is continued in steady regime, be extinguished by frictional dissipation of the corresponding energy. The externally applied force will maintain the vibrational part of the flow; the non-vibrational part is not aided by any force, receives no energy unless a disturbance (a variation of the speed of running or some other irregularity) calls it into existence, when it immediately begins again to die away.

18. Impedance in an electric circuit. Influence of capacity. We may write the solution (58) in the form

$$\gamma = \frac{E_0}{\left\{R^2 + \left(nL - \frac{1}{Cn}\right)^2\right\}^{\frac{1}{2}}} \cos(nt - \theta). \qquad \ldots\ldots\ldots\ldots\ldots(61)$$

The quantity represented by the denominator of the expression for the amplitude on the right is called the *impedance* of the circuit. If C is zero the impedance is $(R^2 + n^2L^2)^{\frac{1}{2}}$; so that it is clear that the effect of the capacity is to counteract the influence of the self-inductance. For a given resistance in circuit and a given self-inductance the current is a maximum when $n^2LC = 1$, that is when capacity $C = 1/n^2L$ is inserted in the circuit.

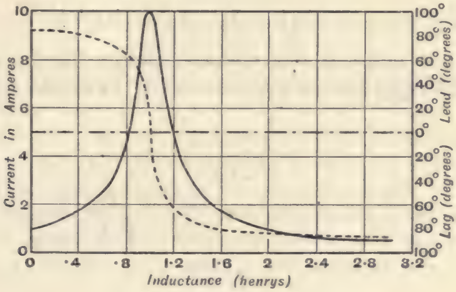

FIG. 72.

The relation between the current and the inductance for a given condenser is shown in Fig. 72, which illustrates the foregoing theory.

Equation (61) gives the interesting result that when the circuit contains a given condenser and a given resistance, the addition of self-inductance, up to a certain point, increases the current, and that the maximum is obtained when $CLn^2 = 1$. This is of importance in the theory of signalling through a cable. There however the capacity is distributed along the cable, and the theory is more complicated, but the general result is the same.

19. Influence of inductance in telephony. The inductance appears in (61) multiplied by n, so that the effect of inductance becomes much greater when the frequency is high. For slow signalling through a submarine cable the inductance may be neglected, but it is a mistake to suppose, as is sometimes done, that it is necessarily deleterious, and that it should always be made as small as possible. In very rapid ordinary working, especially in telephony, the presence of a certain amount of self-inductance improves the clearness of the signals. To see how clearness is brought about the reader has only to observe that for zero inductance the retardation of phase is $\tan^{-1}(nCR)$ which depends on the frequency. In telephony this retardation must produce confusion of the signals, inasmuch as, in a composite sound, vibrations of one pitch have a different retardation from those of another pitch. But with self-inductance L of even moderate amount, and high frequency, such as we have in telephonic sounds, the retardation becomes $\tan^{-1}\{nCR/(1-n^2CL)\}$, that is, approximately, $-nCR/n^2CL$, or zero, so that the retardation is nearly zero for all the actual values of n, and distortion does not occur. Excessive self-inductance however produces attenuation of the signals.

20. Electric resonance. The difference of potential V between the plates of the condenser is, according to (57), given by

$$V = E_0 \sin nt - L\frac{d\gamma}{dt} - R\gamma,$$

or by (61),

$$V = \frac{E_0 \sin(nt-\theta)}{\{(1-n^2CL)^2 + n^2C^2R^2\}^{\frac{1}{2}}}. \qquad \text{(62)}$$

The maximum value of V is obtained when the denominator of the quantity on the right has its smallest value. For this n^2 has the special value

$$\frac{2L - CR^2}{2CL^2}.$$

If R is very small in comparison with L this becomes $1/CL$, so that n is then 2π times the natural frequency of electrical oscillation of the condenser and coil as arranged. Then

$$\frac{\text{amplitude of } V}{E_0} = \frac{1}{R}\sqrt{\frac{L}{C}}, \qquad \text{(63)}$$

which will be much greater than unity, since CR^2 has been supposed to be small compared with $2L$. This is the case of what is called electrical resonance, in which the amplitude of the difference of potential between the terminals of the condenser is greater than E_0, the electromotive force of the alternating machine.

This curious result was first observed in practice in observations by Mr. Ferranti on mains carrying alternating currents between London and a generating station at Deptford. It was found that the square root of the mean of V^2 on the terminals of the alternator, working at its normal speed with a certain exciting current, was increased by connecting the machines to the mains. This result was no doubt due to a partial fulfilment of the conditions necessary for a small value of the denominator of the right-hand side of (62).

21. Primary circuit with a condenser, and a secondary which contains no electromotive force. We now consider a primary circuit arranged as in the case just considered, but with a secondary circuit containing no electromotive force. The equation of the primary is evidently

$$L_1\frac{d\gamma_1}{dt}+M\frac{d\gamma_2}{dt}+R_1\gamma_1+\frac{1}{C}\int\gamma_1 dt=E_0\cos nt, \quad \text{..........}(64)$$

while that of the secondary is

$$M\frac{d\gamma_1}{dt}+L_2\frac{d\gamma_2}{dt}+R_2\gamma_2=0. \quad \text{... ....}(65)$$

Since we propose to consider only the forced electrical oscillations, and these will be simple harmonic and of period $2\pi/n$, we see that

$$\frac{1}{C}\int\gamma_1\,dt=-\frac{1}{Cn^2}\frac{d\gamma_1}{dt}.$$

Hence if we write L'_1, for L_1-1/Cn^2 the first of the above equations will be

$$L'_1\frac{d\gamma_1}{dt}+M\frac{d\gamma_2}{dt}+R_1\gamma_1=E_0\cos nt, \quad \text{..............}(64')$$

and we have for the primary and secondary circuits the equations (64′) and (65).

The problem is therefore that of a primary without condenser and with self-inductance $L'_1=L_1-1/Cn^2$, and a secondary without condenser and without electromotive force. If however the secondary contained a condenser of capacity C_2, we should only have to put for L_2 the expression $L'_2=L_2-1/C_2n^2$, in order to take the condenser into account. If we operate on (64′) by R_2+L_2d/dt and on (65) by Md/dt, we obtain

$$(L'_1L_2-M^2)\frac{d^2\gamma_1}{dt^2}+(L'_1R_2+L_2R_1)\frac{d\gamma_1}{dt}$$
$$+R_1R_2\gamma_1=E_0(R^2+n^2L_2{}^2)^{\frac{1}{2}}\cos(nt+\theta), \quad \text{......}(66)$$

where $\theta=\tan^{-1}(nL_2/R_2)$. By the rule given above the solution of this differential equation is, for forced oscillations,

$$\gamma_1=\frac{(R_2{}^2+n^2L_2{}^2)E_0}{[\{R_1R_2-n^2(L'_1L_2-M^2)\}^2+n^2(L'_1R_2+L_2R_1)^2]^{\frac{1}{2}}}\cos(nt-\theta_1), \quad \text{...}(67)$$

where

$$\theta_1=\tan^{-1}\frac{nR_2(L'_1R_2+L_2R_1)-nL_2\{R_1R_2-n^2(L'_1L_2-M^2)\}}{R_2\{R_1R_2-n^2(L'_1L_2-M^2)\}+n^2L_2(L'_1R_2+L_2R_1)}.$$

Similarly for forced oscillations in the secondary we obtain the differential equation

$$(L'_1L_2 - M^2)\frac{d^2\gamma_2}{dt^2} + (R_1L_2 + R_2L'_1)\frac{d\gamma_2}{dt} + R_1R_2\gamma_2 = MnE_0\cos\left(nt - \frac{\pi}{2}\right). \quad \text{..........(68)}$$

The solution of this equation for forced oscillations only is

$$\gamma_2 = \frac{MnE_0\cos(nt - \theta_2)}{[\{R_1R_2 - n^2(L'_1L_2 - M^2)\}^2 + n^2(R_1L_2 + R_2L'_1)^2]^{\frac{1}{2}}}, \quad \text{......(69)}$$

where $$\theta_2 = \tan^{-1}\frac{n^2(L'_1L_2 - M^2) - R_1R_2}{n(R_1L_2 + R_2L'_1)}.$$

Equation (67) can be written in the form

$$\gamma_1 = \frac{E_0\cos(nt - \theta_1)}{\left\{\left(R_1 + n^2\frac{M^2R_2}{R_2^2 + n^2L_2^2}\right)^2 + n^2\left(L_1 - \frac{1}{Cn^2} - n^2\frac{M^2L_2}{R_2^2 + n^2L_2^2}\right)^2\right\}^{\frac{1}{2}}}, \quad \text{...(70)}$$

which shows that the effect of the secondary has been virtually to increase the resistance of the primary by $n^2M^2R_2/(R_2^2 + n^2L_2^2)$, and to diminish the inductance by $n^2M^2L_2/(R_2^2 + n^2L_2^2)$.

The current in the secondary is the same as it would be if the circuit were independent, and contained a harmonic electromotive force of amplitude $MnE_0'/(R_1^2 + n^2L^2)^{\frac{1}{2}}$, and had a resistance

$$R_1^2 + n^2M^2R_1/(R_1^2 + n^2L_1^2)$$

and a self-inductance $L_2 - n^2M^2L'_1/(R_1^2 + n^2L_1^2)$.

22. Conductors in parallel, containing resistance, inductance and capacity. This discussion may be concluded with a short treatment of the case in which we have conductors in parallel (Fig. 73) which contain resistance, inductance, and capacity without mutual inductance. If L_1, L_2, ... , include the capacity terms, as in (70), the equations are

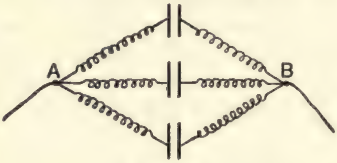

FIG. 73.

$$\left.\begin{aligned} L_1\frac{d\gamma_1}{dt} + R_1\gamma_1 &= V_0\cos nt, \\ L_2\frac{d\gamma_2}{dt} + R_2\gamma_2 &= V_0\cos nt, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ &\cdot\ \cdot\ \cdot\ \cdot \end{aligned}\right\} \quad \text{......................(71)}$$

The typical solution for forced oscillations is

$$\left.\begin{aligned}\gamma&=\frac{V_0}{(R^2+n^2L^2)^{\frac{1}{2}}}\cos(nt-\theta),\\ \text{where}\qquad \theta&=\tan^{-1}\frac{nL}{R}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(72)$$

This solution applied to each of equations (72) gives the currents in the different parallel conductors. Adding these together, we find for the total current, Γ, at any instant entering at one point, A say, and leaving at B,

$$\Gamma=\Sigma\gamma=V_0\Sigma\left\{\frac{1}{(R^2+n^2L^2)^{\frac{1}{2}}}\cos(nt-\phi)\right\}$$

$$\left.\begin{aligned}\text{or}\qquad \Gamma&=\frac{V_0}{(\mathsf{R}^2+n^2\mathsf{L}^2)^{\frac{1}{2}}}\cos(nt-\phi),\\ \text{with}\qquad \phi&=\tan^{-1}\frac{n\mathsf{L}}{\mathsf{R}}\\ \text{and}\qquad \mathsf{R}&=\frac{A}{A^2+n^2B^2},\quad \mathsf{L}=\frac{B}{A^2+n^2B^2},\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(73)$$

where A denotes $\Sigma\{R/(R^2+n^2L^2)\}$, B denotes $\Sigma\{L/(R^2+n^2L^2)\}$.

The total current is thus the same as if the points A, B in the diagram were connected by a single conductor of resistance R and inductance L. These may be called respectively the effective resistance and inductance of the system of parallel conductors. The angle ϕ is the lag in phase of the total current, entering or leaving the parallel system at any instant, behind that of the impressed difference of potential.

The effective capacity of the system of parallels is in general indeterminate, and of no practical importance.

Let us suppose the circuit between the two points AB to be completed by a single conductor of resistance R, inductance L, and containing an electromotive force $E_0\cos(nt+\zeta)$, so that ζ is the lag of the difference of potential $V_0\cos nt$ behind the electromotive force. The solution for the case in which this conductor contains a condenser C will be obtained by putting $L-1/Cn^2$ for L. The current in this conductor is Γ, so that the differential equation is

$$L\frac{d\Gamma}{dt}+R\Gamma+V_0\cos nt=E_0\cos(nt+\zeta).\quad\ldots\ldots\ldots\ldots\ldots(74)$$

Substituting the value of Γ already found, and remembering that the identity thereby obtained must hold for all values of t, we see that it gives the two equations

$$\left.\begin{aligned}\frac{V_0}{E_0}&=\frac{(\mathsf{R}^2+n^2\mathsf{L}^2)^{\frac{1}{2}}}{\{(\mathsf{R}+R)^2+n^2(\mathsf{L}+L)^2\}^{\frac{1}{2}}},\\ \zeta+\phi&=\tan^{-1}\frac{n(\mathsf{L}+L)}{\mathsf{R}+R}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(75)$$

Instead of the first of (73) we therefore have

$$\Gamma = \frac{E_0}{\{(\mathsf{R}+R)^2+n^2(\mathsf{L}+L)^2\}^{\frac{1}{2}}}\cos(nt+\zeta-\phi). \qquad \text{............(76)}$$

23. Alternator with electromotive force any periodic function of t. It is only in exceptional cases that the difference of potential produced between the terminals of an alternator is a simple harmonic function of the time. It is however a periodic function $f(t)$ of the time with a definite period T, such that

$$f(t)=f(t+T)=f(t+2T)=\dots . \qquad \text{.....................(77)}$$

But alternators, since the poles of the field magnets are alternately + and −, give also the equation

$$f(t)=-f(t+\tfrac{1}{2}T)=f(t+T)=-f(t+\tfrac{3}{2}T)=\dots . \qquad \text{.........(78)}$$

In this case Fourier's theorem enables us to write

$$f(t)=A_1\sin(nt+\alpha_1)+A_3\sin(3nt+\alpha_3)+\dots ,$$

where $n=2\pi/T=2\pi\times$ frequency of alternation. The differential equation for the machine has the form

$$L\frac{d\gamma}{dt}+R\gamma=A_1\sin(nt+\alpha_1)+A_3\sin(3nt+\alpha_3)+\dots , \qquad \text{.......(79)}$$

on the supposition, it is to be observed, that L is a constant. But the magnetic induction through the circuit when the current is γ is $L\gamma$, and it is often the case that L is a function of the time, which of course the current always is. Thus the complete equation for this more general case is

$$\frac{d}{dt}(L\gamma)+R\gamma=A_1\sin(nt+\alpha_1)+A_3\sin(3nt+\alpha_3)+\dots .$$

Thus there appears in this case the term $\gamma\, dL/dt$, which cannot be estimated, except as a rough approximation, without exact knowledge of the function which L is of the time. In the present work L is in most cases treated as a constant; and further information must be sought in treatises on Alternate Current Machines and Transformers. We have seen that L is analogous to inertia, so that $L\dot{\gamma}$ is the analogue of the usual term $m\dot{v}$ in the equation of motion of a particle of mass m, while $\gamma\dot{L}$ is the analogue of the term $\dot{m}v$ which appears when the mass of the particle considered undergoes variation, as for example when a growing raindrop falls through a rain cloud, or evaporates as it falls through a stratum of unsaturated air.

Regarding then L as a constant, and applying the rule of 17 above to each term on the right of (79), we get

$$\gamma=\frac{A_1}{(R^2+n^2L^2)^{\frac{1}{2}}}\sin(nt+\alpha_1-\theta_1)+\frac{A_3}{(R^2+3^2n^2L^2)^{\frac{1}{2}}}\sin(3nt+\alpha_3-\theta_3)+\dots ,$$

or, as we may write this solution,

$$\gamma = \sum \frac{A_{2k+1}}{\{R^2+(2k+1)^2n^2L^2\}^{\frac{1}{2}}} \sin\{(2k+1)nt + a_{2k+1} - \theta_{2k+1}\}, \quad \text{...........(80)}$$

where the summation is taken for as many of the successive values 0, 1, 2, 3, ... of k as may be required to express $f(t)$ with sufficient accuracy. The retardations of phase are given by the equation

$$\theta_{2k+1} = \tan^{-1}\frac{(2k+1)nL}{R}. \quad \text{........................(81)}$$

24. Rate of working in the circuit of an alternator. Mean current and mean square of current. We now consider the activity, or power, in an alternating circuit. We shall denote this for a circuit or conductor, according to the case considered, by A. If V be the difference of potential at a given instant between the extremities or terminals of the conductor, or system of conductors in question (which we shall suppose for the present to be a single linear conductor, or a set of linear conductors arranged in series), and γ be the current flowing between the terminals, the rate at which work is being done by the current is $V\gamma$. Thus the instantaneous value of A is $V\gamma$, but in the case of an alternating current what we have to reckon with in practice is the mean value, A_m of $V\gamma$ taken over a period T, that is with

$$A_m = \frac{1}{T}\int_0^T V\gamma\,dt. \quad \text{..............................(82)}$$

It is important to express A_m in terms of the quantities determined by a voltmeter placed across the terminals, and an amperemeter placed so as to carry the current γ. Let us consider what these instruments give. The dynamically effective action on the indicator of the voltmeter may be taken as proportional to the mean square of V, that is to

$$[V^2]_m = \frac{V_0^2}{T}\int_0^T \cos^2 nt\,dt.$$

Similarly the dynamical action in the current-meter is proportional to

$$[\gamma^2]_m = \frac{\gamma_0^2}{T}\int_0^T \cos^2(nt-\theta)\,dt.$$

Now we have for the mean square of V and the mean square of γ,

$$[V^2]_m = \frac{V_0^2}{T}\int_0^T \cos^2 nt\,dt = \tfrac{1}{2}V_0^2, \quad [\gamma^2]_m = \tfrac{1}{2}\gamma_0^2;$$

so that the square roots of these mean squares are

$$V' = \frac{1}{\sqrt{2}}V_0, \quad \gamma' = \frac{1}{\sqrt{2}}\gamma_0. \quad \text{........................(83)}$$

The mean value, A_m, of the activity is given by

$$A_m = \frac{1}{T}\int_0^T V\gamma\, dt = \frac{V_0\gamma_0}{T}\int_0^T \cos nt \cos(nt-\theta)\, dt = \tfrac{1}{2}V_0\gamma_0 \cos\theta_1,$$

that is, by (83),

$$A_m = V'\gamma' \cos\theta. \quad \text{.........................} (84)$$

V', γ' are read off from the voltmeter and amperemeter, so that A_m is the product of these readings by the cosine of the difference of phase angle between the difference of potential and the current. The multiplier $\cos\theta$ is called the power factor.

It is usual to regard V', the effective electromotive force, given by the voltmeter, as made up of two components, $V_1' = V'\cos\theta$ and $V_2' = V'\sin\theta$. Since the unit of power used in practice is the *watt*, and

$$A_m = \gamma' V' \cos\theta = \gamma' V_1', \quad \text{.........................} (85)$$

V_1' is called the "watt electromotive force" and V_2' the "wattless electromotive force."

25. Power factor in an alternating circuit. A more general specification of the power factor which suits other cases in which the wave forms of the current and electromotive force are merely specified as periodic may be given as follows. Of course both have the same period T. As it will be shown that the power factor cannot exceed unity in numerical value, we shall denote it by $\cos\phi$. Then we have

$$\cos\phi = \frac{\frac{1}{T}\int_0^T V\gamma\, dt}{V'\gamma'}, \quad \text{.........................} (86)$$

with

$$[V^2]_m = \underset{k=\infty}{\mathrm{Lt}} \left\{\frac{1}{k}(V_1^2 + V_2^2 + \dots + V_k^2)\right\},$$

$$[\gamma^2]_m = \underset{k=\infty}{\mathrm{Lt}} \left\{\frac{1}{k}(\gamma_1^2 + \gamma_2^2 + \dots + \gamma_k^2)\right\},$$

where $V_1, V_2, \dots, V_k, \gamma_1, \gamma_2, \dots, \gamma_k$ are the lengths of successively equidistant ordinates of a single wave of the potential and current respectively. Thus, clearly, with $k = \infty$,

$$(V_1\gamma_1 + V_2\gamma_2 + \dots + V_k\gamma_k)^2 \\ - (V_1^2 + V_2^2 + \dots + V_k^2)(\gamma_1^2 + \gamma_2^2 + \dots + \gamma_k^2)\cos^2\phi = 0,$$

that is

$$(V_1\gamma_2 - V_2\gamma_1)^2 + (V_1\gamma_3 - V_3\gamma_1)^2 + \dots \\ + (V_1^2 + V_2^2 + \dots + V_k^2)(\gamma_1^2 + \gamma_2^2 + \dots + \gamma_k^2)\sin^2\phi = 0. \quad \text{......} (87)$$

The expression on the left-hand side of the last equation is least when $\sin\phi = 0$, that is when $\cos\phi = \pm 1$. We have then for $\cos\phi = +1$,

$$\frac{V_1}{\gamma_1} = \frac{V_2}{\gamma_2} = \dots = \frac{V_k}{\gamma_k}. \quad \text{.........................} (88)$$

We see thus that the power factor as defined by (86) has the maximum value unity, and that then the wave form of the potential curve is the same as that for the current curve, and by (88) that the two curves have no difference of phase, and have simultaneous zeroes and maxima and minima.

The interpretation of $\cos\phi = -1$ is, by (86), that the potential and current curves are, as in the last case, similar, but have the ordinates turned in opposite directions, that is the curves drawn for the same time axis lie on opposite sides of the axis.

It is usual to call the angle $\cos^{-1}\phi$, the phase difference of the waves of potential and current. We can define in a similar way the phase difference between any two waves of the same period.

For two simple harmonic functions of the same period

$$x = x_0 \overset{\cos}{\underset{\sin}{\cdot}}(nt - a), \quad y = y_0 \overset{\cos}{\underset{\sin}{\cdot}}(nt - \beta),$$

the phase difference is the angle $a - \beta$. The *time-value* of this phase difference is $(a - \beta)/n$.

We have not space in which to dwell on this affair of phase difference and its treatment by a species of vector analysis. The reader is referred to special treatises, such as Russell's on *Alternating Currents*, in which a full analytical treatment, illustrated by graphs, is to be found. But we take as an illustration a single case in which this angle of lag is not equal to the phase difference as defined by equation (86). The angle of lag gives the time interval between the instants at which the values of the ordinates for the two curves pass in the positive direction through zero.

26. Phase difference and time-lag. We consider a periodic curve (not simple harmonic) the ordinate of which for any value of t is $f(t)$, with $f(t) = 0$, for $t = 0$. Since the curve is supposed to alternate, that is to have its alternate halves on opposite sides of the time-axis, and we here suppose further that each half is symmetrical about a middle ordinate, we have

$$y = f(t) = f(\tfrac{1}{2}T - t) = -f(\tfrac{1}{2}T + t). \quad \text{..................}(89)$$

The angle of lag between this curve and a simple harmonic curve, of which the equation is

$$\eta = a \sin(nt - a)$$

and period is T, is a. Now

$$\int_0^{\frac{1}{2}T} f(t) \sin(nt - a)\,dt = \cos a \int_0^{\frac{1}{2}T} f(t) \sin nt\,dt - \sin a \int_0^{\frac{1}{2}T} f(t) \cos nt\,dt.$$

It is easy to show that by the conditions (89) the second integral is zero. Hence

$$\int_0^{\frac{1}{2}T} f(t) \sin(nt - a)\,dt = \cos a \int_0^{\frac{1}{2}T} f(t) \sin nt\,dt.$$

The interpretation of this result is

$$\cos\phi = \cos\phi_0 \cos a, \qquad \text{........................(90)}$$

where ϕ is the phase difference between the two curves and ϕ_0 is the phase difference, as defined by (86), which would exist if the time-lag were zero.

It is to be remarked that if one of the curves is given by a constant ordinate while the other is periodic, fulfilling the condition

$$f(t) = -f(\tfrac{1}{2}T + t),$$

the cosine of the phase difference is zero, since the numerator on the right of (86) is then zero. Thus the phase difference of a constant and an alternating periodic quantity is 90.

CHAPTER IX.

THE DISTRIBUTION OF ALTERNATING CURRENTS IN PARALLEL CONDUCTORS.

1. Flow of alternating currents in a coaxial main. Differential equation. We consider first the flow of an alternating current in the inner conductor of a main consisting of a long right cylindrical conductor surrounded by a coaxial tube of given external and internal radius, which forms the return conductor. We have three regions to consider, that within the surface of the inner conductor, the tubular insulating space between the two conductors, and the return tube of conducting material. We shall suppose that the magnetic permeability of the insulator is μ', and that the radii of the cylindrical surfaces, taken in the outward order, are a, b, c. We shall neglect the capacity current and suppose that there is no leakage current. A stricter discussion will be found in *Bessel Functions* by Gray and Mathews.

In the first instance we shall suppose the conductivity of the inner conductor to be k, and that the outer conductor is a shell of infinite conductivity, so that it may be taken as exceedingly thin.

Take a coaxial tubular part of the inner conductor. Let the radii of this tube be r and $r+dr$ and the current in it $2\pi r\,dr\,.\,q$. The conductivity of a length dx of this tube is $2\pi kr\,dr/dx$, and therefore the electromotive force on the element is $2\pi qr\,dr\,dx/2\pi kr\,dr = q\,dx/k$. This is equal to the difference of potential $-\partial V/\partial x\,.\,dx$ between the extremities of the element *minus* the time rate of increase of the magnetic induction which surrounds the element. If $N\,dx$ be this induction, we have (writing P for $\partial V/\partial x$, as we shall require the symbol x for another purpose)

$$P = \frac{\partial N}{\partial t} + \frac{q}{k}. \qquad \text{.................................}(1)$$

We assume that there is no flow across the surface of the conductor, and therefore no radial flow anywhere, so that the equipotential surfaces in each of the conductors are planes perpendicular to the axis. Hence (1) gives

$$\frac{\partial}{\partial t}\frac{\partial N}{\partial r} + \frac{1}{k}\frac{\partial q}{\partial r} = 0. \qquad \text{...............................}(2)$$

Now consider the lines of magnetic force round a core of the conductor of radius r. Let the field intensity at distance r from the axis be H. As we pass from radius r to radius $r+dr$, the induction $\mu H\,dr$ is lost from N, since we are here considering N as furnished by the magnetic field outside the circle of radius r, and so we have

$$\frac{\partial N}{\partial r} = -\mu H, \quad\text{.............................(3)}$$

and (2) becomes

$$\mu\frac{\partial H}{\partial t} = \frac{1}{k}\frac{\partial q}{\partial r}. \quad\text{.................................(2')}$$

The line integral of magnetic force round the conducting cylinder of radius r which we are here considering is $2\pi rH$, and this is, by the remark made above as to the meaning of N, may be taken as $+4\pi$ times the flow through the circle of radius r. Hence we get

$$-\frac{\partial N}{\partial r} = \frac{4\pi\mu}{r}\int_0^r qr\,dr \quad\text{..............................(3')}$$

and

$$\left.\begin{aligned}\mu\frac{\partial H}{\partial t} &= \frac{4\pi\mu}{r}\int_0^r r\frac{\partial q}{\partial t}\,dr,\\ \text{or}\qquad \frac{1}{k}\frac{\partial q}{\partial r} &= \frac{4\pi\mu}{r}\int_0^r r\frac{\partial q}{\partial t}\,dr.\end{aligned}\right\} \quad\text{.........................(4)}$$

Eliminating H between (2') and (4), we obtain the differential equation of flow

$$\frac{\partial^2 q}{\partial r^2} + \frac{1}{r}\frac{\partial q}{\partial r} = 4\pi\mu k\frac{\partial q}{\partial t}. \quad\text{..........................(5)}$$

2. Integration of the differential equation. This is an equation of diffusion of electric current. The complete equation is one of wave propagation along the conductor as well as of diffusion in the radial direction, but as the speed of wave propagation is very great any ordinary length of conductor may be regarded as characterised at any one instant by the same state at all points on a line parallel to the axis.

The currents diffuse into the conductors just as heat would, if the surfaces of the conductors were subjected to variations of temperature.

We now suppose that q is a simple harmonic function of the time. Hence we may write

$$q = Aue^{int}, \quad\text{...................................(6)}$$

where n is 2π times the frequency of alternation, $i=\sqrt{-1}$, and u is a function of r only. The value of q can be replaced when desired by a real value by means of the theorem $e^{int} = \cos nt + i\sin nt$. The differential equation now becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - 4\pi i\mu nku = 0, \quad\text{.........................(7)}$$

or

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - m^2 iu = 0, \quad\text{............................(7')}$$

where $m^2 = 4\pi\mu nk$. If for mr we write x, this equation becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} - iu = 0. \qquad \text{(7'')}$$

The complete solution of the last equation is

$$u = AI_0(x\sqrt{i}) + BK_0(x\sqrt{i}), \qquad \text{(8)}$$

where I_0, K_0 are the Bessel functions* defined by the series

$$\left.\begin{aligned} I_0(\xi) &= 1 + \frac{\xi^2}{2^2} + \frac{\xi^4}{2^2 . 4^2} + \frac{\xi^6}{2^2 . 4^2 . 6^2} + \cdots, \\ K_0(\xi) &= (\log 2 - C) I_0(\xi) - \log x . I_0(\xi) \\ &\quad + \frac{\xi^2}{2^2} + \left(1 + \frac{1}{2}\right)\frac{\xi^4}{2^2 . 4^2} + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{\xi^6}{2^2 . 4^2 . 6^2} + \cdots . \end{aligned}\right\} \qquad \text{(9)}$$

C is what is known as Euler's constant, and has the numerical value 0·57721 56649 01.... Hence

$$\log 2 - C = 0{\cdot}11593\ 15156\ 58\ldots .$$

This value of the multiplier of $I_0(\xi)$ is required to ensure that the second function $K_0(\xi)$ shall vanish for large values of the argument ξ. The value of $K_0(\xi)$ becomes very great for small values of ξ, and therefore in applying the solution (8) to the inner conductor we must take $B = 0$. We shall return to the second solution presently.

We consider here first the distribution of current in the inner cylindrical solid conductor. For this then the solution is

$$u = AI_0(x\sqrt{i}). \qquad \text{(10)}$$

It is to be observed that this solution holds for every case in which the outer conductor surrounds the inner symmetrically as a sheath, with the insulating coaxial tubular space between. For the only link of connection between the current in the outer conductor and current elsewhere is the magnetic field of the former, and in the present case the outer current produces no magnetic field in the inner space.

The case is different if the return conductor is not a coaxial tube. If it is a wire the return current will affect the distribution of current in the inner conductor, unless the wire is at a very great distance from the latter, when again the solution (10) may be applied.

It is clear that $I_0(x\sqrt{i})$ is complex. We have in fact

$$\begin{aligned} I_0(x\sqrt{i}) &= 1 - \frac{x^4}{2^2 . 4^2} + \frac{x^8}{2^2 . 4^2 . 6^2 . 8^2} - \cdots \\ &\quad + i\left(\frac{x^2}{2^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \frac{x^{10}}{2^2 . 4^2 . 6^2 . 8^2 . 10^2} - \cdots\right). \end{aligned} \qquad \text{(11)}$$

Lord Kelvin wrote this in the form

$$I_0(x\sqrt{i}) = \text{ber}\, x + i\, \text{bei}\, x, \qquad \text{(12)}$$

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* See Gray and Mathews' *Bessel Functions*, chap. vii.

where ber denotes the real part and bei the imaginary part of the Bessel function. This notation is now in common use. On it has been founded a notation for the other function $K_0(x\sqrt{i})$,

$$K_0(x\sqrt{i}) = \operatorname{ker} x + i \operatorname{kei} x, \qquad \text{(13)}$$

where again ker denotes the real part and kei the imaginary part.

When x is small the forms used for the Bessel functions are those given in (9); when x is large the forms employed are*

$$\left.\begin{aligned} I_0(x) &= \frac{e^x}{(2\pi x)^{\frac{1}{2}}}\left\{1 + \frac{1^2}{8x} + \frac{1^2 . 3^2}{2!(8x)^2} + \frac{1^2 . 3^2 . 5^2}{3!(8x)^3} + \dots\right\}, \\ K_0(x) &= \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}\left\{1 - \frac{1^2}{8x} + \frac{1^2 . 3^2}{2!(8x)^2} - \frac{1^2 . 3^2 . 5^2}{3!(8x)^3} + \dots\right\}. \end{aligned}\right\} \qquad \text{(14)}$$

Thus when x is small, we have

$$\left.\begin{aligned} \operatorname{ber} x &= 1 - \frac{x^4}{2^2 . 4^2} + \frac{x^8}{2^2 . 4^2 . 6^2 . 8^2} - \dots, \\ \operatorname{bei} x &= \frac{x^2}{2^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \frac{x^{10}}{2^2 . 4^2 . 6^2 . 8^2 . 10^2} - \dots . \end{aligned}\right\} \qquad \text{(15)}$$

$$\left.\begin{aligned} \operatorname{ker} x &= (\log 2 - C - \log x) \operatorname{ber} x + \tfrac{1}{4}\pi \operatorname{bei} x \\ &\quad - \left(1 + \frac{1}{2}\right)\frac{x^4}{2^2 . 4^2} + \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)\frac{x^8}{2^2 . 4^2 . 6^2 . 8^2} - \dots, \\ \operatorname{kei} x &= (\log 2 - C - \log x) \operatorname{bei} x - \tfrac{1}{4}\pi \operatorname{ber} x \\ &\quad + \frac{x^2}{2^2} - \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{x^6}{2^2 . 4^2 . 6^2} + \dots . \end{aligned}\right\} \qquad \text{(16)}$$

Lastly, when x is large, the formulae are

$$\left.\begin{aligned} \operatorname{ber} x &= \frac{e^{\beta}}{\sqrt{2\pi x}} \cos \alpha, \quad \operatorname{bei} x = \frac{e^{\beta}}{\sqrt{2\pi x}} \sin \alpha, \\ \operatorname{ker} x &= \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{\beta'} \cos \alpha', \quad \operatorname{kei} x = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{\beta'} \sin \alpha', \end{aligned}\right\} \qquad \text{(17)}$$

where

$$\left.\begin{aligned} \left.\begin{matrix}\beta \\ \beta'\end{matrix}\right\} &= \pm \frac{x}{\sqrt{2}} \pm \frac{1}{8\sqrt{2}x} \mp \frac{25}{384\sqrt{2}x^3} - \frac{13}{128x^4} \mp \dots, \\ \left.\begin{matrix}\alpha \\ \alpha'\end{matrix}\right\} &= \pm \frac{x}{\sqrt{2}} - \frac{\pi}{8} \mp \frac{1}{8\sqrt{2}x} - \frac{1}{16x^2} \mp \frac{25}{384\sqrt{2}x^3} \pm \dots . \end{aligned}\right\} \qquad \text{(18)}$$

The upper signs are to be taken for β and α, and the lower signs for β' and α' in the series last written. To pass from β or α to β' or α' it is only necessary to change the sign of x.

3. Solution for outer return as highly conducting thin tube. Returning now to the solution (10) for the inner coaxial conductor, and putting

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* See Gray and Mathews' *Bessel Functions*, second edition.

q_r for the current per unit area of cross section at distance r from the axis, we have

$$q_r = A(\operatorname{ber} mr + i \operatorname{bei} mr)e^{int}. \qquad \text{.......................(19)}$$

This solution splits into a real part and an imaginary part, and a little consideration shows that each part must satisfy the differential equation. Thus we have the two solutions, with two constants A and B.

$$A(\operatorname{ber} mr \,.\, \cos nt - \operatorname{bei} mr \,.\, \sin nt), \quad B(\operatorname{ber} mr \,.\, \sin nt + \operatorname{bei} mr \cos nt).$$

These two particular solutions taken together are the complete solution, so that

$$q_r = (A \operatorname{ber} mr + B \operatorname{bei} mr)\cos nt + (B \operatorname{ber} mr - A \operatorname{bei} mr)\sin nt, \text{ ...(20)}$$

$$\left.\begin{aligned} &\text{or} & q_r &= (A^2 + B^2)^{\frac{1}{2}}(\operatorname{ber}^2 mr + \operatorname{bei}^2 mr)^{\frac{1}{2}} \cos(nt - \epsilon), \\ &\text{with} & \epsilon &= \tan^{-1}\frac{B \operatorname{ber} mr - A \operatorname{bei} mr}{A \operatorname{ber} mr + B \operatorname{bei} mr}. \end{aligned}\right\} \text{.........(21)}$$

We may simplify this by supposing that the current density along the axis is $q_0 \cos nt$. This gives, since $\operatorname{ber} 0 = 1$, $\operatorname{bei} 0 = 0$, $A = q_0$, $B = 0$. Hence we obtain

$$\left.\begin{aligned} & & q_r &= q_0(\operatorname{ber}^2 mr + \operatorname{bei}^2 mr)^{\frac{1}{2}} \cos(nt - \epsilon), \\ &\text{with} & \epsilon &= \tan^{-1}\frac{-\operatorname{bei} mr}{\operatorname{ber} mr}. \end{aligned}\right\} \text{..............(22)}$$

The value of $\operatorname{ber}^2 mr + \operatorname{bei}^2 mr$ increases rapidly with mr, and all the more rapidly the greater m, that is the greater the frequency. Thus the current density along the axis and near the axis is very small in comparison with the density near the surface of the conductor. The current is therefore confined mainly to the outer part of the conductor, or in the ordinary phrase to the "skin" of the conductor.

The angle ϵ in (21) is the difference of phase between the current at distance r from the axis and the axial current. A comparison of the values of the ber and bei functions as given in the table at the end of this chapter shows that this phase difference changes with r in a very remarkable manner, for which it is difficult to suggest any adequate physical reason.

4. Effective resistance and effective conductance of inner conductor. We now calculate the value, N_1, say, of N external to the surface of the inner conductor. It is to be observed that we are building up here the whole value of N at a surface at a given distance from the axis, not passing, as when N was calculated above, from its value at distance r to its value at distance $r + dr$, within the conductor. If we suppose that the outer conductor is a thin coaxial shell of infinite conductivity, there will be no induction external to the surface of radius b to be taken into account. The current crossing the annular area between the circles of radii a and b is $-\gamma$ if γ be the whole current. We have then

$$N_1 = \mu' \int_a^b H\, dr.$$

But if $a < r < b$, we have, from the value of the current crossing the area considered, $2\pi r H = 4\pi\gamma$, where γ is the whole current. Thus

$$N_1 = 2\gamma\mu' \log\frac{b}{a}$$

and

$$\frac{\partial N_1}{\partial t} = 2\mu' \log\frac{b}{a} \cdot \frac{\partial\gamma}{\partial t}.$$

Thus we obtain from (1), putting P for $\partial V/\partial x$, the space-rate of variation of potential V,

$$P = 2\mu' \log\frac{b}{a} \cdot \frac{\partial\gamma}{\partial t} + \frac{q_a}{k}. \qquad \text{.........................(23)}$$

Now we have seen that

$$q_a = q_0(\text{ber}\, ma \cos nt - \text{bei}\, ma \sin nt). \qquad \text{...............(24)}$$

Also, since

$$\left.\begin{aligned} \int_0^r x\, \text{ber}\, mx \,.\, dx &= \frac{r}{m}\, \text{bei}'\, mr, \\ \int_0^r x\, \text{bei}\, mx \,.\, dx &= -\frac{r}{m}\, \text{ber}'\, mr, \end{aligned}\right\} \qquad \text{....................(25)}$$

we get for the whole current γ in the internal conductor

$$\gamma = 2\pi \int_0^a qr\, dr = 2\pi \frac{q_0}{m}(a\, \text{bei}'\, ma \,.\, \cos nt + a\, \text{ber}'\, ma \,.\, \sin nt), \quad \text{...(26)}$$

and therefore

$$\frac{\partial\gamma}{\partial t} = 2\pi q_0 \frac{n}{m}(a\, \text{ber}'\, ma \,.\, \cos nt - a\, \text{bei}'\, ma \,.\, \sin nt). \qquad \text{...........(27)}$$

We solve (26) and (27) for $q_0 \cos nt$ and $q_0 \sin nt$, and substitute the values in (24), and thence in (23) we get

$$\left.\begin{aligned} q_0 \cos nt &= \frac{m}{2\pi a}\, \frac{1}{\text{ber}'^2\, ma + \text{bei}'^2\, ma}\left(\gamma\, \text{bei}'\, ma + \frac{1}{n}\frac{\partial\gamma}{\partial t}\, \text{bei}'\, ma\right), \\ q_0 \sin nt &= \frac{m}{2\pi a}\, \frac{1}{\text{ber}'^2\, ma + \text{bei}'^2\, ma}\left(\gamma\, \text{ber}'\, ma - \frac{1}{n}\frac{\partial\gamma}{\partial t}\, \text{ber}'\, ma\right), \end{aligned}\right\} \quad \text{...(28)}$$

and from (23), noticing that $m^2 = 4\pi\mu nk$,

$$P = \frac{m}{2\pi ak}\, \frac{\text{ber}\, ma\, \text{bei}'\, ma - \text{bei}\, ma\, \text{ber}'\, ma}{\text{ber}'^2\, ma + \text{bei}'^2\, ma}\, \gamma$$

$$+ \left\{2\mu' \log\frac{b}{a} + \frac{2\mu}{ma}\, \frac{\text{bei}\, ma\, \text{bei}'\, ma + \text{ber}\, ma\, \text{ber}'\, ma}{\text{ber}'^2\, ma - \text{bei}'^2\, ma}\right\}\frac{\partial\gamma}{\partial t}. \quad \text{....(29)}$$

5. Special notation for functions. The following notation for the functions which appear in (29), and some others, is employed by writers on this subject [see Russell's *Alternating Currents*, vol. i. 2nd edition]:

$X(x) = \text{ber}^2\, x + \text{bei}^2\, x.$ $\qquad V(x) = \text{ber}'^2\, x + \text{bei}'^2\, x.$

$Z(x) = \text{ber}\, x\, \text{ber}'\, x + \text{bei}\, x\, \text{bei}'\, x.$ $\qquad W(x) = \text{ber}\, x\, \text{bei}'\, x - \text{bei}\, x\, \text{ber}'\, x.$

$X_1(x) = \text{ker}^2\, x + \text{kei}^2\, x.$ $\qquad V_1(x) = \text{ker}'^2\, x + \text{kei}'^2\, x.$

$S(x) = \text{ber}'\, x\, \text{ker}'\, x + \text{bei}'\, x\, \text{kei}'\, x.$ $\qquad T(x) = \text{bei}'\, x\, \text{ker}'\, x - \text{ber}'\, x\, \text{kei}'\, x.$

The last four have been tabulated for values of the argument x from 0 to 30, by Mr. Harold G. Savidge [*Phil. Mag.* 19, 1916], the functions

$$\frac{x}{2}\frac{W(x)}{V(x)}, \quad \frac{4}{x}\frac{Z(x)}{V(x)},$$

have been computed by the Bureau of Standards, at Washington, for values of x proceeding by steps of 0·1 from 0 to 5, and then by increasing steps from 5 to 100. An abridgment of the tables of the latter functions is given at the end of this chapter.

Using the functions Z, V, W, we write (29) in the form

$$P = \frac{mW(ma)}{2\pi akV(ma)}\gamma + \left\{2\mu' \log\frac{b}{a} + \frac{2\mu Z(ma)}{maV(ma)}\right\}\frac{\partial\gamma}{\partial t}, \quad \text{......(29')}$$

or

$$P = R\gamma + L\frac{\partial\gamma}{\partial t}. \quad \text{..........................(30)}$$

Thus

$$R = \frac{mW(ma)}{2\pi akV(ma)}, \quad L = 2\mu' \log\frac{b}{a} + \frac{2\mu Z(ma)}{maV(ma)}, \quad \text{...........(31)}$$

are the virtual resistance and self-inductance of the inner conductor per unit length in both cases. The special values of $W(ma)/V(ma)$ and $Z(ma)/V(ma)$, for a small and for a large, are given by Russell; but the table in the appendix enables R and L to be calculated with extremely little trouble for a large range of cases.

6. General case of coaxial main. Outside conductor of finite thickness. We now consider the more general case in which the external conductor is not of infinite conductivity, and therefore is not infinitely thin. Putting q' for the current density in the outer conductor at distance r from the axis, and using N', P' instead of N, P, we get the equation

$$P' = \frac{\partial N'}{\partial t} + \frac{q'}{k}, \quad \text{............................(32)}$$

which corresponds to (1) above. Also we obtain an equation

$$\frac{\partial^2 q}{\partial r^2} + \frac{1}{r}\frac{\partial q}{\partial r} = 4\pi\mu k\frac{\partial q'}{\partial t}, \quad \text{.........................(33)}$$

which, if $q' = u'e^{int}$, can be written

$$\frac{\partial^2 u'}{\partial r^2} + \frac{1}{r}\frac{\partial u'}{\partial r} - m^2 iu' = 0. \quad \text{.......................(33')}$$

Proceeding as before we get as a solution of (33),

$$q_r' = \{A(\operatorname{ber} mr + i \operatorname{bei} mr) + B(\operatorname{ker} mr + i \operatorname{kei} mr)\}(\cos nt + i \sin nt).$$

This resolves itself into a real part and an imaginary part which satisfy (33) separately, and therefore yield two real solutions. The sum of these two solutions, each multiplied by a constant, is the complete solution. Thus if, without regard to the use already made of the letters A, B, we take four independent constants A, B, C, D, we can write the complete solution in the form

$$\begin{aligned} q' = (A \operatorname{ber} mr + B \operatorname{bei} mr + C \operatorname{ker} mr + D \operatorname{kei} mr) q_0 \cos nt \\ + (-A \operatorname{bei} mr + B \operatorname{ber} mr - C \operatorname{kei} mr + D \operatorname{ker} mr) q_0 \sin nt, \quad \text{...(34)} \end{aligned}$$

where as before q_0 denotes the current density at the axis of the inner conductor.

By the same reasoning as before we have for a point at distance r from the axis

$$r\frac{\partial N}{\partial r} = -\mu r H, \quad \text{.......................(35)}$$

where N is the magnetic induction (per unit length) external to the cylinder of radius r and H is the magnetic field-intensity at distance r from the axis. But if $\gamma_r{}'$ be the part of the return current external to this cylinder, and γ, as before, be the whole current in the inner conductor, we have

$$H = 2\frac{\gamma - \gamma_r{}'}{r},$$

and therefore $$r\frac{\partial N'}{\partial r} = 2\mu(\gamma - \gamma_r{}') = -4\pi\mu\int_r^c rq'\,dr. \quad \text{..............(35')}$$

Hence by (33) $$\left.\begin{aligned} \frac{1}{k}\frac{\partial q'}{\partial t} &= -\frac{4\pi\mu}{r}\int_r^c r\frac{\partial q'}{\partial t}\,dr, \\ \text{also} \quad \frac{\partial N}{\partial t} &= -2\mu\frac{\partial}{\partial t}\int_r^c \frac{\gamma - \gamma_r}{r}\,dr. \end{aligned}\right\} \quad \text{.....................(36)}$$

7. Determination of constants in the general solution. Similar relations to those expressed in (25) above hold for the ker and kei functions: these may be written

$$\left.\begin{aligned} \int r \operatorname{ker} mr\,dr &= \frac{r}{m}\operatorname{kei}' mr, \\ \int r \operatorname{kei} mr\,dr &= -\frac{r}{m}\operatorname{ker}' mr. \end{aligned}\right\} \quad \text{.....................(37)}$$

The value of q' given by (34), used in (36), gives by (37) the equations for the constants

$$\left.\begin{aligned} A \operatorname{bei}' mc - B \operatorname{ber}' mc + C \operatorname{kei}' mc - D \operatorname{ker}' mc = 0, \\ A \operatorname{ber}' mc + B \operatorname{bei}' mc + C \operatorname{ker}' mc + D \operatorname{kei}' mc = 0, \end{aligned}\right\} \quad \text{.........(38)}$$

which enable us to express A and B each in terms of C and D. We get, writing V_{mc} for $\operatorname{ber}'^2 mc + \operatorname{bei}'^2 mc$ and using also the functions S and T of 5 above,

$$AV(mc) = -CS(mc) + DT(mc), \quad BV(mc) = -CT(mc) + DS(mc). \quad \text{...(39)}$$

For the complete determination of the constants two more relations are required. These are supplied by the condition that the return current must be equal to the total outward current. For the latter we have by (27)

$$\gamma = 2\pi\int_0^a qr\,dr = 2\pi a\frac{q_0}{m}(\mathrm{bei}'\,ma\cos nt + \mathrm{ber}'\,ma\sin nt),$$

and for the former

$$\gamma = 2\pi\int_b^c rq'\,dr,$$

where q' has the value stated in (34). Performing the integration and using (37) and (38), we obtain

$$\left.\begin{aligned}-A\,\mathrm{bei}'\,mb + B\,\mathrm{ber}'\,mb - C\,\mathrm{kei}'\,mb + D\,\mathrm{ker}'\,mb &= \frac{a}{b}\,\mathrm{bei}'\,ma,\\ -A\,\mathrm{ber}'\,mb - B\,\mathrm{bei}'\,mb - C\,\mathrm{ker}'\,mb - D\,\mathrm{kei}'\,mb &= \frac{a}{b}\,\mathrm{ber}'\,ma:\end{aligned}\right\}\ \ldots(40)$$

Substituting in (40) the values of A, B from (39), we find

$$C\{V(mc)\,S(mb) - V(mb)\,S(mc)\} + D\{V(mb)\,T(mc) - V(mc)\,T(mb)\}$$
$$= -\frac{a}{b}\,V(mc)(\mathrm{ber}'\,ma\,\mathrm{ber}'\,mb + \mathrm{bei}'\,ma\,\mathrm{bei}'\,mb),\ \ldots\ldots(41)$$

$$-C\{V(mb)\,T(mc) - V(mc)\,T(mb)\} + D\{V(mc)\,S(mb) - V(mb)\,S(mc)\}$$
$$= -\frac{a}{b}\,V(mc)(\mathrm{ber}'\,ma\,\mathrm{bei}'\,mb - \mathrm{bei}'\,ma\,\mathrm{ber}'\,mb).\ \ldots\ldots(42)$$

These equations give C and D, and then (39) give A and B.

8. Final result in the general case. By the second of (36) and (32), we have

$$P' = -2\mu\frac{\partial}{\partial t}\int_b^c \frac{\gamma - \gamma_r}{r}\,dr + \frac{q_b'}{k},$$

and as before,

$$P = 2\mu'\log\frac{b}{a}\cdot\frac{\partial\gamma}{\partial t} - 2\mu\frac{\partial}{\partial t}\int_b^c \frac{\gamma - \gamma_r}{r}\,dr + \frac{q_a}{k}.\ \ldots\ldots\ldots\ldots(43)$$

Hence, by subtraction,

$$P - P' = \frac{q_a - q_b'}{k} + 2\mu'\log\frac{b}{a}\cdot\frac{\partial\gamma}{\partial t},\ \ldots\ldots\ldots\ldots\ldots\ldots(44)$$

and using the values of $\cos nt$, $\sin nt$, found in (28), we get

$$P - P' = R_1\gamma + L_1\frac{\partial\gamma}{\partial t},$$

with

$$R_1 = R + \frac{m}{2\pi ak\,V(ma)}\{\mathrm{bei}'\,ma(A\,\mathrm{ber}\,mb + B\,\mathrm{bei}\,mb + C\,\mathrm{ker}\,mb + D\,\mathrm{kei}\,mb)$$
$$- \mathrm{ber}'\,ma(A\,\mathrm{bei}\,mb - B\,\mathrm{ber}\,mb + C\,\mathrm{kei}\,mb - D\,\mathrm{ker}\,mb)\},\ \ldots(44')$$

$$L_1 = L + \frac{2\mu}{ma\,V(ma)}\{\mathrm{ber}'\,ma(A\,\mathrm{ber}\,mb + B\,\mathrm{bei}\,mb + C\,\mathrm{ker}\,mb + D\,\mathrm{kei}\,mb)$$
$$+ \mathrm{bei}'\,ma(A\,\mathrm{bei}\,mb - B\,\mathrm{ber}\,mb + C\,\mathrm{kei}\,mb - D\,\mathrm{ker}\,mb)\},\ \ldots(45)$$

where R, L are the virtual resistance and self-inductance found for the inner conductor alone [(31) above]. Equations (44) and (45) show the terms contributed by the outside conductor.

Russell [*Kelvin Lecture*, 1916] has determined the constants A, B, C, D, and has put (44′), (45) in the form

$$\begin{aligned} R_1 = \frac{m}{2\pi ak}\frac{W(ma)}{V(ma)} + \frac{m}{2\pi bk\Delta(b,\ c)} \{ & 2S(mc)(\operatorname{ber} mb \operatorname{kei}' mb - \operatorname{kei} mb \operatorname{ber}' mb) \\ & + 2T(mc)(\operatorname{ber} mb \operatorname{ker}' mb - \operatorname{ker} mb \operatorname{ber}' mb) \\ & + \frac{T(mc)}{mb} - V_1(mc)\, W(mb) - V(mc)\, W_1(mb)\}, \quad \ldots\ldots(44'') \end{aligned}$$

and

$$\begin{aligned} L_1 = 2\mu' \log\frac{b}{a} + \frac{2\mu Z(ma)}{ma V(ma)} & \\ + \frac{2\mu}{mb\Delta(b,\ c)} \{ & 2S(mc)(\operatorname{ker} mb \operatorname{ber}' mb + \operatorname{bei} mb \operatorname{kei}' mb) \\ & + 2T(mc)(\operatorname{ker} mb \operatorname{bei}' mb - \operatorname{ber} mb \operatorname{kei}' mb) \\ & - \frac{S(mc)}{mb} - V_1(mc)\, Z(mb) - V(mc)\, Z_1(mb)\}, \quad \ldots\ldots\ldots(45') \end{aligned}$$

where

$$\Delta(b,\ c) = V_1(mc)\, V(mb) + V(mc)\, V_1(mb) - 2S(mb)\, S(mc) - 2T(mb)\, T(mc).$$

9. Special cases: low frequency and high frequency. We shall consider the relative values of these terms for the two cases, (1) that in which m is small, that is the case of low frequency; (2) that in which m is great, the case of high frequency. The calculations have been made by Russell [*loc. cit.* above], and the results are given in his treatise on *Alternating Currents*, vol. i., and also more fully in his paper, *Phil. Mag.*, April 1909. We quote here the most important. Their verification by the reader will be a somewhat long, but not difficult exercise.

Taking then (1), that is m small, we obtain by (34) and the values of the constants for the approximate value of the ratio of the current density in the outer conductor, at distance r from the axis, to the axial current density,

$$\frac{q'}{q_0} = \frac{a^2}{c^2 - b^2}(1 - \tfrac{1}{2}m^2c^2 \log mr) \cos nt.$$

Thus, as r is increased the current density is diminished.

For case (2), that is m great,

$$\left.\begin{aligned} \frac{q'}{q_0} = \frac{e^{ma/\sqrt{2}}}{\sqrt{2\pi mr}}\left(\frac{a}{b}\right)^{\frac{1}{2}}\left[\frac{\cosh\{\sqrt{2}m(c-r)\} + \cos\{\sqrt{2}m(c-r)\}}{\cosh\{\sqrt{2}m(c-r)\} - \cos\{\sqrt{2}m(c-r)\}}\right]^{\frac{1}{2}} \cos(nt+\theta), \\ \theta = \tan^{-1}\frac{\sin\psi_1 \,.\, e^{-m(c-r)/\sqrt{2}} + \sin\psi_2 \,.\, e^{m(e-r)/\sqrt{2}}}{\cos\psi_1 \,.\, e^{-m(c-r)/\sqrt{2}} + \cos\psi_2 \,.\, e^{m(c-r)/\sqrt{2}}}, \end{aligned}\right\} \quad (46)$$

where

$$\psi_1 = \tan^{-1}\frac{B}{A} - \frac{mr}{\sqrt{2}} + \frac{\pi}{8}, \quad \psi_2 = \tan^{-1}\frac{D}{C} + \frac{mr}{\sqrt{2}} + \frac{\pi}{8}. \quad \ldots\ldots\ldots(47)$$

The quantity

$$\left(\frac{1}{r}\right)^{\frac{1}{2}}[\cosh\{m(c-r)\sqrt{2}\}+\cos\{m(c-r)\sqrt{2}\}]^{\frac{1}{2}}$$

diminishes as r increases, and so when the frequency is very high the current is almost entirely confined to a thin layer on the outside of the inner conductor and a thin layer on the inside of the outer conductor.

10. Case of *mc* not greater than 2. When mc is not greater than 2 the value of R_1 is given by the following approximate formula. Writing

$$\sigma=\frac{(7c^2-b^2)(c^2-b^2)}{192},\quad \sigma_1=\frac{b^2c^4}{8(c^2-b^2)},\quad \sigma_2=-\frac{b^2c^6}{4(c^2-b^2)^2},$$
$$\xi=\log\frac{c}{b},\ \text{.................................}(48)$$

we have in this case

$$R_1=\frac{1}{\pi a^2k}\left(1+\frac{m^4a^4}{192}\right)+\frac{1}{\pi k(c^2-b^2)}\{1+m^4(\sigma+\sigma_1\xi+\sigma_2\xi^2)\}.\ \text{...}(49)$$

Again, if we write

$$\left.\begin{aligned}&\sigma'=\frac{19c^6+103c^4b^2-41b^4c^2+3b^2}{2^2.\,4^2.\,6^2(c^2-b^2)},\quad \sigma_1'=-\frac{14b^2c^4(2c^2-b^2)}{2^2.\,4^2.\,3(c^2-b^2)^2},\\&\sigma_2'=-\frac{b^4c^6}{4(c^2-b^2)^3},\quad \sigma_3'=\frac{b^4c^8}{2(c^2-b^2)^4},\quad \xi=\log\frac{c}{b},\end{aligned}\right\}\ \text{...}(50)$$

we have $$L_1=L_d-\tfrac{1}{2}\mu\frac{a^4m^4}{192}-\mu(\sigma'+\sigma_1'\xi+\sigma_2'\xi^2-\sigma_3'\xi^3),\ \text{............}(51)$$

where L_d is the value of L for steady currents in the two conductors.

For steady ("direct") currents

$$\left.\begin{aligned}&R_d=\frac{1}{\pi a^2k}+\frac{1}{\pi(c^2-b^2)k},\\&L_d=2\mu'\log\frac{b}{a}+\tfrac{1}{2}\mu+\frac{2\mu c^4}{(c^2-b^2)^2}\log\frac{c}{b}-\mu\frac{3c^2-b^2}{2(c^2-b^2)}.\end{aligned}\right\}\ \text{.........}(52)$$

[See Chap. XIII. below.]

In cables for power transmission the section of the inner conductor is made equal to that of the outer, that is $a^2=c^2-b^2$. In this case an increase in the value of b means a diminution of $\sigma+\sigma_1\xi+\sigma_2\xi^2$, in fact the return current is on the whole removed to a greater distance from the direct current, and if the frequency is not too high, say less than 50, the increase in the resistance of unit length of the outer conductor, represented by $m^4(\sigma+\sigma_1\xi+\sigma_2\xi^2)/\pi a^2k$ is negligibly small. It will be noticed however that the term in m^2 has disappeared from the first part of R_1.

11. "Skin effect" in practical cases. Using this formula we can find the magnitude of the so-called "skin effect" in a practical case. For simplicity we take the radius of the inner conductor as 1 centimetre, and if the cable be worked at high "pressure," b may be taken as 2·4 and c as 2·6. This gives for unit length of the cable

$$R=\frac{1}{\pi k}\left(1+\frac{m^4}{192}\right)+\frac{1}{\pi k}\left(1+\frac{0{\cdot}0072m^4}{192}\right). \quad \text{..............(53)}$$

A frequency of 20 gives with high conductivity copper, for which we take $\mu=1$, $m=1$ very approximately. Then

$$R=\frac{1}{mk}\left(1+\frac{1}{192}\right)+\frac{1}{\pi k}\left(1+0{\cdot}000037\right). \quad \text{..............(54)}$$

Thus the increase of resistance is 0·5 p.c. in the inner conductor combined with an increase of about ·0037 p.c. in that of the outer conductor.

For a low voltage cable b and c might be 4/3 and 5/3 respectively. The first part of R would be the same as here given, the percentage increase in the second part would be raised to about 0·03.

12. Case of *ma* greater than 5. When ma is greater than 5, the formulae are

$$\begin{aligned}R_1=&\frac{m}{2\pi ak}\left(\frac{1}{\sqrt{2}}+\frac{1}{2ma}+\frac{3}{8\sqrt{2}m^2a^2}\right)\\&+\frac{m}{2\sqrt{2}\pi bk}\,\frac{\sinh\{\sqrt{2}m(c-b)\}+\sin\{\sqrt{2}m(c-b)\}}{\cosh\{\sqrt{2}m(c-b)\}-\cos\{\sqrt{2}m(c-b)\}}, \quad \text{...(55)}\end{aligned}$$

$$\begin{aligned}L_1=&2\mu'\log\frac{b}{a}+\frac{2\mu}{ma}\left(\frac{1}{\sqrt{2}}-\frac{3}{8\sqrt{2}m^2a^2}-\frac{3}{8m^3a^3}\right)\\&+\frac{2\mu}{mb\sqrt{2}}\,\frac{\sinh\{\sqrt{2}m(c-b)\}-\sin\{\sqrt{2}m(c-b)\}}{\cosh\{\sqrt{2}m(c-b)\}-\cos\{\sqrt{2}m(c-b)\}}. \quad \text{......(56)}\end{aligned}$$

For ma very great and $f=$ frequency these formulae may be replaced by

$$\left.\begin{aligned}R_1&=\frac{m}{2\sqrt{2}k\pi}\left(\frac{1}{a}+\frac{1}{b}\right)=\sqrt{\frac{\mu f}{k}}\left(\frac{1}{a}+\frac{1}{b}\right),\\L_1&=2\mu'\log\frac{b}{a}+\frac{\mu}{2\pi}\frac{1}{\sqrt{\mu kf}}\left(\frac{1}{a}+\frac{1}{b}\right).\end{aligned}\right\} \quad \text{..............(57)}$$

Thus, as f increases, R_1 increases but L_1 diminishes towards the value $2\mu'\log(b/a)$, which is continually approached without limit of closeness as f is increased without limit. The state thus approached is one of concentration of the currents in the surface strata, on the outside of the inner and the inside of the outer conductor.

13. Inner conductor a hollow tube. The case in which the inner conductor is a hollow tube is interesting, but its complete solution is somewhat complicated. Let a_2 be the outer radius of the inner conductor and a_1 its inner radius; we get easily the differential equation

$$r\frac{\partial q}{\partial r}=\frac{m^2}{n}\int_{a_1}^{r} r\frac{\partial q}{\partial t}\,dr. \quad \text{.......... ....(58)}$$

If we suppose that $q=q_0\cos nt$ is the current density at the inner surface of the tube, we have for the density at distance r $(a_1 < r < a_2)$ from the axis,

$$q = (A \operatorname{ber} mr + B \operatorname{bei} mr + C \operatorname{ker} mr + D \operatorname{kei} mr) q_0 \cos nt,$$
$$+ (-A \operatorname{bei} mr + B \operatorname{ber} mr - C \operatorname{kei} mr + D \operatorname{ker} mr) q_0 \sin t, \quad \ldots\ldots(59)$$

with the conditions

$$\left.\begin{aligned} A \operatorname{ber} ma_1 + B \operatorname{bei} ma_1 + C \operatorname{ker} ma_1 + D \operatorname{kei} ma_1 = 1, \\ A \operatorname{bei} ma_1 - B \operatorname{ber} ma_1 + C \operatorname{kei} ma_1 - D \operatorname{ker} ma_1 = 0. \end{aligned}\right\} \quad \ldots\ldots(60)$$

From these we obtain

$$\left.\begin{aligned} A \operatorname{ber}' ma_1 + B \operatorname{bei}' ma_1 + C \operatorname{ker}' ma_1 + D \operatorname{kei}' ma_1 = 0, \\ A \operatorname{bei}' ma_1 - B \operatorname{ber}' ma_1 + C \operatorname{kei}' ma_1 - D \operatorname{ker}' ma_1 = 0. \end{aligned}\right\} \quad \ldots\ldots(61)$$

Equations (60) and (61) determine the constants A, B, C, D. Their values are found [Russell, *Eighth Kelvin Lecture*, 1916] to be

$$A = -ma_1 \operatorname{ker}' ma_1, \quad B = ma_1 \operatorname{kei}' ma_1, \quad C = ma_1 \operatorname{ber}' ma_1,$$
$$D = -ma_1 \operatorname{bei}' ma_1. \quad \ldots\ldots(62)$$

Thus q can be found at once from (59).

The following result may be verified: when $(r - a_1)/a_1$ is not greater than $\frac{1}{4}$, and mr is not greater than 2, the density q_r of the current at distance r from the axis may be taken as given by

$$q_r = q_{a_1} \left\{ 1 + \frac{m^4 (r - a_1)^4}{12} \right\}, \quad \ldots\ldots(63)$$

so that q_r increases as r increases.

Russell has also found that if R denote effective resistance of a length l of the hollow conductor, we have

$$R = \frac{ml}{2\pi a_2 k \Delta(a_1, a_2)} [2S(ma_1)(\operatorname{ber} ma_2 \operatorname{kei}' ma_2 - \operatorname{kei} ma_2 \operatorname{ber}' ma_2)$$
$$+ 2T(ma_1)(\operatorname{ber} ma_2 \operatorname{ker}' ma_2 - \operatorname{ker} ma_2 \operatorname{ber}' ma_2)$$
$$+ \frac{T(ma_1)}{ma_2} - V_1(ma_1) W(ma_2) - V(ma_1) W_1(ma_2)], \quad \ldots\ldots(64)$$

where S, T, V, V_1, W, W_1 (and Z, Z_1 below) are the functions defined in 5 above, and

$$\Delta(a_1, a_2) = V_1(ma_2) V(ma_1) + V(ma_2) V_1(ma_1)$$
$$- 2S(ma_1) S(ma_2) - 2T(ma_1) T(ma_2). \quad \ldots\ldots(65)$$

He has likewise given, for the part of the effective inductance for a length l of this conductor due to magnetic induction in the inner conductor, the equation

$$L_1 = \frac{2\mu l}{ma_2 \Delta(a_1, a_2)} [2Sma_1 (\operatorname{ker} ma_2 \operatorname{ber}' ma_2 + \operatorname{bei} ma_2 \operatorname{kei}' ma_2)$$
$$+ 2T(ma_1)(\operatorname{ker} ma_2 \operatorname{bei}' ma_2 - \operatorname{ber} ma_2 \operatorname{kei}' ma_2)$$
$$- \frac{S(ma_1)}{ma_2} - V_1(ma_1) Z(ma_2) - V(ma_1) Z_1(ma_2)]. \quad \ldots\ldots(66)$$

If mc is not greater than 2 and $(a_2 - a_1)/a_1$ is not greater than $\frac{1}{4}$, the effective resistance of a length l of the double conductor is given by

$$R = \frac{l}{\pi k(a_2{}^2 - a_1{}^2)}\left\{1 + \frac{m^4(a_2 - a_1)^4}{45}\right\} + \frac{l}{\pi k(c^2 + b^2)}\left\{1 + \frac{m^4(c - b)^4}{45}\right\} \text{.....(67)}$$

14. Two parallel wires. The case of two parallel wires, an outward and a return conductor, is important, but its exact solution is a matter of considerable difficulty.* But for a considerable range of practical cases the following approximate solution is sufficient. Let first the wires be of equal radius a, and their axes be at a distance c apart; then if c/a be great enough the current in each conductor may be regarded as symmetrically distributed about the axis of the conductor according to the law expressed in (24) above. In what follows the capacity current is neglected. To find the self-inductance of the circuit per unit length we have only to add to the magnetic induction through the circuit the correction given by the second term of the value of L set forth in (31). The magnetic induction through the circuit is shown by XIII. 16 below to be $4\mu'\gamma^2 \log (c/a)$. Thus

$$L_1 = 4\mu' \log \frac{c}{a} + \frac{4\mu}{ma}\frac{Z(ma)}{V(ma)}. \text{ ....(68)}$$

The resistance per unit length, for the two wires, is given by

$$R_1 = \frac{mW(ma)}{\pi akV(ma)}. \text{ ....(69)}$$

For solid wires of radii a, b, at distance c measured between their axes, the formulae are

$$\left.\begin{aligned} & L_1 = 2\mu' \log \frac{c^2}{ab} + \frac{2\mu}{ma}\frac{Z(ma)}{V(ma)} + \frac{2\mu}{mb}\frac{Zmb}{Vmb} \\ \text{and}\qquad & R_1 = \frac{mW(ma)}{2\pi akV(ma)} + \frac{mW(mb)}{2\pi bkV(mb)}. \end{aligned}\right\} \text{ ....(70)}$$

Formulae of correction given in the paper of Nicholson, cited above, have been used at the Bureau of Standards at Washington for the computation of the error involved in using these approximate formulae for L_1 and R_1. It was found that for two equal wires of radius 0·1 cm, and a distance of 1 cm between the axes, the change of inductance produced by a frequency of 10^6 was $-8{\cdot}5$ per cent., and that Nicholson's correction reduced this by only 9 parts in 10000.†

In the same case the ratio of the virtual resistance to the steady current resistance, which by the formula was 7·56, was reduced to 7·55.

For very high frequencies and equal radii the formula

$$L_1 = \tfrac{1}{4}\mu' \log \frac{c + (c^2 - 4a^2)^{\frac{1}{2}}}{2a} \text{ ....(71)}$$

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* See a paper by Professor J. W. Nicholson, *Phil. Mag.* 19 (1909), also *Note*, p. 284.

† Rosa and Grover, *B.B.S.W.* Vol. 8, No. 1.

may be regarded as exact. It will be observed that when $c=2a$, that is when the wires are in contact, L_1 is zero. The currents are now collected in two infinitely thin filaments along the line of contact, so that there is no magnetic induction enclosed by them.

15. A ring conductor of circular section. For the case of a ring conductor of circular section we can only give results for the extreme cases of very low and very high frequency. If the ring (of mean radius a) be a tube bent into a circle, that is if the cross section is an annulus of inner radius ρ_1 and outer radius ρ_2, the self-inductance of the conductor for uniformly distributed current (which can be found by integration of Maxwell's formula, VI. 18 (53)) is

$$L=4\pi a\left\{\left(1+\frac{\rho_1^2+\rho_2^2}{8a^2}\right)\log\frac{8a}{\rho_2}-\frac{7}{4}+\frac{2\rho_2^2+\rho_1^2}{32a^2}-\frac{\rho_1^2}{2(\rho_2^2-\rho_1^2)}\right.$$
$$\left.+\frac{\rho_1^4}{(\rho_2^2-\rho_1^2)^2}\left(1+\frac{\rho_1^2}{8a^2}\right)\log\frac{\rho_2}{\rho_1}-\frac{\rho_1^4+\rho_1^2\rho_2^2+\rho_2^4}{48a^2(\rho_2^2-\rho_1^2)}\right\}, \quad \text{.....(72)}$$

if terms of higher order than $(\rho_2/a)^2$ and $(\rho_1/a)^2$ are rejected.

This gives for a tube with infinitely thin walls

$$L=4\pi a\left\{\left(1+\frac{\rho^2}{4a^2}\right)\log\frac{8a}{\rho}-2\right\}. \quad \text{...................(73)}$$

For a solid ring of radius ρ in which an alternating current of infinite frequency is flowing, and in which therefore the current is confined to an infinitely thin stratum of the surface, L is approximately the self-inductance. The value of a/ρ must however be so great that the difference of current densities between the maximum and minimum circles may be neglected. [For steady currents, see XV. 22 (30).]

If such a ring conductor revolve uniformly about a diameter in a uniform magnetic field, alternating currents following the simple harmonic law of variation will be produced in it. If it revolve with extremely great rapidity L will be given by (73). If there is no flow of current from one coaxial filament to another, and the ring revolve very slowly, the current will be distributed over the whole cross section so that the current density is proportional to the distance of the filament considered from the axis. In this case we have

$$L=4\pi a\left\{\left(1+\frac{3}{8}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-0{\cdot}092\frac{\rho^2}{a^2}-\frac{7}{4}\right\}. \quad \text{.............(74)}$$

16. Main consisting of two flat conducting strips with insulating separator. We now consider the case of two coaxial cylindrical conductors of equal thickness and of radii differing by a small fraction of either radius, and so large that the influence of the curvature on the current, etc., at any point may be neglected. This is the case of two parallel infinitely long, infinitely broad, and equally thick plane strips of conducting material, facing one another and containing between them a stratum of uniform dielectric. We suppose one to carry the outward current, the other to be the return conductor.

If we neglect, as we have done hitherto, all condenser action, we get at once the differential equation which holds in each strip by putting in the second term on the left of (5) $\rho = \infty$, so that we obtain

$$\frac{\partial^2 q}{\partial r^2} = 4\pi\mu k \frac{\partial q}{\partial t}. \quad \text{......(75)}$$

We must now regard any small step dr along r as a space step at right angles to the plane faces of the strips, and so we may take r as the distance of the point considered from a chosen plane of reference taken parallel to these faces. We shall take as this plane of reference the plane midway between the opposed faces of the two slabs.

If we write $q = ue^{int}$, where u is a function of r only, we have

$$\frac{\partial q}{\partial t} = inq,$$

so that (65) becomes

$$\left.\begin{aligned} \frac{\partial^2 q}{\partial r^2} &= 4\pi\mu kinq = m_1{}^2 q, \\ \text{if} \quad m_1{}^2 &= 4\pi\mu kin. \end{aligned}\right\} \quad \text{......(76)}$$

It will be observed that $m_1{}^2 = m^2 i$.

It will be convenient to take as the solution of (76)

$$q = Ae^{m_1(r-a)} + Be^{-m_1(r-a)}, \quad \text{......(77)}$$

where A and B are constants to be determined by the conditions of the problem.

We consider two points on one of the slabs, a point on the face opposed to the other slab, and a point on the back of the slab. At the latter point the magnetic field-intensity H, which, in the slab, is everywhere parallel to the slab faces, and at right angles to the current, is zero, while at the former point we have the condition

$$P - \frac{\partial N}{\partial t} = \frac{q_a}{k}, \quad \text{......(78)}$$

where P is, as before, the electromotive intensity at the point.

Differentiating with respect to r we get

$$-\frac{\partial}{\partial t}\frac{\partial N}{\partial r} = \frac{1}{k}\frac{\partial q}{\partial r},$$

that is

$$\frac{\partial H_a}{\partial t} = \frac{1}{k}\frac{\partial q}{\partial r} = \frac{m_1}{k}(A - B).$$

But, if γ as usual denote the total current,

$$H_a = 4\pi\mu\gamma, \quad \text{......(79)}$$

and therefore

$$\frac{\partial H_a}{\partial t} = 4\pi\mu\frac{\partial \gamma}{\partial t} = 4\pi\mu in\gamma, \quad \text{......(80)}$$

since everywhere the current is a simple harmonic function of the time.

Hence

$$4\pi\mu\frac{\partial \gamma}{\partial t} = \frac{m_1}{k}(A - B). \quad \text{......(81)}$$

Now at the point on the back of the slab we have, since there and at all other outside points $H=0$,

$$\frac{\partial H}{\partial t}=\frac{1}{k}\frac{\partial q}{\partial r}=\frac{m_1}{k}\left(Ae^{m_1 b}-Be^{-m_1 b}\right)=0. \qquad\qquad (82)$$

17. The complete solution and its realization. Equations (81) and (82) give for the constants the values

$$\left.\begin{aligned} A&=\frac{4\pi\mu k}{m_1}\frac{e^{-m_1 b}}{e^{m_1 b}-e^{-m_1 b}}\frac{\partial\gamma}{\partial t},\\ B&=\frac{4\pi\mu k}{m_1}\frac{e^{m_1 b}}{e^{m_1 b}-e^{-m_1 b}}\frac{\partial\gamma}{\partial t}.\end{aligned}\right\} \qquad\qquad (83)$$

Now, returning to the other face of the slab, we have for N at the point the equation

$$N=\int_{+a}^{-a}H\,dr+\int_{-a}^{-(b+a)}H\,dr=8\pi\mu' a\gamma+\int_{-a}^{-(b+a)}H\,dr.$$

Thus

$$\frac{\partial N}{\partial t}=8\pi\mu' a\frac{\partial\gamma}{\partial t}+\int_{-a}^{-(b+a)}\frac{\partial H}{\partial t}dr \qquad\qquad (84)$$

and

$$P_a=8\pi\mu' a\frac{\partial\gamma}{\partial t}+\int_{-a}^{-(b+a)}\frac{\partial H}{\partial t}dr+\frac{q_a}{r}. \qquad\qquad (85)$$

With this we get for the corresponding point on the face of the opposite slab

$$P_{-a}=\int_{-a}^{-(b+a)}\frac{\partial H}{\partial t}dr+\frac{q_{-a}}{k}, \qquad\qquad (86)$$

and therefore

$$P_a-P_{-a}=8\pi\mu' a\frac{\partial\gamma}{\partial t}+\frac{q_a-q_{-a}}{k}, \qquad\qquad (87)$$

But, clearly $P_{-a}=-P_a$ and $q_{-a}=-q_a$, so that we obtain

$$P_a=4\pi\mu' a\frac{\partial\gamma}{\partial t}+\frac{q_a}{k}. \qquad\qquad (88)$$

Now, from (77),

$$\frac{q_a}{k}=\frac{1}{k}(A+B).$$

Using the values of A, B, from (83), we get instead of (88)

$$P_a=\left(4\pi\mu' a+\frac{4\pi\mu k}{m_1}\frac{e^{m_1 b}+e^{-m_1 b}}{e^{m_1 b}-e^{-m_1 b}}\right)\frac{\partial\gamma}{\partial t}. \qquad\qquad (89)$$

The quantity in brackets on the right is partly real, partly imaginary, since m_1 is complex. The equation may of course be written

$$P_a=in\left(4\pi\mu' a+\frac{4\pi\mu k}{m_1}\frac{e^{m_1 b}+e^{-m_1 b}}{e^{m_1 b}-e^{-m_1 b}}\right)\gamma. \qquad\qquad (90)$$

But $m_1=m\sqrt{i}=m(1+i)/\sqrt{2}$. Using this value of m_1 and reducing, we get

$$\frac{1}{m_1}\frac{e^{m_1 b}+e^{-m_1 b}}{e^{m_1 b}-e^{-m_1 b}}=\frac{\sinh(\sqrt{2}mb)-\sin(\sqrt{2}mb)-i\{\sinh(\sqrt{2}mb)+\sin(\sqrt{2}mb)\}}{\sqrt{2}m(\cosh\sqrt{2}mb-\cos\sqrt{2}mb)},$$

and therefore

$$P_a = \left\{ 4\pi\mu' a + \sqrt{\frac{2\pi\mu}{nk}}\, k \frac{\sinh(\sqrt{2}mb) - \sin(\sqrt{2}mb)}{\cosh(\sqrt{2}mb) - \cos(\sqrt{2}mb)} \right\} \frac{\partial\gamma}{\partial t}$$
$$+ \sqrt{2\pi\mu k n}\, \frac{\sinh(\sqrt{2}mb) + \sin(\sqrt{2}mb)}{\cosh(\sqrt{2}mb) - \cos(\sqrt{2}mb)}\, \gamma. \quad \text{.........(91)}$$

The effective resistance and inductance per unit length for each slab are therefore

$$\left. \begin{aligned} R_1 &= \sqrt{2\pi\mu n k}\, \frac{\sinh(\sqrt{2}mb) + \sin(\sqrt{2}mb)}{\cosh(\sqrt{2}mb) - \cos(\sqrt{2}mb)}, \\ L_1 &= 4\pi\mu' a + \sqrt{\frac{2\pi\mu}{nk}}\, k \frac{\sinh(\sqrt{2}mb) - \sin(\sqrt{2}mb)}{\cosh(\sqrt{2}mb) - \cos(\sqrt{2}mb)}. \end{aligned} \right\} \quad \text{......(92)}$$

18. Particular cases: high frequency and low frequency. The chief interest of the problem lies in the comparison of the values of R_1, L_1 for very slow and very rapid alternations. Taking $\mu' = 1$ we find by expansion of the exponentials in (80) that when n is small, the resistance and inductance for a length l of the slab are given by

$$R_1\gamma + L_1 \frac{\partial\gamma}{\partial t} = \frac{l}{bk}\gamma + inl\left(4\pi a + \frac{4}{3}\pi\mu b\right)\gamma, \quad \text{...........(93)}$$

so that
$$R_1 = \frac{l}{bk}, \quad L_1 = l\left(4\pi a + \frac{4}{3}\pi\mu b\right). \quad \text{..............(94)}$$

If the strips be very close so that $a = 0$, we have

$$P = \frac{l}{bk}\gamma + \tfrac{4}{3} in\pi\mu b l\gamma, \quad \text{.................... ......(95)}$$

and the resistance is the same as before. The self-inductance is $\frac{4}{3}\pi\mu bl$ for a strip of unit breadth, so that for a strip of breadth $2\pi r$ it is

$$L_1 = \tfrac{2}{3}\mu\frac{bl}{r}, \quad \text{.................................(96)}$$

which is half the result obtainable by direct calculation, as in XIII. 7 below, for steady currents in two close coaxial cylinders at a distance apart small in comparison with their radii. This is as it should be, since there the self-inductance would be found for what in the present reckoning is a length $2l$, viz., a length l in the outgoing, and an equal length in the return strip.

Now let n be very great. Then
$$(e^{m_1 b} + e^{-m_1 b})/(e^{m_1 b} - e^{-m_1 b}) = (1 + e^{-2m_1 b})/(1 - e^{-2m_1 b}) = 1,$$

so that for this case

$$R_1\gamma + L_1 \frac{\partial\gamma}{\partial t} = \frac{l}{k}\sqrt{2\pi\mu n k}\,\gamma + inl\left(4\pi a + \sqrt{\frac{2\pi\mu}{nk}}\right)\gamma. \quad \text{.........(97)}$$

Thus
$$\left.\begin{aligned} R_1 &= \frac{l}{k}\sqrt{2\pi\mu nk}, \\ L_1 &= l\left(4\pi a + \sqrt{\frac{2\pi\mu}{nk}}\right). \end{aligned}\right\} \quad \text{......................(98)}$$

We conclude that as n is increased the resistance is increased without limit, while L_1 diminishes towards the limit $4\pi al$. The result shows moreover that the thickness of the strip which would give the same resistance is $1/\sqrt{2\pi\mu nk}$, which agrees with the result obtained at p. 270 above. This may be taken as the effective thickness of the conductor. It diminishes as $\sqrt{n}$ increases.

When the thickness is great so that b may be regarded as infinite, we have by (77), since $-4\pi q = \partial H/\partial r$,

$$\begin{aligned} q &= me^{-m(r-a)}\gamma \\ &= \alpha(1+i)e^{-\alpha(1+i)(r-a)}, \end{aligned}$$

if we denote $\sqrt{2\pi\mu nk}$ by α. This may be written

$$q = \sqrt{2}\alpha\gamma e^{-\alpha(r-a)}\left[\cos\left\{\tfrac{1}{4}\pi - \alpha(r-a)\right\} + i\sin\left\{\tfrac{1}{4}\pi - \alpha(r-a)\right\}\right]. \quad (99)$$

19. General dynamical theory of effects of constraints. Lord Rayleigh[*] showed that the restriction of a rapidly varying current to the outer strata of the conductor is a consequence of a general dynamical principle which regulates the effects of constraints on the motion of a material system. This principle is embodied in two general theorems due to Thomson and Bertrand respectively. Thomson's theorem asserts that if any material system given at rest be suddenly set in motion with any specified velocities (possible under the kinematical conditions of the system) imposed on certain parts of the system, the other parts being left free to take such velocities as result from the connections, the resulting motion is that for which the kinetic energy has the smallest possible value consistent with fulfilment of the prescribed velocity conditions. Bertrand's theorem, on the other hand, asserts that if the impulses applied to certain parts of the system be specified, the resulting motion is that for which the kinetic energy has the greatest value consistent with the prescribed condition as to impulses.

Taking any case in which we consider a system impulsively set into motion with a single specified velocity, or with a specified impulse of the same type. Let Φ denote the impulsive force, then the impulse is the time-integral

$$\int_0^\tau \Phi\, dt = \Phi;$$

and the corresponding velocity generated is $\dot{\phi}$. The resulting kinetic energy T is $\frac{1}{2}\dot{\phi}\Phi$.

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[*] *Phil. Mag.* May 1886.

According to Thomson's theorem the introduction of any constraint limiting the freedom of the system causes an increase of T if $\dot{\phi}$ be given. On the other hand, according to Bertrand's theorem, if Φ_1 be given, the effect of the constraint will be to diminish T. In both cases the ratio $2T/\dot{\phi}^2$, or $\Phi/\dot{\phi}$, is increased, for in the former case Φ is increased, and in the latter $\dot{\phi}$ is diminished. Thus the effect of constraint is in each case to increase the generalized inertia-coefficient corresponding to the coordinate in question.

Consider now a system in which a force Φ_1 of type corresponding to the coordinate $\dot{\phi}_1$, and varying according to a simple harmonic function of the time, is applied to the system. Suppose the system to have no potential energy, and to be subject to dissipative forces given (according to the rule in VIII. 3 above) by a dissipation function F, which is a homogeneous quadratic function of the generalized velocities of the system. Let, further, the remaining coordinates $\dot{\phi}_2, \dot{\phi}_3, \dots \dot{\phi}_m$ of the system be so chosen that no product of them enters into the expressions of T and F.

$$\left.\begin{aligned} T &= \tfrac{1}{2}(a_{11}\dot{\phi}_1{}^2 + a_{22}\dot{\phi}_2{}^2 + \dots + 2a_{12}\dot{\phi}_1\dot{\phi}_2 + 2a_{13}\dot{\phi}_1\dot{\phi}_2 + \dots),\\ F &= \tfrac{1}{2}(b_{11}\dot{\phi}_1{}^2 + b_{22}\dot{\phi}_2{}^2 + \dots + 2b_{12}\dot{\phi}_1\dot{\phi}_2 + 2b_{13}\dot{\phi}_1\dot{\phi}_2 + \dots).\end{aligned}\right\} \quad \dots(100)$$

But by Lagrange's equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}_1} + \frac{\partial F}{\partial \dot{\phi}_1} = \Phi_1,$$

and hence by (91)

$$\left.\begin{aligned} &a_{11}\ddot{\phi}_1 + a_{12}\ddot{\phi}_2 + a_{13}\ddot{\phi}_3 + \dots + b_{11}\dot{\phi}_1 + b_{12}\dot{\phi}_2 + \dots = \Phi_1,\\ &a_{12}\ddot{\phi}_1 + a_{22}\ddot{\phi}_2 + b_{12}\dot{\phi}_1 + b_{22}\dot{\phi}_2 = 0,\\ &a_{13}\ddot{\phi}_1 + a_{33}\ddot{\phi}_3 + b_{13}\dot{\phi}_1 + b_{33}\dot{\phi}_3 = 0.\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned}\right\} \quad \dots\dots(101)$$

Let now the whole motion be simple harmonic in the period of the force Φ_1. Representing the latter by e^{int} we get instead of (92)

$$\left.\begin{aligned} &(ina_{11} + b_{11})\dot{\phi}_1 + (ina_{12} + b_{12})\dot{\phi}_2 + (ina_{13} + b_{13})\dot{\phi}_3 + \dots = \Phi_1,\\ &(ina_{12} + b_{12})\dot{\phi}_1 + (ina_{22} + b_{22})\dot{\phi}_2 = 0,\\ &(ina_{13} + b_{13})\dot{\phi}_1 + (ina_{33} + b_{33})\dot{\phi}_3 = 0.\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned}\right\} \quad \dots(102)$$

The second and following equations of (102) give $\dot{\phi}_2$, $\dot{\phi}_3$, etc., in terms of $\dot{\phi}_1$, and these values substituted in the first equation of (102) yield

$$\frac{\Phi_1}{\dot{\phi}_1} = ina_{11} + b_{11} - \frac{(ina_{12} + b_{12})^2}{ina_{22} + b_{22}} - \frac{(ina_{13} + b_{13})^2}{ina_{33} + b_{33}} - \dots \quad \dots(103)$$

20. Effective "resistance," and "inertia" (or "inductance") of system. Since to a constant factor $\dot{\phi}_1$ is represented by e^{int} it is clear that $(\partial F/\partial \dot{\phi})/\dot{\phi}$ is the real part of $\Phi/\dot{\phi}_1$, and therefore corresponds to the dissipative force. Calling this quantity R' and the other inL', we have instead of (103)

$$\Phi_1 = (R' + inL')\dot{\phi}_1. \qquad \text{.........................(104)}$$

R' may be called the resistance of the system, and L' the generalized inertia-coefficient, or what corresponds to the self-inductance in the electrical theory.

To calculate R' we have to find the real parts of the successive terms in (94). Now

$$\text{real part of } \frac{(ina_{12} + b_{12})^2}{ina_{22} + b_{22}} = \text{real part of } \frac{(ina_{12} + b_{12})^2 (b_{22} - ina_{22})}{b_{22}^2 + n^2 a_{22}^2}$$

$$= \frac{b_{12}^2}{b_{22}} - n^2 \frac{(a_{12}b_{22} - a_{22}b_{12})^2}{b_{22}(b_{22}^2 + n^2 a_{22}^2)},$$

and similarly the real parts of the other terms may be found. Hence

$$R' = b_{11} - \sum_{i=2}^{i=m} \frac{b_{1i}^2}{b_{ii}} + n^2 \sum_{i=2}^{i=m} \frac{(a_{1i}b_{ii} - a_{ii}b_{1i})^2}{b_{ii}(b_{ii}^2 + n^2 a_{ii})^2}. \qquad \text{............(105)}$$

It is clear that each term of the second series in this expression increases as n increases, that is, as the frequency increases. It follows that the value of R' increases with the frequency. When n is very great R' approaches the limiting value

$$b_{11} - \sum_{i=2}^{i=m} \frac{b_{ii}^2}{b_{ii}^2} - \sum_{i=2}^{i=m} \frac{(a_{1i}b_{ii} - a_{ii}b_{1i})^2}{b_{ii}a_{ii}^2}.$$

When n is small R' is approximately equal to the first two terms in (105), and is an absolute minimum for steady and for continuous non-periodic motions.

The imaginary part of (103) is easily found to be

$$in\left\{a_{11} - \sum_{i=2}^{i=m} \frac{a_{1i}^2}{a_{ii}^2} + \sum_{i=2}^{i=m} \frac{(a_{1i}b_{ii} - a_{ii}b_{1i})^2}{a_{ii}(b_{ii}^2 + n^2 a_{ii}^2)}\right\}.$$

Hence

$$L' = a_{11} - \sum_{i=2}^{i=m} \frac{a_{ii}^2}{a_{ii}^2} + \sum_{i=2}^{i=m} \frac{a_{1i}b_{ii} - a_{ii}b_{1i})^2}{a_{ii}(b_{ii}^2 + n^2 a_{ii}^2)}. \qquad \text{...............(106)}$$

Each term of the second series in (106) is positive, and continually diminishes as n increases. Hence as n increases L' approaches more and more nearly the value

$$a_{11} - \sum_{i=2}^{i=m} \frac{a_{ii}^2}{a_{ii}^2},$$

which is independent of dissipative terms.

In these results we have, as Lord Rayleigh pointed out, an analogue to Thomson's theorem. In the absence of constraints, R_1 is great and L_1 a minimum when the vibrations are very rapid, and on the other hand L_1 is great and R_1 a minimum when the vibrations are very slow.

21. Electrical problems. Primary and secondary circuits. The application of these results to electrical problems is obvious. Let us take the case, already treated in VIII. 8 above, of a primary and secondary circuit. If R_1, L_1 be the resistance and self-induction of the primary, L_2, R_2 those of the secondary, and M the mutual inductance, we have

$$a_{11}=L_1,\quad a_{12}=M,\quad a_{22}=L_2,$$
$$b_{11}=R_1,\quad b_{12}=0,\quad b_{22}=R_2.$$

Then, for the resultant resistance and self-inductance of the primary,

$$\left.\begin{aligned} R_1' &= R_1+\frac{n^2M^2R_2}{R_2{}^2+n^2L_2{}^2}, \\ L_1' &= L_1-\frac{n^2M^2L_2}{R_2{}^2+n^2L_2{}^2}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(107)$$

It follows that if the alternations be very slow, the secondary has no effect on the primary. On the other hand, if they are very rapid, R' approaches the limit $R_1+M^2R_2/L_2{}^2$, and L' the limit L_1-M^2/L_2.

The resultant resistance and self-inductance of the secondary will be given in the same way. An inductive electromotive force of frequency $n/2\pi$ acts in the secondary. The reaction of the primary will therefore give for the secondary

$$\left.\begin{aligned} R_2' &= R_2+\frac{n^2M^2R_1}{R_1{}^2+n^2L_1{}^2}, \\ L_2' &= L_1-\frac{n^2M^2L_1}{R_1{}^2+n^2L_1{}^2}, \end{aligned}\right\}$$

as given in VIII. 21 above.

22. System of primary, secondary, tertiary, etc., circuits. Another interesting application given by Lord Rayleigh is to a series of conductors forming primary, secondary, tertiary, etc., circuits, but such that no mutual induction exists except between the primary and the secondary, the secondary and the tertiary, etc. Taking, for example, four circuits in the series, the current in the fourth is due to the inductive action of the third. The reaction on the third causes the latter to have a resultant resistance R'_3 and self-inductance L'_3, at once calculable from (108) by substituting for R_1, L_1, R_2, L_2, M, the quantities R_3, L_3, R_4, L_4, M_{34}. If R'_3, L'_3 are used as the resistance and self-inductance of the third circuit, the fourth circuit may be ignored. Then the resultant resistance and self-inductance of the second circuit due to the action of the third can be found in the same way, and (the third then being also ignored) used to obtain those of the primary circuit. The effect on the primary is to increase its effective resistance and diminish its effective self-inductance in a degree which is greater the greater the frequency of alternation.

It can be shown that the phases of the currents in the different circuits of the series depend in the case of very rapid alternation on the induction coefficients only, and differ successively by half a period.

A very important example, which will be given later in connection with the measurement of activity in alternating circuits, is that of two conductors in parallel, and a related one, which may be worked out easily by the reader from the general formulae given above, is the case already treated, VIII. 22, of a number of conductors joining two points in parallel, but so arranged as not to exert on one another any mutual induction.

23. Numerical example of use of tables. Three numerical tables of the values of certain functions are here appended. Tables of ber x, bei x, ker x, kei x, have been compiled by Mr. Harold G. Savidge, and will be found in Russell's *Alternating Currents*, vol. i. chap. vii. Tables of ber x, bei x, ber$'x$ and bei$'x$ have been computed to nine significant figures, and are given in the *British Association Report*, 1912.

The following numerical example of the utility of the tables here appended is given by Russell. Two parallel cylindrical wires of radius 0·125 cm at a distance (between the axes) apart 1·5 cm are used as leads. The material is high conductivity annealed copper for which $k=5{\cdot}811\times10^{-4}$. For this case at a frequency of 1000 $m=6{\cdot}774$ and $ma=0{\cdot}8468$. The table on p. 283 gives

$$\frac{ma}{2}\frac{W(ma)}{V(ma)}=1{\cdot}003, \quad \frac{4}{ma}\frac{Z(ma)}{V(ma)}=0{\cdot}9987.$$

Hence for unit length

$$R_1=1{\cdot}003R, \quad L_1=9{\cdot}9396+0{\cdot}9987,$$

where R is the resistance of unit length for a steady current.

For a frequency of 500,000, $ma=18{\cdot}93$ and

$$R_1=6{\cdot}950R, \quad L_1=9{\cdot}9396+0{\cdot}1492.$$

The results for this case, but with wires of different materials, are as given in the following table:

		Copper.	Manganin.	Iron ($\mu=100$).
R_1/R	$f=1000$	1·003	1·000	1·385
	$f=500{,}000$	6·950	1·476	25·55
$L_1-9{\cdot}936$	$f=1000$	0·9987	1·0000	81·39
	$f=500{,}000$	0·1492	0·7726	3·593

TABLES OF THE FUNCTIONS

$X_1(x)$, $V_1(x)$, $S(x)$, $T(x)$.

[Computed by HAROLD G. SAVIDGE.]

$X_1(x) = \mathrm{ker}^2 x + \mathrm{kei}^2 x$, $V_1(x) = \mathrm{ker}'^2 x + \mathrm{kei}'^2 x$,

$S(x) = \mathrm{ber}'x\,\mathrm{ker}'x + \mathrm{bei}'x\,\mathrm{kei}'x$, $T(x) = \mathrm{bei}'x\,\mathrm{ker}'x - \mathrm{ber}'x\,\mathrm{kei}'x$.

x	$X_1(x)$	$V_1(x)$	$S(x)$	$T(x)$
0	∞	∞	0	$-0\cdot5$
1	$3\cdot272 \times 10^{-1}$	$6\cdot066 \times 10^{-1}$	$2\cdot186 \times 10^{-1}$	$-3\cdot235 \times 10^{-1}$
2	$4\cdot270 \times 10^{-2}$	$5\cdot968 \times 10^{-2}$	$2\cdot541 \times 10^{-1}$	$1\cdot063 \times 10^{-2}$
3	$7\cdot106 \times 10^{-3}$	$8\cdot933 \times 10^{-3}$	$4\cdot733 \times 10^{-2}$	$1\cdot634 \times 10^{-1}$
4	$1\cdot314 \times 10^{-3}$	$1\cdot563 \times 10^{-3}$	$-1\cdot104 \times 10^{-1}$	$5\cdot949 \times 10^{-2}$
5	$2\cdot577 \times 10^{-4}$	$2\cdot962 \times 10^{-4}$	$-6\cdot254 \times 10^{-2}$	$-7\cdot801 \times 10^{-2}$
6	$5\cdot250 \times 10^{-5}$	$5\cdot901 \times 10^{-5}$	$5\cdot501 \times 10^{-2}$	$-6\cdot263 \times 10^{-2}$
7	$1\cdot099 \times 10^{-5}$	$1\cdot214 \times 10^{-5}$	$6\cdot086 \times 10^{-2}$	$3\cdot741 \times 10^{-2}$
8	$2\cdot344 \times 10^{-6}$	$2\cdot560 \times 10^{-6}$	$-2\cdot347 \times 10^{-2}$	$5\cdot793 \times 10^{-2}$
9	$5\cdot078 \times 10^{-7}$	$4\cdot491 \times 10^{-7}$	$-5\cdot421 \times 10^{-2}$	$-1\cdot217 \times 10^{-2}$
10	$1\cdot113 \times 10^{-7}$	$1\cdot195 \times 10^{-7}$	$2\cdot910 \times 10^{-3}$	$-4\cdot992 \times 10^{-2}$
11	$2\cdot464 \times 10^{-8}$	$2\cdot628 \times 10^{-8}$	$4\cdot521 \times 10^{-2}$	$-4\cdot684 \times 10^{-3}$
12	$5\cdot500 \times 10^{-9}$	$5\cdot833 \times 10^{-9}$	$1\cdot087 \times 10^{-2}$	$4\cdot022 \times 10^{-2}$
13	$1\cdot236 \times 10^{-9}$	$1\cdot305 \times 10^{-9}$	$-3\cdot506 \times 10^{-2}$	$1\cdot582 \times 10^{-2}$
14	$2\cdot792 \times 10^{-10}$	$2\cdot936 \times 10^{-10}$	$-1\cdot967 \times 10^{-2}$	$-2\cdot981 \times 10^{-2}$
15	$6\cdot341 \times 10^{-11}$	$6\cdot646 \times 10^{-11}$	$2\cdot456 \times 10^{-2}$	$-2\cdot254 \times 10^{-2}$
16	$1\cdot446 \times 10^{-11}$	$1\cdot512 \times 10^{-11}$	$2\cdot451 \times 10^{-2}$	$1\cdot939 \times 10^{-2}$
17	$3\cdot311 \times 10^{-12}$	$3\cdot452 \times 10^{-12}$	$-1\cdot438 \times 10^{-2}$	$2\cdot566 \times 10^{-2}$
18	$7\cdot608 \times 10^{-13}$	$7\cdot912 \times 10^{-13}$	$-2\cdot607 \times 10^{-2}$	$-9\cdot593 \times 10^{-3}$
19	$1\cdot753 \times 10^{-13}$	$1\cdot820 \times 10^{-13}$	$5\cdot085 \times 10^{-3}$	$-2\cdot582 \times 10^{-2}$
20	$4\cdot051 \times 10^{-14}$	$4\cdot197 \times 10^{-14}$	$2\cdot498 \times 10^{-2}$	$9\cdot118 \times 10^{-4}$
21	$9\cdot383 \times 10^{-15}$	$9\cdot704 \times 10^{-15}$	$2\cdot883 \times 10^{-3}$	$2\cdot363 \times 10^{-2}$
22	$2\cdot178 \times 10^{-15}$	$2\cdot250 \times 10^{-15}$	$-2\cdot185 \times 10^{-2}$	$6\cdot261 \times 10^{-3}$
23	$5\cdot068 \times 10^{-16}$	$5\cdot226 \times 10^{-16}$	$-9\cdot195 \times 10^{-3}$	$-1\cdot970 \times 10^{-2}$
24	$1\cdot181 \times 10^{-16}$	$1\cdot216 \times 10^{-16}$	$1\cdot726 \times 10^{-2}$	$-1\cdot166 \times 10^{-2}$
25	$2\cdot757 \times 10^{-17}$	$2\cdot836 \times 10^{-17}$	$1\cdot366 \times 10^{-2}$	$1\cdot461 \times 10^{-2}$
26	$6\cdot447 \times 10^{-18}$	$6\cdot625 \times 10^{-18}$	$-1\cdot182 \times 10^{-2}$	$1\cdot517 \times 10^{-2}$
27	$1\cdot510 \times 10^{-18}$	$1\cdot550 \times 10^{-18}$	$-1\cdot621 \times 10^{-2}$	$-8\cdot948 \times 10^{-3}$
28	$3\cdot540 \times 10^{-19}$	$3\cdot631 \times 10^{-19}$	$6\cdot073 \times 10^{-3}$	$-1\cdot679 \times 10^{-2}$
29	$8\cdot312 \times 10^{-20}$	$8\cdot517 \times 10^{-20}$	$1\cdot693 \times 10^{-2}$	$3\cdot253 \times 10^{-3}$
30	$1\cdot954 \times 10^{-20}$	$2\cdot000 \times 10^{-20}$	$-3\cdot196 \times 10^{-4}$	$1\cdot666 \times 10^{-2}$
∞	0	0	0	0

TABLES OF FUNCTIONS USED IN CALCULATING RESISTANCES AND INDUCTANCES OF CONDUCTORS CARRYING RAPIDLY ALTERNATING CURRENTS.

[Computed by BUREAU OF STANDARDS, Washington.]

x	$\frac{x}{2}\frac{W(x}{V(x)}$	$\frac{4}{x}\frac{W(x)}{V(x)}$	x	$\frac{x}{2}\frac{W(x)}{V(x)}$	$\frac{4}{x}\frac{W(x)}{V(x)}$	x	$\frac{x}{2}\frac{W(x)}{V(x)}$	$\frac{4}{x}\frac{W(x)}{V(x)}$
0·0	1·00000	1·00000	4·0	1·67787	0·68632	12·5	4·67993	0·22567
·1	1·00000	1·00000	4·1	1·71516	0·67135	13·0	4·85631	0·21703
·2	1·00001	1·00000	4·2	1·75233	0·65677	13·5	5·03272	0·20903
·3	1·00004	0·99998	4·3	1·78933	0·64262	14·0	5·20915	0·20160
·4	1·00013	0·99993	4·4	1·82614	0·62890	14·5	5·38560	0·19468
0·5	1·00032	0·99984	4·5	1·86275	0·61563	15·0	5·56208	0·18822
·6	1·00067	0·99966	4·6	1·89914	0·60281	16·0	5·91509	0·17649
·7	1·00124	0·99937	4·7	1·93533	0·59044	17·0	6·26817	0·16614
·8	1·00212	0·99894	4·8	1·97131	0·57852	18·0	6·62129	0·15694
·9	1.00340	0·99830	4·9	2·00710	0·56703	19·0	6·97446	0·14870
1·0	1·00519	0·99741	5·0	2·04272	0·55597	20·0	7·32767	0·14128
1·1	1·00758	0·99621	5·2	2·11353	0·53506	21·0	7·68091	0·13456
1·2	1·01071	0·99465	5·4	2·18389	0·51566	22·0	8·03418	0·12846
1·3	1·01470	0·99266	5·6	2·25393	0·49764	23·0	8·38748	0·12288
1·4	1·01969	0·99017	5·8	2·32380	0·48086	24·0	8·74079	0·11777
1·5	1·02582	0·98711	6·0	2·39359	0·46521	25·0	9·09412	0·11307
1·6	1·03323	0·98342	6·2	2·46338	0·45056	26·0	9·44748	0·10872
1·7	1·04205	0·97904	6·4	2·53321	0·43682	28·0	10·15422	0·10096
1·8	1·05240	0·97390	6·6	2·60313	0·42389	30·0	10·86101	0·09424
1·9	1·06440	0·96795	6·8	2·67312	0·41171	32·0	11·56785	0·08835
2·0	1·07816	0·96113	7·0	2·74319	0·40021	34·0	12·27471	0·08316
2·1	1·09375	0·95343	7·2	2·81334	0·38933	36·0	12·98160	0·07854
2·2	1·11126	0·94482	7·4	2·88355	0·37902	38·0	13·68852	0·07441
2·3	1·13069	0·93527	7·6	2·95380	0·36923	40·0	14·39545	0·07069
2·4	1·15207	0·92482	7·8	3·02411	0·35992	42·0	15·10240	0·06733
2·5	1·17538	0·91347	8·0	3·09445	0·35107	44·0	15·80936	0·06427
2·6	1·20056	0·90126	8·2	3·16480	0·34263	46·0	16·51634	0·06148
2·7	1·22753	0·88825	8·4	3·23518	0·33460	48·0	17·22333	0·05892
2·8	1·25620	0·87451	8·6	3·30557	0·32692	50·0	17·93032	0·05656
2·9	1·28644	0·86012	8·8	3·37597	0·31958	60·0	21·46541	0·04713
3·0	1·31809	0·84517	9·0	3·44638	0·31257	70·0	25·00063	0·04040
3·1	1·35102	0·82975	9·2	3·51680	0·30585	80·0	28·53593	0·03535
3·2	1·38504	0·81397	9·4	3·58723	0·29941	90·0	32·07127	0·03142
3·3	1·41999	0·79794	9·6	3·65766	0·29324	100·0	35·60666	0·02828
3·4	1·45570	0·78175	9·8	3·72812	0·28731	∞	∞	0
3·5	1·49202	0·76550	10·0	3·79857	0·28162			
3·6	1·52879	0·74929	10·5	3·97477	0·26832			
3·7	1·56587	0·73320	11·0	4·15100	0·25622			
3·8	1·60314	0·71729	11·5	4·32727	0·24516			
3·9	1·64051	0·70165	12·0	4·50358	0·23501			

VALUE OF THE ARGUMENT m FOR COPPER WIRES OF CONDUCTIVITY $5{\cdot}811 \times 10^{-4}$ c.g.s. UNITS.

[BUREAU OF STANDARDS, Washington.]

('High conductivity annealed copper' at 20° C.)

$m^2 = 4\pi\mu nk = 8\pi^2 f \times 5{\cdot}811 \times 10^{-4}$. f = frequency.

f	m	f	m	f	m
25	1·071	6,000	16·59	200,000	95·79
50	1·515	7,000	17·92	250,000	107·1
100	2·142	8,000	19·16	300,000	117·3
200	3·029	9,000	20·32	333,333	123·7
300	3·710	10,000	21·42	375,000	131·2
400	4·284	15,000	26·23	428,570	140·2
500	4·790	20,000	30·29	500,000	151·5
600	5·247	30,000	37·10	600,000	165·9
700	5·667	40,000	42·84	700,000	179·2
800	6·058	50,000	47·90	750,000	185·5
900	6·426	60,000	52·47	800,000	191·6
1,000	6·774	70,000	56·67	900,000	203·2
2,000	9·579	80,000	60·58	1,000,000	214·2
3,000	11·73	90,000	64·26	1,500,000	262·3
4,000	13·55	100,000	67·74	3,000,000	371·0
5,000	15·15	150,000	82·96	6,000,000	524·7

NOTE.—A new discussion of the derivation of the alternating-current resistance and inductance of conductors has been given by Mr. Harvey L. Curtis, of the Bureau of Standards, Washington (B.B.S.W. No. 374, April 7, 1920). This includes a special treatment of parallel conductors at different distances apart.

CHAPTER X.

THE MEASUREMENT OF ACTIVITY IN ELECTRIC CIRCUITS.

1. Activity in circuit of generator and motor. When a circuit in which a current of electricity is flowing contains a motor, or machine by which work is done in virtue of electromagnetic action, the whole electrical work done in the circuit consists, as was first shown by Joule, of two parts, work spent in heat in the generator and motor and in the conductors connecting them, and work done in moving the motor against external resistance. We consider here in the first place a system in which the current, γ, is constant and neglect loss of energy due to local currents, etc., in the motor. Information regarding practical motors and their action must be sought in the treatises on *Dynamo-electric Machinery* and on *Transmission of Power by Electricity*.

The total rate at which electrical energy is given out in the circuit is $E\gamma$ watts, where E is the total electromotive force of the generator in volts, and γ is the number of amperes of current flowing. The rate at which work is spent in heat is in watts, by Joule's law, γ^2R, where R is the total resistance in circuit in ohms; hence, if we call W the rate at which work is done in the motor, we have,

$$E\gamma = \gamma^2R + W. \quad \text{..................................(1)}$$

We may write this equation in the form,

$$\gamma = \frac{E - W/\gamma}{R}, \quad \text{..................................(2)}$$

which shows that the current flowing is equal to that which would flow in the circuit if, the resistance remaining the same, the motor were held at rest, and the electromotive force diminished by an amount equal to W/γ. This is what is called the *back electromotive force* of the motor, and is due to the action of the motor in setting up an electromotive force tending to send a current through the circuit in the opposite direction to that of the current by which the motor is driven. We shall denote the back electromotive force by E_1. Hence equation (2) becomes,

$$\gamma = \frac{E - E_1}{R} \quad \text{..................................(3)}$$

and the rate at which work is spent in driving the motor is $E_1\gamma$.

To determine E we have simply to measure with a potential galvanometer or voltmeter, the difference of potential between the two terminals of the generator. Calling this V, and R_1 the effective resistance of the generator, we have plainly,

$$E = V + \gamma R_1 \quad \text{...................................(4)}$$

Again, since γ and also the total resistance R in the circuit can be found by measurement, we find by (93)

$$E_1 = E - \gamma R, \quad \text{...................................(5)}$$

where all the quantities on the right-hand side are known.

2. Electrical efficiency of arrangement of generator and motor. The ratio or $E_1\gamma$, the electrical energy spent per unit of time in the circuit otherwise than in heating the conductors, to the whole electrical energy $E\gamma$ spent in the circuit per unit of time, that is the ratio of E_1 to E, we may call the electrical efficiency of the arrangement. Denoting this efficiency by e, we find, by equation (4),

$$e = \frac{E_1}{E} = 1 - \frac{\gamma R}{E} = 1 - \frac{E - E_1}{E}. \quad \text{..................(6)}$$

Hence the smaller γ is made, that is, the slower the energy is given out, the value of the efficiency of the arrangement is the more nearly equal to *unity*, the value of the efficiency of an arrangement in which the work is done in the motor against external resistance is exactly equal to the whole electrical energy given out in the circuit.

When energy is spent at the maximum rate in working the motor, $E\gamma_1$ has its greatest value. But by (5)

$$E_1\gamma = E\gamma - \gamma^2 R = W,$$

from which it will be seen that the activity $E_1\gamma$ in the motor does not include the activity spent in heating its coils. This equation may be written,

$$\gamma^2 R - E\gamma + W = 0,$$

a quadratic of which the solution is,

$$\gamma = \frac{E \pm (E^2 - 4RW)^{\frac{1}{2}}}{2R}. \quad \text{.......................(7)}$$

Now in order that these values of γ may be *real*, $4RW$ cannot be greater than E^2. Hence the greatest value W can have is $E^2/4R$. When W has this maximum value, γ is equal to $E/2R$, and therefore E_1 equal to $E/2$. Hence the electrical efficiency is $\frac{1}{2}$. It is to be very carefully observed that although in this case the arrangement is that of *greatest electrical activity*, it is *not that of greatest electrical efficiency*, as it has only about one-half the efficiency of one in which energy is given out at a very slow rate. The case is exactly analogous to that of a battery arranged so as to give maximum current through a given external resistance, which is far from being an economical or efficient arrangement.

All that has been stated above is applicable to the case of a motor fed by any kind of generator whatever. The generator employed however is generally some form of dynamo- or magneto-electric machine driven by an external motor, such as a steam- or gas-engine or a water-wheel, and a few of the results obtained below apply only to such cases, which will be indicated as they occur. It must be carefully observed that the efficiency considered above is only the efficiency of the arrangement of generator and motor. It is not at all the absolute efficiency of production of the electrical power. For that we shall have to consider a series of other efficiencies and combine them to obtain the final result. There is the efficiency which involves the ratio of the power production in the prime mover to the rate of consumption of the energy value of the fuel, whatever it may be, spent in driving the engine, the efficiency of the utilization of this power in driving the electrical generator, and so on.

3. Case when generator and motor are similar machines. When the generator and motor are exactly similar machines, and the same current passes through both, we shall assume that the ratio of E_1 to E is that of

$$nAf(\gamma) \text{ to } n'Af(\gamma),$$

where n and n' are the speeds of the machines, A a constant depending on the form and disposition of the magnets, and $f(\gamma)$ a function of the current. Hence in this case the efficiency is measured simply by the ratio of the speed of the motor to that of the dynamo. The more nearly therefore the speed of the motor approaches to that of the generator, the greater is the efficiency. It is to be observed however that two machines identically alike will not in practice be perfectly similar in their action, even with the same currents flowing in their armatures and field-magnet coils. The armature currents tend to weaken the field in the generator, and to strengthen the field in the motor. There is also alternation of currents in sections of the armatures, which, going on at different rates, must make $f(\gamma)$ different for the two machines.

In general, the higher the speed at which the motor is run, the greater is the electrical efficiency of any arrangement, for it is obvious that the higher the speed the more nearly does E_1 approach to E, and therefore the values of E_1/E, the measure of efficiency, to unity.

4. Electrical efficiency increased by increasing e.m.f. in circuit. For a constant difference $E - E_1$, the ratio of the energy spent in heating the conductors by the current to the whole energy expended in the circuit, may be reduced by increasing the electromotive force E of the circuit. If E be increased to nE while E_1 is changed to E'_1, so that $nE - E'_1 = E - E_1$, the electrical efficiency, as will be shown immediately, is raised to $(n-1)/n + E_1/nE$, or $(n-1)/n + 1/n^{\text{th}}$ of the former efficiency. Clearly, as n is increased this approaches more and more nearly to unity.

The energy spent in heat is $\gamma^2 R$, or $(E - E_1)^2/R$, and the ratio of this to $E\gamma$ is $\gamma R/E$. But γR is equal to the constant difference $E - E_1$, hence the ratio is $(E - E_1)/E$, and this becomes smaller as E is increased. A greater efficiency is therefore obtained by using high potentials than by using low potentials. Hence a greater electrical efficiency is realized, with a given magneto- or dynamo-electric machine used as generator and a given motor, when both generator and motor are run at higher speeds. Consequently the generator should be run as fast as possible, and the motor loaded lightly, or the speed with which the working resistance is overcome reduced by gearing between it and the motor.

When high potentials are obtained by the use of machines wound with fine wire, or by using as generator a battery of a large number of cells joined in series to drive a high potential motor, the gain of electromotive force is accompanied by an increase of resistance in the circuit. But if we suppose the speed of the motor to be so regulated that the difference between the total electromotive force in the circuit and the back electromotive force of the motor remains the same in the different cases, it is easy to show that the electrical efficiency of the arrangement is greater for high electromotive forces than for low. If, as supposed, $E - E_1$ remains constant, while E is changed to nE, we have for the total activity of the motor $nE\gamma - (E - E_1)\gamma$. Dividing this by $nE\gamma$ we get for the electrical efficiency,

$$e = \frac{n-1}{n} + \frac{1}{n}\frac{E_1}{E}. \quad \text{..............................} (6')$$

As n is made greater and greater, the first term on the right becomes more and more nearly equal to unity, and the last term to zero. Hence, on the supposition made, the efficiency is increased by increasing the working electromotive forces. Taking as a particular case $n = 2$, we see that the efficiency is $\frac{1}{2}$ together with one-half of the former efficiency; if $n = 4$, the efficiency is $\frac{3}{4}$ together with one-fourth of the former efficiency, and so on for other values of n. This result holds for any case whatever in which the condition that $E - E_1$ should remain constant is fulfilled; and hence it is independent of any change that may have been made in the resistance of the generator or motor in order to obtain the higher electromotive force nE. For example, it is plain that no sensible change in the actual rate of loss by heating of the conductors by the current will be produced by increasing the resistances of the generator and motor, if these be very small in comparison with the remainder of the resistance in circuit; as, since $E - E_1$ remains constant and the resistance is practically the same as before, the current strength will not be perceptibly altered. The ratio, however, of the activity wasted in heating to the total activity will be only $1/n^{\text{th}}$ of what it was before. In the opposite extreme case, in which the generator and motor have practically all the resistance in circuit, the current, $\gamma (= (E - E_1)/R)$, is diminished in the ratio in which the resistance is increased; and the actual rate of loss by heat according to Joule's law, $(E - E_1)^2/R$, is diminished in the same ratio, so that, as in the former case, its ratio to the total activity $nE\gamma$ is $1/n^{\text{th}}$ of what it was for the electromotive force E. We see, therefore, that here also the efficiency must be the same in both cases.

We have called E_1/E *the electrical efficiency of the arrangement*, but this is not to be confounded with the efficiency of the motor itself. The activity $E_1\gamma$ includes the wasted activity, or rate at which work is done against frictional resistances in the motor itself, and in the gearing which connects it with its load, as well as the useful activity or rate at which it performs useful work. Hence, although the electrical efficiency of the arrangement be very great, it does not follow that a comparatively large amount of the energy given to the motor is usefully expended; that will depend on circumstances. Hence we define the efficiency of a motor at any given speed as the ratio of the useful activity to the whole activity, taking as the latter the total rate at which electrical energy is expended in the motor; that is, $E_1\gamma + \gamma^2 R_1$, or, which is the same, $V\gamma$, where V is the difference of potential between the terminals of the motor. Accordingly, if A be the useful activity, we have for the efficiency of the motor the ratio $A/V\gamma$. We may call this the working efficiency of the motor.

5. Measurement of working efficiency of motor. To determine this ratio in any particular case the motor is run at the required speed, V is measured with a potential galvanometer, and γ with a current galvanometer, and their product taken, or $V\gamma$ is determined with some form of electrical activity-meter, while A is determined by means of a suitable ergometer. A very convenient and accurate friction ergometer may be formed by passing a cord once completely round the pulley of the motor, and hanging a weight on the downward end, while the other is made to pull on a spiral spring fixed at its upper end and provided with an index to show its extension. The weight is adjusted so that the motor runs at the required speed, while wasting all its work in overcoming the friction of the cord, and the extension of the spring is noted, and the corresponding pull found in the same units of force as those used in estimating the downward pull due to the weight. Let the weight used in any experiment be taken in grammes, and be denoted by w, and let w' be the number of grammes required to stretch the spring by gravity to the same amount, then the total force overcome is in dynes $(w-w')g$, where g is the acceleration, in centimetres per second per second, produced by gravity at the place of experiment (at London $g=981\cdot17$ nearly). If n be the number of revolutions per second, and c the circumference in cm of the pulley at the part touched by the rope, the speed with which this force is overcome is nc, and therefore the activity in ergs per second is $nc(w-w')g$. If A is reckoned in watts, we have the equation,

$$A = \frac{1}{10^7} nc(w-w')g. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

If $w-w'$ be taken in pounds, and c in feet, and n be the number of revolutions per *minute*, the activity in horse-power is given by

$$A=\frac{1}{33000}nc(w-w'), \qquad \text{.........................(9)}$$

and in watts approximately by

$$A=\cdot 0226nc(w-w'). \qquad \text{.........................(10)}$$

6. Generator charging storage battery. We have now considered cases in which electrical energy is transformed into mechanical work by means of motors working by electromagnetic action, and have seen that the whole electrical activity $E\gamma$ in the circuit is equal to the useful activity of the motor together with the unavailable part spent in heating the conductors in circuit, and in overcoming the frictional resistances opposing the motion of the motor. Part of the electrical energy developed by a generator may however be spent in effecting chemical decompositions in electrolytic cells placed in the circuit, as, for example, in charging a secondary battery or "accumulator." Each cell in which electrolytic action takes place, so that the result is chemical separation at the plates of the constituents of the solution acted on, opposes a counter electromotive force to that causing the current, and the work done per second in each cell, over and above that spent in heat according to Joule's law, is equal to the product of this counter electromotive force into the strength of the current. In most cases the counter electromotive force exceeds the electromotive force required to effect the chemical decompositions, and the energy corresponding to the difference of electromotive force appears in the form of what has been called *local heat* in the electrolytic cells.

In the case of a secondary battery charged by the current from an electrical generator, which is the only case we shall here consider, the activity spent in the battery while it is being charged is equal to the product of the difference of potential existing between the terminals of the battery while the current is flowing, multiplied by the strength of the current. Let V be this difference of potential in volts, and γ the current strength in amperes, then $V\gamma$ joules is the whole work per unit of time spent in the battery. The whole activity spent in the circuit is $E\gamma$, or $V\gamma+\gamma^2R$, where E is the total electromotive force of the generator, and R is the resistance of the generator and the wires connecting it with the secondary. Again, if E_1 volts be the electromotive force of the secondary battery, which may be measured by removing the charging battery for an instant and applying a potential galvanometer to the terminals of the secondary, the activity actually spent in charging the battery may be taken as $E_1\gamma$ watts. Hence the ratio of the activity spent in charging the battery to the whole activity in the circuit is $E_1/(V+R\gamma)$ or E_1/E, and the activity wasted in heating the conductors in circuit is $(E-E_1)\gamma$. This ratio E_1/E is the same as that found above in the case of a generator and a motor, and may be called as before the electrical efficiency of the arrangement.

7. Arrangement of maximum electrical efficiency. Effect of increased e.m.f. in circuit. Hence, in order that as nearly as possible the whole electrical energy given out in the circuit may be spent in charging the battery, as many cells should be placed in circuit as suffice nearly to balance the electromotive force E of the generator, that is, the charging should be made to proceed as slowly as possible. In practice, however, a very slow rate of charging is not economical, as the work spent in driving the generator, if a dynamo- or magneto-electric machine, against frictional resistances might be comparable with or even greater than the useful work done in the circuit; and if the speed of the generator slackened for a little the battery would tend to discharge through it.

As in the case of the motor the electrical efficiency of the arrangement can be increased by increasing E and E_1, so that $E - E_1$ is maintained constant. E may, in the present case, be increased by running the generator faster, or by using a machine adapted to give higher potentials. As before, if E be increased to nE, while E_1 is changed to E' so that $nE - E' = E - E_1$, the electrical efficiency becomes $(n-1)/n + E_1/nE$ as in (7) above.

8. Measurement of energy spent in charging. The electromotive force of a Faure or storage cell is rather over 2 volts when fully charged, but is considerably less when nearly discharged. When the cell is placed in the charging circuit, the counter electromotive force which it gives rises quickly to a little less than the full value, and thereafter gradually increases, while the charging current falls in strength. In order to measure, therefore, the whole energy spent in charging a secondary battery, we must either use some form of integrating energy-meter which gives accurate results, or measure, at short intervals of time, V with a potential galvanometer, and γ with a current galvanometer placed permanently in the circuit. After the battery has been charged, the total number of joules spent is obtained by multiplying each value of $V\gamma$ by the number of seconds between the instant at which the corresponding readings were taken and that at which the next pair of readings were taken, and adding all the results. Or, more exactly, values V and γ are obtained for each interval by finding the arithmetic means of the values of V and of γ at the beginning and end of each interval, and taking the product of these two means as the value of the activity for that interval. Each product is multiplied by the number of seconds in the corresponding interval, and the sum of the products is the whole energy spent. The integral work in joules having been thus estimated, the efficiency of the battery may be obtained by finding in the same manner the total number of joules given out in the external working circuit while the battery is discharging. The ratio of the useful work thus obtained to the whole work spent in charging is the efficiency of the battery. In discharging in an electric light circuit, the greatest economy is obtained when the resistance of the working part of the circuit is very great in comparison with that of the battery and main conductors. Neglecting the latter part of the resistance, we see that, if a large number of lamps are arranged in parallel, a large number of cells should also be joined in parallel, so that, while the requisite difference of potential is obtained, the resistance of the battery is still small in comparison with that of the external circuit.

As regards the measurement of energy spent in electric light circuits, in which direct currents are flowing, we have already sufficiently indicated above how this may be done. To find the activity, or work spent per unit of time, in any part of a circuit, we have only to find the difference of potential, V, in volts between its extremities with a potential galvanometer, and the current, γ, in amperes flowing through it with a current instrument. If the activity be constant, we have simply to multiply $V\gamma$ by the number of seconds in any interval of time to find the number of joules spent in that time in the part of the circuit in question. If the activity is variable, the whole energy spent in any time may be estimated by finding $V\gamma$ at short intervals of time, and calculating the integral as just explained.

9. Electrical activity in alternating current circuits. Activity-meters. So far we have been considering only measurements made in the circuits of batteries or of direct-current generators. Alternating machines in which the direction of the current is reversed two or three hundred times a second are, however, frequently employed, especially in electric-light circuits, and it is necessary therefore to consider the methods of electrical measurement available in such cases.

The only electromagnetic instruments which can be used in alternating circuits are such as depend on the mutual force between two current-carrying conductors. Electrodynamometers generally, and current weighers such as those described in Chap. XII., are instruments which act on this principle, and can be used both in alternating and in continuous-current circuits. We have only to indicate here how they can be applied to measure currents, differences of potential, and activity in constant or alternating-current circuits.

In practical work the instruments on this principle usually employed are such as require to have their constants determined by comparison with standard instruments, such as a standard tangent galvanometer, or a standard dynamometer. An early form was Siemens' electrodynamometer, in which a suspended coil is acted on by a fixed coil, and the strength of the current deduced by means of a table of values for different angles, from the torsion which must be given to a spiral spring to bring the coil back to the zero position.

When an instrument on this principle is arranged for use as an activity-meter, one set of coils, the fixed or the movable, is made of thick wire so as to carry the whole current in the circuit, while the other set is made of high resistance and is connected to the two ends of the part of the circuit in which the electrical activity is to be measured. In this case the force or couple required to restore the movable coils to the zero position is proportional to the product $V\gamma$ of the difference of potential and current, that is to the activity, for that part of the circuit; and if the instrument has been properly graduated this can be at once read off in watts, or in any other chosen units of activity. Many practical commercial instruments of this kind have been invented and made, and will be found described in books on Electrical Engineering.

10. Differences of potential and currents in alternating current circuits. We shall now consider the measurement of current and differences of potential, and therefore also of electrical energy in the circuits of alternating machines or of transformers. Some account of the theory of alternating currents is given in Chapter VIII. In all such circuits the march of the current in each complete alternation may be stated roughly as a rise from zero to maximum in one direction, then a diminution to zero, then a change of sign and a rise to maximum in the opposite direction followed by a diminution again to zero. The law according to which these changes take place is more or less complex in the various cases, and the complete mathematical representation of the current strength at any time would require an application of Fourier's method of representing any arbitrary periodic function by means of an infinite series of simple harmonic terms of the form $A_k \sin(knt - e_k)$, where n is 2π divided by the maximum period T of alternation, A_k and e_k are constants and k is any integer. It has been found experimentally that the variation of electromotive force in some alternating machines can be expressed with a fair degree of approximation by the single harmonic term $E \sin nt$, where we reckon t from the instant at which the electromotive force was zero when changing from the direction reckoned as negative to that reckoned as positive. The values of $E \sin nt$ are shown graphically by the ordinates of the curve in Fig. 74, t being measured from A along AB. The maximum and minimum ordinates CE, DF are in length numerically equal to the electromotive force E. We shall assume the truth of this law in most of what follows, and shall thereby obtain results which will help in the tracing out of what happens in more complicated cases. Information regarding such cases will be

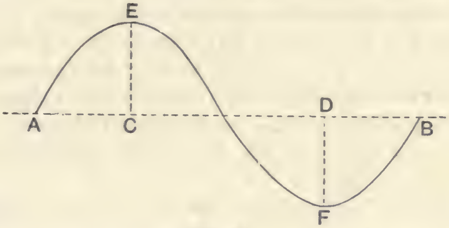

FIG. 74.

found in special treatises on Electrical Engineering. By means of a proper contact arrangement, which makes connection with an electrometer at different instants during an alternation, the values of the difference of potential between the terminals at these instants can be obtained. If the difference of potential does not follow the simple law of signs, the simple harmonic constituents can be deduced by some satisfactory method of combining the results of observations. [See Appendix.]

The current strength is affected by the action of self-induction to a greater or less extent in all such machines independently of the disposition of the external circuit, especially if the revolving armature contains iron; but, as shown below, it follows, with a difference in phase, the same law as does the electromotive force. The effect of variations in the field-magnets produced by the rotating armature has also in a rigorous theory to be taken into account, but this effect in well-designed machines without iron in their armatures is not great, and where experiments have been made to detect it, has been found to be slight, and we shall therefore neglect it.

11. Mean current and square root of mean square [R.M.S.] of current. Writing then γ for the current, at a time t, reckoned from the instant at which the current was zero, we have

$$\gamma = A \sin nt. \quad \text{......}(11)$$

The whole quantity of electricity generated in a half period $T/2$ is therefore

$$\int_0^{T/2} \gamma\, dt = A \int_0^{T/2} \sin nt\, dt = \frac{AT}{\pi}. \quad \text{......}(12)$$

Hence if γ_m denote the mean current in that time, we have

$$\gamma_m = \frac{2A}{\pi}. \quad \text{......}(13)$$

Here we have used γ_m to denote the mean current, which is not the same thing as the square root of the mean square of the current for which the symbol γ' was used in VIII. 24 above. We denoted the mean square of the current by $[\gamma^2]_m$, and used γ' for the positive square root of this. Hence we may legitimately write γ'^2 instead of $[\gamma^2]_m$, and similarly V'^2 for $[V^2]_m$. The reader must observe that γ' is not identical with γ_m nor V' with V_m. The relation between them is given in the next paragraph.

Now if an electrodynamometer be placed in the circuit so that the same current passes through both its fixed and movable coils, the current in both will be reversed at the same instant, and their mutual action will be the same for the same current strength, and will be instantaneously proportional to γ^2, that is to $A^2 \sin^2 nt$. If the period of the alternation be small in comparison with the period of free oscillation of the movable coil system of the dynamometer, the mutual action of the fixed and movable coil will be the same as if a continuous current γ' given by the equation

$$\gamma'^2 = \frac{1}{T}\int_0^T \gamma^2 dt = \frac{A^2}{T}\int_0^T \sin^2 nt\, dt \quad \text{.................}(14)$$

were kept flowing through them. But by integration

$$\gamma'^2 = \frac{A^2}{2}, \quad \text{..............................}(15)$$

and substituting from (13) in this equation, we get

$$\gamma_m = \frac{2\sqrt{2}}{\pi}\gamma' = \cdot 9003\,\gamma'. \quad \text{.......................}(16)$$

In order therefore to find the actual mean current strength in the circuit of an alternating machine from the value of γ' (the R.M.S. of the current) given by a current dynamometer we must multiply the latter by ·9; in other words the mean current strength is 9/10 of the strength of the continuous current which would give the same deflection. The product, if γ' has been taken in amperes, multiplied by the number of seconds in any interval of time during which the machine has been working uniformly on the same circuit, will give the number of coulombs of electricity which that continuous current would have carried in that time.

12. Measurement of difference of potential by idiostatic electrometer. The measurement of differences of potential is however attended with more difficulty on account of the effect of the self-induction of any electromagnetic instrument which can be applied to the circuit for this purpose. The following method of employing a quadrant electrometer for this purpose was used in the early days of the testing of electrical generators by M. Joubert* and by the author in various trials of Ferranti alternating machines. The needle of the instrument is left uncharged, and the charging rod connected with it and used as a third electrode. If the needle be connected to a point in the circuit at which the potential is V relatively to the outside case, one pair of quadrants at a point at which the potential is V_1, and the other pair at a third point where the potential is V_2, and if D be the deflection of the spot of light corresponding to the angle (supposed small) through which the needle has been turned against the bifilar suspension, then, subject to the caution below, we have

$$D = k(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right), \quad \text{.................}(17)$$

where k is a constant. The needle (and case) is connected to the pair of quadrants at potential V_1, so that

$$D = \frac{k}{2}(V_1 - V_2)^2. \quad \text{.........................}(18)$$

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* *Comptes Rendus*, July 1880. *Annales de Chimie et de Physique*, May 1883.

This equation is applicable, whatever the law of the electrometer, provided k be determined by a process of calibration with known differences of potential.

It was found by Ayrton and Perry and Sumpner* that when a quadrant electrometer is used idiostatically the metallic cheeks left where the guard-tube is cut away for the needle exert an influence on the needle in its unsymmetrical position when deflected, which renders the formula (17) seriously inaccurate. It may be used however without correction for values of V up to about 100 volts. In quadrant electrometers manufactured since 1892, the guard-tube is dispensed with.†

Any carefully graduated electrostatic voltmeter‡ may (preferably) be used instead of the quadrant electrometer, except when three points at different potentials are to be connected to the electrometer at the same time. Any doubt as to the applicability of the expression on the right of (18), with k a constant, is avoided, for in these instruments the values of different deflections on the scale have been fixed by experiment.

If the terminals of the electrometer employed be connected to any two points in the circuit of a machine in which the period of alternation is short in comparison with the free period of the needle, the couple acting on the needle will be at each instant proportional to the second power of the difference $V_1 - V_2$ of potential existing between these two points at that instant. Also, as in the similar case of the dynamometer above, the deflection of the needle will be the same as that which would be produced by a constant difference of potential V' given by the equation

$$[V^2]_m = V'^2 = \frac{1}{T}\int_0^T (V_1 - V_2)^2 dt. \quad \dots\dots\dots\dots\dots\dots(19)$$

If we denote the actual mean difference of potential by V_m, for V positive, in the half period from $V=0$ to $V=0$ again, then, since the difference of potential follows the same law of variation as the current, we get also

$$V_m = \cdot 9003\, V'. \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(20)$$

13. Enhanced resistance due to alternation: distinguished from impedance. It is to be noticed that the resistance of the conductors in circuit is greater the greater the frequency of alternation. This variation, as explained in Chap. IX. above, is due to the fact that as the alternation increases in rapidity the current is more and more confined by inductive action to the outer strata of the conductor, which is therefore virtually reduced in section. This is not to be confounded with the fictitious increase of resistance seen in the expression $(R^2 + n^2L^2)^{\frac{1}{2}}$

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* *Phil. Trans. R.S.*, A. 1891.

† The distance, if any, of the quadrants apart for which the formula is correct should be found by experiment for each electrometer used in this manner.

‡ See Chapter XVII. below.

(see 15 below), which arises directly from the electromotive force of self-induction; but is a real increase of the value of R for the current in question. (See Chapter IX. and Appendix for resistances and self-inductances of conductors at different frequencies of alternation.)

14. Measurement of activity. We shall find that the true mean value of the electrical activity is equal to the product of the square root of the mean square of the difference of potential, by the square root of the mean square of the current strength. [" The square root " in all such cases means the positive square root.] It can, as we shall see, be determined by means of an electrometer and an electrodynamometer or alternate-current ammeter without its being necessary to know the resistance.

Let R be the total resistance in the circuit, γ the current flowing in it at the time t, E the total electromotive force of the machine, and L the inductance for the whole circuit, that is, the number which multiplied into $d\gamma/dt$ gives the electromotive force opposing the increase or diminution of the current. We shall suppose L a constant, although there can be no doubt that in some alternating machines its value is different in different positions of the armature. The iron cores of the field magnets act to a greater or less extent as cores for the armature coils, and as the magnetic susceptibility of iron is a function of the strength of the magnetizing current, L, which is the magnetic induction through the armature produced per unit of its own current, must vary accordingly.

15. Circuit containing simple harmonic e.m.f. Still for certain alternators the variation of L with the position of the armature is not very great. It will also be assumed that there are no masses of metal in which local currents can be generated moving in the field. On these assumptions the equation of the current is

$$R\gamma = E - L\frac{d\gamma}{dt}. \quad \text{.........................}(21)$$

But by the law which we have assumed for the machine,

$$E = n\eta \sin nt = E_0 \sin nt, \quad \text{.......................}(22)$$

where η is a constant such that E_0 is the maximum value of E for the given speed. Substituting in (21) we get

$$L\frac{d\gamma}{dt} + R\gamma = E_0 \sin nt, \quad \text{.........................}(23)$$

which integrated becomes

$$\gamma = Ae^{-\frac{R}{L}t} + \frac{E_0}{(R^2+n^2L^2)^{\frac{1}{2}}}\sin(nt-\epsilon), \quad \text{..................}(24)$$

where
$$\sin\epsilon = \frac{nL}{(R^2+n^2L^2)^{\frac{1}{2}}}, \quad \cos\epsilon = \frac{R}{(R^2+n^2L^2)^{\frac{1}{2}}}. \quad \text{...........}(25)$$

The term $Ae^{-\frac{R}{L}t}$ is only important immediately after the circuit is closed, and when any notable variation of speed takes place, and will therefore be neglected.

We may remark that if L were equal to zero (24) would reduce to $\gamma = E_0/R \,.\, \sin nt$.

From (24) we get for the true mean current

$$\gamma_m = \frac{2E_0}{T(R^2+n^2L^2)^{\frac{1}{2}}}\int_{\epsilon/n}^{\epsilon/n+T/2} \sin\,(nt-\epsilon)\,dt = \frac{2E_0}{\pi(R^2+n^2L^2)^{\frac{1}{2}}}. \quad \ldots\ldots(26)$$

Also for the mean square of the current strength as given directly by an electrodynamometer we have by (24) the equation

$$\gamma'^2 = \frac{E_0{}^2}{T(R^2+n^2L^2)}\int_0^T \sin^2(nt-\epsilon)\,dt = \frac{1}{2}\,\frac{E_0{}^2}{R^2+n^2L^2}, \quad \ldots\ldots\ldots(27)$$

and we have therefore, as before, the relation

$$\gamma_m = \cdot 9003\gamma'.$$

The current in, and the difference of potential between the terminals of, a conductor may also be the subjects of measurement. The same results follow, *mutatis mutandis*, that is to say the equations already found, and those which follow, may be applied to this case if V_0 be the amplitude of the applied difference of potentials, and L and R be the self-inductance and resistance *of the conductor*.

16. Difference of phase of current and e.m.f. We see that the effect of self-induction is to diminish every value of the current in the ratio of $E_0/(R^2+n^2L^2)^{\frac{1}{2}}$ to E_0/R [or, for a conductor of $V_0/(R^2+n^2L^2)^{\frac{1}{2}}$, to V_0/R], and to produce a retardation of phase which measured in time is e/n seconds; that is, the resistance is virtually increased in the ratio $(R^2+n^2L^2)^{\frac{1}{2}}/R$, and the current in following the law of sines passes through any value e/n seconds after it would have passed through the corresponding value if there had been no self-induction. If in Fig. 74 above the ordinates of the curve of sines represent the values of the current at different instants of time, when L is zero, the current would be represented for any given value of L by diminishing the ordinates of the curve all in the proportion of R to $(R^2+n^2L^2)^{\frac{1}{2}}$, and shifting the curve along AB from left to right through a distance equal to η. It is plain also that, for any finite resistance R, by diminishing T, that is, by increasing the speed of the machine, the current can, by (24), be made to approach the limiting value

$$\gamma = \frac{\eta}{L}\sin\left(nt-\frac{\pi}{2}\right), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

which is independent of the resistance, and has a retardation of phase of $T/4$ seconds, a quarter period of a complete alternation. Hence integrating over a half period from zero current to zero current again, and dividing by $T/2$ we get for the maximum mean current

$$\gamma_m = \frac{2\eta}{\pi L}. \quad \text{............................} (29)$$

17. Mean electrical activity in circuit. To find the mean value A_m of the total electrical activity in the circuit, we have by (22) and (24),

$$A_m = \frac{1}{T}\int_0^T E\gamma\, dt = \frac{E_0^{\,2}}{(R^2+n^2L^2)^{\frac{1}{2}}}\int_0^T \sin(nt-\epsilon)\sin nt\, dt$$

$$= \frac{1}{2}\,\frac{E_0^{\,2}R}{R^2+n^2L^2}. \quad \text{....................................} (30)$$

Hence by (27) if the activity is spent in heat

$$A_m = \gamma'^2 R, \quad \text{..............................} (31)$$

that is, the true mean value of the total electrical activity is equal to the mean square of the current strength multiplied by the total resistance in circuit. This also applies to part of a circuit.

It may easily be shown, from (30), by the ordinary method that the total activity in the circuit is greatest when $R = nL$, that is, *for a given speed and a given value of* L, the activity is a maximum when $R = nL$. It must be observed however that *for a given resistance* R the activity is greater the smaller the value of T, that is, the greater the speed. When R has the value nL we have, by (24), $e = \pi/4$; that is, the retardation of phase is then one-eighth of the whole period.

18. Compound periodic e.m.f. Supposing the electromotive force and current, though periodic, not to follow the simple sine law, then, as we have seen above, we may represent either by a Fourier series. Thus

$$E = \Sigma E_k \sin(knt - e_k), \quad \text{.........................} (32)$$

where k is an integer, and takes all the values required for the simple components which make up the periodic function which E is of the time.

For the equation of current we have now instead of (23),

$$L\frac{d\gamma}{dt} + R\gamma = \Sigma E_k \sin(knt - e_k), \quad \text{.................} (33)$$

of which the solution is

$$\gamma = A\epsilon^{-\frac{R}{L}t} + \sum \frac{E_k}{(R^2+k^2n^2L^2)^{\frac{1}{2}}} \sin(knt - e_k - \phi_k), \quad \text{........} (34)$$

where
$$\sin\phi_k = \frac{knL}{(R^2+k^2n^2L^2)^{\frac{1}{2}}},\ \cos\phi_k = \frac{R}{(R^2+k^2n^2L^2)^{\frac{1}{2}}}. \quad \text{..........} (35)$$

As before we may neglect the exponential term in the solution.

To find the mean square of the current strength we have only to square the series on the right of (34), and integrate over the whole compound period, $2\pi/n$, that is, over an interval which is the least common multiple of the periods of the components. Now it can be easily shown that an integral, of the form

$$\int \sin (jnt - e_j - \phi_j) \sin (knt - e_k - \phi_k) dt,$$

vanishes when taken over an interval $2\pi/n$, unless $j = k$. For the product under the integral sign can by elementary trigonometry be resolved into the difference of two cosines, each yielding a simple integral, which obviously vanishes.

Then, since

$$\frac{1}{2\pi/n} \frac{E_k^2}{R^2 + k^2n^2L^2} \int_0^{2\pi/n} \sin^2 (knt - e_k - \phi_k)\, dt = \frac{1}{2} \frac{E_k^2}{R^2 + k^2n^2L^2},$$

we get for the mean square of the current

$$\gamma'^2 = \tfrac{1}{2} \sum \frac{E_k^2}{R^2 + k^2n^2L^2}, \quad \text{.........................(36)}$$

that is, the mean square of the current is the sum of the mean squares of the currents which would be given by the components of γ if each existed separately.

19. Mean activity = sum of mean activities of the components. The mean activity is given by the equation

$$A_m = \frac{1}{2\pi/n} \int_0^{2\pi/n} \Sigma E_k \sin (knt - e_k) \sum \frac{E_k}{(R^2 + k^2n^2L^2)^{\frac{1}{2}}} \sin (knt - e_k - \phi_k)\, dt.$$

If the multiplication of the two series on the right is performed, a number of integrals of the form

$$\frac{1}{2\pi/n} \int_0^{2\pi/n} \sin (jnt - e_j - \phi_j) \sin (knt - e_k)\, dt$$

are obtained, all of which vanish as before, except those for which $j = k$. But we have

$$\frac{1}{2\pi/n} \frac{E_k^2}{(R^2 + k^2n^2L^2)^{\frac{1}{2}}} \int_0^{2\pi/n} \sin (knt - e_k) \sin (knt - e_k - \phi_k)\, dt$$

$$= \frac{E_k^2 \cos \phi_k}{(R^2 + k^2n^2L^2)^{\frac{1}{2}}} = \frac{1}{2} \frac{E_k^2 R}{R^2 + k^2n^2L^2},$$

by (35). Hence

$$A_m = \tfrac{1}{2} R \sum \frac{E_k^2}{R^2 + k^2n^2L^2}, \quad \text{.........................(37)}$$

that is the mean activity is the sum of the mean activities which the component currents would give separately.

Also, by (36) and (37),

$$A_m = \gamma'^2 R. \quad \text{.................................(38)}$$

The practical importance of this result lies in the fact, that it proves that any method of measuring power which is demonstrated for a current following the simple sine law of variation with the time, is also true for any periodic current whatever, inasmuch as such a current can be regarded as made up of simple sine currents of different periods. For example, the generality of the method, given below, of measuring power in the circuit of a transformer can be inferred from this result.

20. Circuit with two e.m.f.s of the same period. If in the circuit there be two sources of electromotive force of the same period T, but of different phases—for example, two machines driven so as to have the same period of alternation—the solution here given applies. For the two electromotive forces combine to give a single electromotive force of the same period as the components, but differing in phase from either; so that, to use the solution it is only necessary to take this resultant electromotive force as $E_0 \sin nt$, reckoning the time from an instant at which $\sin nt$ is zero and increasing. If the difference of phases be 2ϕ reckoned in angle, the interval between the successive instants at which a component is increasing through zero is $2\phi/n$. Hence taking the zero of reckoning of time midway between these two instants, we may denote the two components by $E_1 \sin(nt+\phi)$, $E_2 \sin(nt-\phi)$. Calling their resultant $E_0 \sin(nt-\psi)$, we have

$$E_0 \sin(nt-\psi) = E_1 \sin(nt+\phi) + E_2 \sin(nt-\phi). \quad \text{.........(39)}$$

By elementary trigonometry, we get

$$\left.\begin{aligned} &E_0^2 = E_1^2 + E_2^2 + 2E_1E_2\cos 2\phi, \\ \text{and} \qquad &\tan\psi = \frac{E_2 - E_1}{E_1 + E_2}\tan\phi. \end{aligned}\right\} \quad \text{..................(40)}$$

When $\phi = 0$, $\psi = 0$, and $E_0 = E_1 + E_2$, as is evident without calculation, since the machines are then in the same phase. If $E_1 = E_2$, that is if the machines are equal, the resultant is in phase halfway between its components. When this is the case we have also

$$E_0 = 2E_1 \cos\phi, \quad \text{..............................(41)}$$

which when $\phi = 0$ gives, as it ought, $E_0 = 2E_1$.

21. Two unequal e.m.f.s of different phase. Considering still two unequal machines, and remembering that when the value of the resultant electromotive force is increasing through zero, the value of the current is given by (24), that then the electromotive force of the leading machine is $E_1 \sin(nt+\phi+\psi)$, and that of the following machine $E_2 \sin(nt-\phi+\psi)$, we have, for the mean activity A_{1m} of the leading machine,

$$\begin{aligned} A_{1m} = \frac{1}{T}\int_0^T E\gamma\,dt &= \frac{E_0E_1}{T(R^2+n^2L^2)^{\frac{1}{2}}}\int_0^T \sin(nt-e)\sin(nt+\phi+\psi)\,dt \\ &= \frac{1}{2}\,\frac{E_0E_1}{(R^2+n^2L^2)^{\frac{1}{2}}}\cos(\phi+\psi+e) \\ &= \frac{1}{2}\,\frac{E_0E_1}{R^2+n^2L^2}\{R\cos(\phi+\psi) - nL\sin(\phi+\psi)\}. \quad \text{...(42)} \end{aligned}$$

To find the mean activity of the following machine we have only to change the sign of ϕ in this expression. We get

$$A_{2m}=\frac{1}{2}\frac{E_0E_2}{R^2+n^2L^2}\{R\cos(\phi-\psi)+nL\sin(\phi-\psi)\}. \quad \text{.................(43)}$$

If the machines be equal $E_1=E_2$, and $\psi=0$, so that

$$A_{1m}=\frac{E_1{}^2\cos\phi}{R^2+n^2L^2}(R\cos\phi-nL\sin\phi), \quad \text{...............(44)}$$

$$A_{2m}=\frac{E_1{}^2\cos\phi}{R^2+n^2L^2}(R\cos\phi+nL\sin\phi). \quad \text{...............(45)}$$

22. Tendency of equal machines to opposition of phase. Parallel alternators. Since ϕ is less than $\pi/2$, both $\cos\phi$ and $\sin\phi$ are positive, and therefore the following machine does more work than the leading machine. Hence, unless each is completely controlled by the prime-mover, the leading machine will increase its lead, and this will go on until $2\phi=\pi$, when the two machines will be in exactly opposite phases, and will exactly neutralise one another. This tendency to assume opposition of phase depends on the difference $A_{2m}-A_{1m}$, and this having the factor $nL/(R^2+n^2L^2)$, has a maximum value, for a given resistance and a given period of alternation, when $nL=R$.

The machines thus arrange themselves so that no current passes in the wires joining their terminals, and these wires alternate in relative potential with the period of the machines, and each is at any instant very approximately at one potential throughout. It might therefore be inferred that if a working circuit be joined from one wire to the other, a current will pass through that circuit, and that the two machines will control one another so as to keep in the same phase in supplying it. We shall consider this case as a further example of the theory.

Let 2ϕ be the difference of phase with reference to the external circuit, so that at time t, $E\sin(nt+\phi)$, $E\sin(nt-\phi)$ are the electromotive forces of the two machines, γ_1, γ_2 the currents, L the coefficient (supposed constant) of self-induction for each, r the resistance of each machine from one point of attachment to the other point, and R the resistance of the external circuit. We shall suppose that the external circuit has no sensible self-induction, and that the whole work there developed is spent in overcoming resistance, for example, in lighting glow-lamps. By considering the circuit through each machine and the external resistance, according to Kirchhoff's rule, taking into account the electromotive force of self-induction in each circuit, remembering that the current in the external resistance is $\gamma_1+\gamma_2$, and therefore the difference of potential between the terminals $R(\gamma_1+\gamma_2)$, we find the equations

$$\left.\begin{aligned}L\frac{d\gamma_1}{dt}+r\gamma_1+R(\gamma_1+\gamma_2)&=E\sin(nt+\phi),\\ L\frac{d\gamma_2}{dt}+r\gamma_2+R(\gamma_1+\gamma_2)&=E\sin(nt-\phi).\end{aligned}\right\} \quad \text{.............(46)}$$

Adding and subtracting, we get

$$\left.\begin{aligned} L\frac{d}{dt}(\gamma_1+\gamma_2)+(2R+r)(\gamma_1+\gamma_2)&=2E\cos\phi\,.\,\sin nt,\\ L\frac{d}{dt}(\gamma_1-\gamma_2)+r(\gamma_1-\gamma_2)&=2E\sin\phi\,.\,\cos nt.\end{aligned}\right\} \quad \text{......(47)}$$

Solving these we find, as in (24),

$$\gamma_1+\gamma_2=\frac{2E\cos\phi}{\{(2R+r)^2+n^2L^2\}^{\frac{1}{2}}}\sin(nt-e), \quad \text{............(48)}$$

$$\gamma_1-\gamma_2=\frac{2E\sin\phi}{\{r^2+n^2L^2\}^{\frac{1}{2}}}\cos(nt-e'), \quad \text{......................(49)}$$

where
$$\tan e=\frac{nL}{2R+r}, \quad \tan e'=\frac{nL}{r}. \quad \text{......................(50)}$$

Hence, if A_{1m} be the activity of the leading machine,

$$\begin{aligned} A_{1m}&=\frac{E}{2T}\int_0^T(\gamma_1+\gamma_2+\gamma_1-\gamma_2)\sin(nt+\phi)\,dt\\ &=\frac{E^2}{T}\left\{\frac{\cos\phi}{\{(2R+r)^2+n^2L^2\}^{\frac{1}{2}}}\int_0^T\sin(nt-e)\sin(nt+\phi)\,dt\right.\\ &\qquad\left.+\frac{\sin\phi}{(r^2+n^2L^2)^{\frac{1}{2}}}\int_0^T\cos(nt-e')\sin(nt+\phi)\,dt\right\}\\ &=\frac{1}{2}\,\frac{E^2}{(2R+r)^2+n^2L^2}\{(2R+r)\cos^2\phi-nL\sin\phi\cos\phi\}\\ &\qquad+\frac{1}{2}\,\frac{E^2}{r^2+n^2L^2}(r\sin^2\phi+nL\sin\phi\cos\phi). \quad \text{.........(51)}\end{aligned}$$

The mean activity A_{2m} of the following machine may be got from A_{1m} by altering the sign of ϕ throughout the expression on the right. Hence

$$\begin{aligned} A_{2m}&=\frac{1}{2}\,\frac{E^2}{(2R+r)^2+n^2L^2}\{(2R+r)\cos^2\phi+nL\sin\phi\cos\phi\}\\ &\qquad+\frac{1}{2}\,\frac{E^2}{r^2+n^2L^2}(r\sin^2\phi-nL\sin\phi\cos\phi). \quad \text{.........(52)}\end{aligned}$$

$A_{1m}-A_{2m}$ is positive, that is more work is done by the leading than by the following machine. The lead will therefore tend to zero, and the machines to settle down into coincidence of phase with reference to the external circuit, that is, into opposite phases with reference to their own circuit, which agrees with the result already obtained.

23. Theory of alternating motor. We shall consider only one more case of this theory, that of an alternating motor connected by its terminals to two conductors upon which an alternating difference of potential is impressed by other machines. Let the motor be started so as to have the same period of alternation. Then denoting by R the resistance of the motor-armature and the leads, up to the point at which the difference of potential is impressed, by L the self-inductance for the same part of the circuit, by $E_1 \sin(nt+\phi)$ the impressed difference of potential at time t, by $E_2 \sin(nt-\phi)$ the back electromotive force of the motor at the same instant, we have for the equation of the current,

$$L\frac{d\gamma}{dt}+R\gamma=E_1\sin(nt+\phi)-E_2\sin(nt-\phi). \quad \text{............(53)}$$

This equation differs only in the sign of E_2 from (39) from which (42) and (43) above are deduced. Hence taking the value of A_2m in (43) we have for the mean electric activity received by the motor

$$A_{2m}=\frac{1}{2}\frac{E_0E_2}{R^2+n^2L^2}\{R\cos(\phi-\psi)+nL\sin(\phi-\psi)\}, \quad \text{.......(54)}$$

where

$$\left.\begin{aligned} E_0&=(E_1{}^2+E_2{}^2-2E_1E_2\cos 2\phi)^{\frac{1}{2}},\\ \tan\psi&=-\frac{E_1+E_2}{E_1-E_2}\tan\phi. \end{aligned}\right\} \quad \text{..................(55)}$$

The second of (55) gives

$$\cos\psi=(E_1-E_2)\cos\phi/E_0, \quad \sin\psi=-(E_1+E_2)\sin\phi/E_0,$$

and these values substituted in (54) yield

$$A_{2m}=\frac{1}{2}\frac{E_2}{R^2+n^2L^2}\{E_1(R\cos 2\phi+nL\sin 2\phi)-E_2R\}. \quad \text{......(56)}$$

24. Maximum activity of motor. Explanation of self-synchronizing action. Now 2ϕ being the difference of phase cannot be numerically greater than π, and therefore the work *received* by the motor is less when 2ϕ is negative than when it is positive, that is, less when the motor is leading than when it is following. Hence the motor will tend to run slower when leading and faster when following ; or the difference of phase will tend towards zero. Also so long as 2ϕ is not far from zero A_{2m} is less the greater the lead, and greater the greater the lag, and in nearly the same proportion. Hence when the machines are once in phase any small deviation is opposed by a proportional corrective tendency. This depends almost entirely on the term involving the factor $nL/(R^2+n^2L^2)$ in the value of A_{2m} given in (56), and therefore for a given resistance R, and period of alternation T, has its greatest value when $nL=R$, or $L/R=T/2\pi$.

Writing in (56)

$$\sin 2\phi'=R/(R^2+n^2L^2)^{\frac{1}{2}}, \quad \cos 2\phi'=nL/R^2+n^2L^2)^{\frac{1}{2}}, \quad \text{.........(57)}$$

we get $$A_{2m}=\frac{1}{2}\frac{E_2}{R^2+n^2L^2}\{E_1(R^2+n^2L^2)^{\frac{1}{2}}\sin 2(\phi+\phi')-E_2R\}, \quad \text{.....(58)}$$

which is obviously a maximum when $\phi + \phi' = \pi/4$. We have then

$$A_{2m} = \frac{1}{2} \frac{E_2}{R^2 + n^2L^2} \{E_1 (R^2 + n^2L^2)^{\frac{1}{2}} - E_2R\}. \quad \text{.............(59)}$$

The value of A_{2m} is positive if

$$\frac{E_1}{E_2} > \frac{R}{(R^2 + n^2L^2)^{\frac{1}{2}}},$$

which may be the case even if $E_2 > E_1$. Hence we have the curious result that an alternating machine may act as a motor even if its electromotive force be greater than the impressed or driving electromotive force.

A qualitative explanation of the results given above for two alternators can be given graphically by taking the areas of curves drawn to represent the activity at each instant. From these it will at once appear which machine is doing the greater amount of work. The reader may easily construct these curves by drawing for each machine, from the curves giving the current and electromotive force at each instant a new curve, the ordinates of which are the products of the corresponding ordinates of the former.

The theory just given of the working of alternating machines in the same circuit is (apart from notation and mode of statement) substantially that due to Dr. J. Hopkinson. [See his *Collected Papers.*] Its conclusions were verified by him in 1884, in experiments made with two De Meritens machines made for the lighthouse at Tino. Some very striking experiments are described by Mr. Mordey in a paper on alternate-current working, which contains moreover much interesting practical information on this subject. In discussion some difference of opinion was expressed as to whether Mr. Mordey's results were in accordance with the mathematical theory. It is to be remembered however that the theory does not take into account the action of the armature currents in the field-magnets, nor of the variation of self-induction. The subject is further dealt with in treatises on Alternating Currents.

25. Treatment of part of a circuit. We may apply, as we have already done repeatedly above (see for example VIII. 22), the mode of treatment adopted for the whole circuit to a part of it, taking for E the impressed electromotive force on the part of the circuit considered, and for R and L the proper values for that part only. We find that the effect of self-induction is virtually to increase the resistance from R to $(R^2 + n^2L^2)^{\frac{1}{2}}$, that is to substitute impedance for resistance, and to produce a difference of phase between the current and the impressed electromotive force given also by (24) and (25). But the resistance of a conductor is the activity spent in it by unit current in producing heat; hence the resistance in this sense is not increased.

26. Effect of impedance of measuring instruments. Phase differences. The impedance of a current electrodynamometer or current balance or alternate-current ammeter, through both coil systems of which flows the whole current in the main circuit, cannot, if it be low (as it generally is) in comparison with that of the rest of the circuit, affect appreciably the strength of the current by its introduction ; and since the whole current passes through both sets of coils, the instrument will give the mean square of the current passing.

It may be otherwise however with a fine wire instrument used as a shunt to measure very accurately the difference of potential between two points of the circuit. The inductance of such an instrument may be considerable, and if it be used alone its impedance will affect the result, though of course the effect of inductance is kept relatively small by the high value of the resistance. Since the value of the impedance depends on the period of alternation, it will have different values when the instrument is connected to circuits in which the periods are different. To obviate the uncertainty and inconvenience arising from this cause, the instrument is made sensitive enough to allow a considerable non-inductive resistance to be joined in series with its own coils. This makes the value of $R/(R^2+n^2L^2)^{\frac{1}{2}}$ approximately unity. Some calculations made by Prof. T. Gray, for Lord Kelvin's vertical scale voltmeter, give for this ratio with only the resistance of the instrument (640 ohms) included, and a period of alternation of $\frac{1}{100}$ of a second, the value ·9976, which is within $\frac{1}{4}$ per cent. of unity. Plainly the error caused by the impedance in this case is small with any period commonly employed, and can be made still smaller by the introduction of non-inductive resistance. The difference of phase between the currents through the coils of the instrument, and the difference of potential [given by (25) above] is therefore small. This difference of phase, it is to be remembered, does not affect the value of the mean square of the difference of potential, provided the amplitude be corrected for the effect of inductance. It is to be noticed that it is here supposed that the two coils of the wattmeter are so placed that the mutual inductance between them may be taken as zero.

It is however of importance in the action of a wattmeter, of which one coil is placed in the main circuit, and the other as a shunt between the extremities of the portion of the circuit in which the activity is to be estimated. For let the circuit divide into two parts, each forming a derived current with the other, and L_1, L_2, R_1, R_2, ${}_m\gamma_1$, ${}_m\gamma_2$, be the inductances, the resistances and the maximum currents in the two parts, ${}_m\gamma$ the maximum total current in the circuit, and e_1', e_2', the difference of phase between ${}_m\gamma$ and ${}_m\gamma_1$, ${}_m\gamma_2$, respectively, then the general formula of VIII. 22, above, for the difference of phase between the total current in the circuit and the applied electromotive force at the common terminals of two parallel conductors, gives in this case

$$\tan\theta = \frac{n\{L_1(R_2^2+n^2L_2^2)+L_2(R_1^2+n^2L_1^2)\}}{R_1(R_2^2+n^2L_2^2)+R_2(R_1^2+n^2L_1^2)}\,. \quad \ldots\ldots(60)$$

and by (25), the difference of phase e_1 between the impressed electromotive force and the current γ_1 is given by

$$\tan e_1 = \frac{nL_1}{R_1}.$$

Hence for the lag in phase e_1' $(=e_1-\theta)$ of the current γ_1 behind the main current we have

$$\tan e_1' = \tan(e_1-\theta) = \frac{n(L_1R_2 - L_2R_1)}{R_2(R_1+R_2)+n^2L_2(L_1+L_2)}. \quad \text{......(61)}$$

An interchange of suffixes in this result of course gives $\tan e_2'$.

A method of determining the difference of phase between the currents in two branches of the same circuit, or between two currents of the same period, will presently be explained.

The value of the square of the maximum total current is easily found to be

$${}_m\gamma^2 = E^2\frac{(R_1+R_2)^2+n^2(L_1+L_2)^2}{(R_1{}^2+n^2L_1{}^2)(R_2{}^2+n^2L_2{}^2)}, \quad \text{..............(62)}$$

and by (36) $${}_m\gamma_1{}^2 = \frac{E^2}{R_1{}^2+n^2L_1{}^2}, \quad {}_m\gamma_2{}^2 = \frac{E^2}{R_2{}^2+n^2L_2{}^2}. \quad \text{..............(63)}$$

so that $$\frac{{}_m\gamma_1{}^2}{R_2{}^2+n^2L_2{}^2} = \frac{{}_m\gamma_2{}^2}{R_1{}^2+n_2L_1{}^2} = \frac{{}_m\gamma^2}{(R_1+R_2)^2+n^2(L_1+L_2)^2}. \quad \text{......(64)}$$

The difference of phase ϕ between the two currents $\gamma_1+\gamma_2$ can be found as follows. Let $\gamma=\gamma_1+\gamma_2$. Use three alternate-current direct-reading ammeters to measure γ, γ_1, γ_2, and let A, A_1, A_2 be their readings. Then, since

$$\gamma^2 = \gamma_1{}^2+\gamma_2{}^2+2\gamma_1\gamma_2,$$

we get $$A^2 - A_1{}^2 - A_2{}^2 = 2A_1A_2\cos\phi.$$

27. Condition that difference of phase between currents in parallel may be insensible. If either L_1, L_2, be both small or $L_1/L_2 = R_1/R_2$, the difference of phase between the two currents ${}_m\gamma_1$, ${}_m\gamma_2$, will be insensible. If the first condition is fulfilled both parts of the circuit will have currents agreeing in phase with the difference of potential between the terminals, and on the usually allowable supposition of negligible mutual inductance, a wattmeter whose coils are included in them will measure accurately the power expended. It will, on the same supposition, also measure accurately the power expended *while the wattmeter is on circuit*, if the ratio $R/(R^2+n^2L^2)^{\frac{1}{2}}$ be approximately unity for the fine wire circuit, since the main current passes through the other coil, and it can be shown that the deflection will be the same as would be produced by a constant activity A_m given by the equation

$$A_m = \frac{1}{T}\int_0^T V\gamma\,dt \quad \text{..............................(65)}$$

where V, γ, are the values of the difference of potential and the current at time t. If also $(R^2+n^2L^2)^{\frac{1}{2}}$ for the thick wire coil be small in comparison with the same quantity for the part of the main circuit in which the activity is being measured, the inclusion of the wattmeter will not affect the circuit, and the activity shown by the instrument may be taken as that existing when it is not applied.

28. Apparent and true mean activity. The general problem of finding the ratio of the apparent to the true mean activity as shown by the wattmeter can now be solved with great ease. For let A, B, be the points at which the terminals of the fine wire coil system are attached to the main circuit; let R_1, R_2, L_1, L_2, be the resistances and inductances of the fine wire and thick wire circuits between A, B, and γ_1, γ_2, the currents in them; then by (24), if the difference of potentials between the terminals AB is $E_0 \sin nt$,

$$\left.\begin{aligned} \gamma_1 &= \frac{E_0}{(R_1{}^2+n^2L_1{}^2)^{\frac{1}{2}}} \sin(nt-e_1), \\ \gamma_2 &= \frac{E_0}{(R_2{}^2+n^2L_2{}^2)^{\frac{1}{2}}} \sin(nt-e_2), \end{aligned}\right\} \qquad \text{(66)}$$

with $\tan e_1 = nL_1/R_1$, $\tan e_2 = nL_2/R_2$. The former [$\tan e_1$] is usually, as remarked below, negligibly small.

The current through the fine wire coil is therefore the same as if the resistance in its circuit between the points A, B, were without inductance, and the difference of potential had the value obtained by multiplying the above value of γ_1 by R_1. Hence if A'_m be the apparent activity,

$$\begin{aligned} A'_m &= \frac{1}{T}\frac{E_0{}^2R_1}{(R_1{}^2+n^2L_1{}^2)^{\frac{1}{2}}(R_2{}^2+n^2L_2{}^2)^{\frac{1}{2}}}\int_0^T \sin(nt-e_1)\sin(nt-e_2)\,dt \\ &= \frac{1}{2}\frac{E_0{}^2R_1\cos(e_1-e_2)}{(R_1{}^2+n^2L_1{}^2)^{\frac{1}{2}}(R_2{}^2+n^2L_2{}^2)^{\frac{1}{2}}}, \end{aligned} \qquad \text{(67)}$$

that is the apparent activity is $\frac{1}{2}$ the product of the maximum values of the two currents by the resistance R_1 of the fine wire branch, and by the cosine of the phase-angle between the currents.

The true mean activity A_m would be obtained if the current through the fine wire branch had the value $E_0 \sin nt/R_1$. In that case the phase-angle between the two currents would be e_2. Hence as before

$$A_m = \frac{1}{2}\frac{E_0{}^2\cos e_2}{(R_2{}^2+n^2L_2{}^2)^{\frac{1}{2}}}. \qquad \text{(68)}$$

Hence

$$\frac{A_m}{A'_m} = \frac{(R_1{}^2+n^2L_1{}^2)^{\frac{1}{2}}}{R_1}\,\frac{\cos e_2}{\cos(e_1-e_2)}. \qquad \text{(69)}$$

The angle e_1 is the angular phase difference between V and the current γ_1 in the fine wire coil, and we have

$$\cos e_1 = R_1/(R_1{}^2+n^2L_1{}^2)^{\frac{1}{2}}.$$

Thus for the true activity, in terms of the apparent activity, we get the working formula

$$A_m = \frac{\cos e_2}{\cos e_1 \cos (e_1 - e_2)} A'_m.$$

Since $\sin e_1 = nL_1/(R_1^2 + n^2L_1^2)^{\frac{1}{2}}$, $\cos e_1 = R_1/(R_1^2 + n^2L_1^2)^{\frac{1}{2}}$, we may calculate $\cos e_2/\cos (e_1 - e_2)$ in terms of L_1, L_2, R_1, R_2. Thus we obtain

$$\frac{A_m}{A'_m} = \frac{R_2}{R_1}\frac{R_1^2 + n^2L_1^2}{R_1R_2 + n^2L_1L_2} = \frac{1 + n^2\tau_1^2}{1 + n^2\tau_1\tau_2}, \quad \text{..............(70)}$$

where τ_1, τ_2, are written for L_1/R_1, L_2/R_2, respectively, the so-called time constants of the two parts of the circuit. It is to be observed that the voltmeter coil of the current-meter is for practical purposes non-inductive. For the wire being long and thin, nL_1/R_1 is very small, and so the instrument gives nearly correct readings.

Now in general $\tau_1 < \tau_2$, hence as a rule the wattmeter will give too high a result.

29. Measurement of difference of phase between alternating currents. The angle ϕ of difference of phase between the currents in two such branches may be measured as follows (Blakesley, *Phil. Mag.* April 1888). We have seen that a current-dynamometer in any branch measures the mean square of the current in that branch. This has the value ${}_m\gamma^2/2$, where ${}_m\gamma$ denotes the maximum value of the current in the branch. Now let ${}_m\gamma_1$, ${}_m\gamma_2$ be the maximum currents in the two branches, and let two electrodynamometers, or alternate-current meters, be arranged one to measure ${}_m\gamma_1^2/2$, and the other ${}_m\gamma_2^2/2$, and let a third be placed, with one coil in one, and the other coil in the other of the two branches in question. The action on the third electrodynamometer, or alternate-current meter, at any instant will be proportional to $\gamma_1\gamma_2 \cos \phi$. Hence the instrument will give a reading proportional to $\frac{1}{2}{}_m\gamma_1{}_m\gamma_2 \cos \phi$. If then D_1, D_2, D_3, be the readings of the current-meters, A, B, C, their constants, so that

$$D_1/A = {}_m\gamma_1^2/2, \quad D_2/B = {}_m\gamma_2^2/2, \quad D_3/C = \tfrac{1}{2}{}_m\gamma_1{}_m\gamma_2 \cos \phi,$$

we get

$$\cos \phi = \frac{D_3}{\sqrt{D_1D_2}} \cdot \frac{\sqrt{AB}}{C}. \quad \text{............ ....(71)}$$

If the current-meters are direct reading so that $A = B = C = 1$, the factor $\sqrt{AB}/C$ is unity.

Of course if three current-meters are not available a single current-meter may be used to take the three readings in succession (or to eliminate error several sets of readings may be taken and combined). In that case $A = B = C$ and

$$\cos \phi = \frac{D_3}{\sqrt{D_1D_2}}. \quad \text{........... ....(72)}$$

Blakesley* has also given a very simple method of measuring the total activity spent in the primary circuit of a transformer, that is of finding the whole electrical work done per unit of time in feeding the secondary, and directly or indirectly in dissipation.

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* *Phil. Mag.* 1891 or *Proc. Phys. Soc.* 11, pt. 2, 1891.

30. Transformer. Measurement of activity. A transformer consists, as is well known, of a primary and secondary circuit wound round a core of laminated iron, in general in such a manner that as nearly as possible all lines of magnetic induction, which pass through any spire of one of the coils, also pass through every other spire of the same or the other coil. It is not however safe to assume that this is always the case, and serious errors may arise through making the assumption in all circumstances.

Now let a current electrodynamometer, or alternate-current meter, be placed in the primary circuit, and another be arranged with one coil in the primary and the other coil in the secondary circuit. Then if D_1 be the deflection-reading of the first instrument, A the constant of reduction of the readings to (current)^2, D_{12} and B the corresponding quantities for the other instrument (both deflections being taken positive), N_1, N_2, the number of turns in the primary and secondary respectively, R_1, R_2, their resistances, and A_m the mean activity to be measured,

$$A_m = R_1 \frac{D_1}{A} + R_2 \frac{N_1}{N_2} \frac{D_{12}}{B} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(73)$$

under certain assumptions.

This method is applicable whatever may be the law of variation of current.

To prove this relation the equations of a primary and secondary circuit given in (11) and (12), p. 234, may be used, and may be modified by writing E for $E_0 \sin nt$, since we make here no assumption as to the mode of variation of the current or electromotive force, since these equations hold for any primary or secondary whether or not containing iron. We shall first also write B_1, B_2, for the total inductions through a single turn of the primary and the secondary respectively, and N_1, N_2, for the number of turns in the coils. Thus, since $E_2 = 0$, we can write the equations referred to in the form

$$\left.\begin{aligned} R_1\gamma_1 + N_1 \frac{d\mathsf{B}_1}{dt} &= E, \\ R_2\gamma_2 + N_2 \frac{d\mathsf{B}_2}{dt} &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(74)$$

Then we have

$$R_1 \frac{D_1}{A} = \frac{R_1}{T}\int_0^T \gamma_1{}^2 dt = \frac{1}{T}\int_0^T E\gamma_1 dt - \frac{N_1}{T}\int_0^T \gamma_1 \frac{d\mathsf{B}_1}{dt} dt$$

or

$$\frac{1}{T}\int_0^T E\gamma_1 dt = R_1 \frac{D_1}{A} + \frac{N_1}{T}\int_0^T \gamma_1 \frac{d\mathsf{B}_1}{dt} dt, \quad \ldots\ldots\ldots\ldots\ldots(75)$$

by the first of (74).

If D_{12} be, in the same way, the reading of the second instrument, taking account of the sign of the deflection, and B its constant, we have

$$R_2 \frac{D_{12}}{B} = \frac{R_2}{T} \int_0^T \gamma_1 \gamma_2 dt = -\frac{N_2}{T} \int_0^T \gamma_1 \frac{d\mathsf{B}_2}{dt} dt, \quad \text{............(76)}$$

by the second of (74).

If now we assume that $\mathsf{B}_1 = \mathsf{B}_2$, we get from the last equation

$$\frac{N_1}{T} \int_0^T \gamma_1 \frac{d\mathsf{B}_1}{dt} dt = -R_2 \frac{N_1}{N_2} \frac{D_{12}}{B}.$$

Substituting from this in (75) we find

$$\frac{1}{T} \int_0^T E\gamma_1 dt = R_1 \frac{D_1}{A} - R_2 \frac{N_1}{N_2} \frac{D_{12}}{B}, \quad \text{..................(77)}$$

and the quantity on the left is the mean value of the total activity. Thus the total activity is given by the expression on the right in terms of the readings of the electrodynamometers, or the alternate-current ammeters.

It is to be noticed that since γ_1, γ_2, are on the whole in opposite directions, the sign of D_{12} must be opposite to that of D_1. Thus $R_2 N_1 D_{12}/N_2 B$ is really negative, and the total rate of working is greater than the first term, which represents the activity spent in heat in the circuit. Hence if we agree to take the positive numerical value of the reading of the second instrument for D_{12}, we may, putting A_{1m} for the mean activity on the primary, write (76) in the form

$$A_{1m} = R_1 \frac{D_1}{A} + R_2 \frac{N_1}{N_2} \frac{D_{12}}{B}. \quad \text{....................(78)}$$

This method and result were given by Mr. T. H. Blakesley for a transformer on the assumption that the currents followed the simple sine law of variation: in the demonstration here given no assumption at all is made except that $\mathsf{B}_1 = \mathsf{B}_2$. The method is therefore applicable to any transformer, whatever the law of variation followed by the current, provided B_1 may be taken as equal to B_2. This was first pointed out by Ayrton and J. F. Taylor,* whose method of proof is similar to that here given.

31. Proof of validity of method on assumption of constant permeability. This method would hold even if B_1 were not equal to B_2, provided we could suppose the permeability constant during a cycle. In this case the equations of current could be written in the form,

$$\left.\begin{aligned} R_1\gamma_1 + L_1 \frac{d\gamma_1}{dt} + M \frac{d\gamma_2}{dt} &= E, \\ R_2\gamma_2 + L_2 \frac{d\gamma_2}{dt} + M \frac{d\gamma_1}{dt} &= 0, \end{aligned}\right\} \quad \text{....................(79)}$$

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* *Proc. Phys. Soc.* Dec. 1891.

since L_1, L_2, M, do not vary in a cycle if the permeability does not. Hence multiplying the first of these by γ_1, and calculating the mean value of each quantity by integrating over a whole period, we get

$$A_{1m} = \frac{1}{T}\int_0^T E\gamma_1 dt = \frac{R_1}{T}\int_0^T \gamma_1{}^2 dt + \frac{M}{T}\int_0^T \gamma_1 \frac{d\gamma_2}{dt} dt, \quad \text{.........(80)}$$

since the integral of $\gamma_1 d\gamma_1/dt \,.\, dt$ over a period is zero.

But if we multiply the second of (79) by γ_1 and take mean values as before, we find

$$\frac{R_2}{T}\int_0^T \gamma_1\gamma_2 dt + \frac{L_2}{T}\int_0^T \gamma_1 \frac{d\gamma_2}{dt} dt = 0,$$

since the last integral vanishes as before. Thus

$$\frac{M}{T}\int_0^T \gamma_1 \frac{d\gamma_2}{dt} dt = -\frac{M}{L_2}\frac{R_2}{T}\int_0^T \gamma_1\gamma_2 dt. \quad \text{...............(81)}$$

Substituting in (80) we get

$$A_{1m} = \frac{R_1}{T}\int_0^T \gamma_1{}^2 dt - \frac{M}{L_2}\frac{R_2}{T}\int_0^T \gamma_1\gamma_2 dt, \quad \text{...............(82)}$$

or putting in the readings of the dynamometers (taking D_{12} positive as before),

$$A_{1m} = R_1\frac{D_1}{A} + R_2\frac{M}{L_2}\frac{D_{12}}{B} = R_1\frac{D_1}{A} + R_2\frac{N_1}{N_2}\frac{D_{12}}{B}, \quad \text{.........(83)}$$

since approximately $M = N_1N_2$, $L_2 = N_2{}^2$. This is the same result as before, but obtained under a different assumption, not however involving any hypothesis as to the mode of variation of the current.

32. Constant permeability involves zero dissipation in iron core of transformer. It is to be observed that this supposition of no variation of L_1, L_2, or M, is equivalent to supposing that all the activity is employed in generating heat in the two circuits. For if the second of (79) be multiplied by γ_2, and then integrated for mean values, it gives

$$\frac{R_2}{T}\int_0^T \gamma_2{}^2 dt + \frac{M}{T}\int_0^T \gamma_2 \frac{d\gamma_1}{dt} dt = 0.$$

This added to (80) gives

$$A_{1m} = \frac{R_1}{T}\int_0^T \gamma_1{}^2 dt + \frac{R_2}{T}\int_0^T \gamma_2{}^2 dt + \frac{M}{T}\int_0^T \frac{d(\gamma_1\gamma_2)}{dt} dt,$$

and the last term vanishes since the integration is round a closed cycle. Thus

$$A_{1m} = \frac{R_1}{T}\int_0^T \gamma_1{}^2 dt + \frac{R_2}{T}\int_0^T \gamma_2{}^2 dt, \quad \text{...............(84)}$$

and all that is measured is the mean value of $R_1\gamma_1{}^2$ *plus* that of $R_2\gamma_2{}^2$, or the total mean activity is equal to the rate of generation of heat in the secondary plus that in the primary. The activity could in this case be equally well measured by placing an alternate-current ammeter in the primary, and another in the secondary, as by Blakesley's method.

The supposition thus made above therefore excludes all dissipation of energy otherwise than by direct heating of the circuits by the currents. It has been urged that on the analogy of the behaviour of ordinary bodies under strain produced by stress varied in rapid cycles, there ought to be no dissipation of energy due to lagging of the magnetization behind the magnetic force in the cycle, as explained in II. 27 above, or, as it is now called, *hysteresis* action, in iron subjected to rapid cycles of magnetic stress. On this view magnetic like elastic hysteresis is only important in slow cycles. This analogy appears a plausible one, but any opinion founded on it must be tested by direct experiment, and it would appear that the results of such experiments are adverse to the view here indicated. Now it has been given as the result of experiment by several observers* that there is in rapid cycles dissipation of energy in the core of the same order of magnitude as in slow cycles; but that there is much less when the transformer is loaded by closing the secondary circuit through a low resistance, than when the secondary circuit is open.

This result is questioned by Ewing, who gives as the result of experiments on a transformer core, an anchor ring made of iron wire insulated to prevent eddy currents, that, for the same frequency of reversal and limits of magnetization, the loss by magnetic hysteresis is just as great when the transformer is heavily loaded, as when its secondary circuit is open.†

The rate of loss by hysteresis is however in all cases small in comparison with the whole activity.

Assuming the truth of Blakesley's formula as deduced from the hypothesis of no magnetic leakage, we can find the amount of energy spent in eddy currents and magnetic hysteresis in the iron.

33. Energy spent in hysteresis. Assuming for simplicity that the electrodynamometers are direct-reading instruments, or if not that D_1, D_{12}, are reduced readings expressing each a mean square of a current measured in amperes, so that the constants $A=B=1$, then R_1, R_2, being taken in ohms, A_{1m} will be given in watts. If now we suppose a third electrodynamometer placed in the secondary circuit, and D_2 in like manner be its reading, we shall have

$$R_2 D_2 = \frac{R_2}{T}\int_0^T \gamma_2{}^2 dt.$$

Thus we have $$A_{1m} = R_1 D_1 + R_2 D_2 + R_2\left(\frac{N_1}{N_2} D_{12} - D_2\right). \quad \text{..............} (85)$$

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* Warburg and Hönig, *Wied. Ann.* 20, 1883.

† Tanakadate, *Phil. Mag.* Sept. 1889. See also Ewing, *Magnetism in Iron and other Metals*, §§ 83, 180.

The two first terms on the right express the whole work done in heating the wires of the primary and secondary, the third term that spent in heating the iron by eddy currents and hysteresis.

If R_2' be the resistance of the external part of the secondary, and the work done in that be wholly spent in heat, the energy there spent is $R_2'D_2$. Thus if e be the electrical efficiency of the transformer

$$e=\frac{R_2'D_2}{R_1D_1+R_2\frac{N_1}{N_2}D_{12}}. \qquad \text{(86)}$$

34. Difference of Potential between terminals of primary. From the expression for A_{1m} can be found at once the difference of potential between the terminals of the primary. For if R_1' be the external resistance of the primary circuit between its terminals, we have instead of (74)

$$\left.\begin{aligned} R_1'\gamma_1+N_1\frac{d\mathsf{B}_1}{dt}&=V,\\ R_2\gamma_2+N_2\frac{d\mathsf{B}_2}{dt}&=0. \end{aligned}\right\} \qquad \text{(87)}$$

Squaring the first of these we get

$$V^2=R_1'^2\gamma_1^2+N_1^2\left(\frac{d\mathsf{B}_1}{dt}\right)^2+2R_1'N_1\gamma_1\frac{d\mathsf{B}_1}{dt}.$$

Hence if V'^2 be the mean square of the difference of potential V,

$$V'^2=\frac{R_1'^2}{T}\int_0^T\gamma_1^2dt+\frac{N_1^2}{T}\int_0^T\left(\frac{d\mathsf{B}_1}{dt}\right)^2dt+\frac{2R_1'N_1}{T}\int_0^T\gamma_1\frac{d\mathsf{B}_1}{dt}dt.$$

The first integral is, as we have seen above, $R_1'^2D_1$, and the third is $2R_1'R_2D_{12}N_1/N_2$. The second integral can be found by the second of (87) since B_2 is taken as equal to B_1. Thus

$$\frac{1}{T}\int_0^T\left(\frac{d\mathsf{B}_1}{dt}\right)^2dt=\frac{R_2^2}{N_2^2}\frac{1}{T}\int_0^T\gamma_2^2dt=\frac{R_2^2}{N_2^2}D_2.$$

Substituting these values for the integrals we get

$$V'^2=R_1'^2D_1+\frac{R_2^2}{N_2^2}D_2+2R_1'R_2\frac{N_1}{N_2}D_{12}. \qquad \text{(88)}$$

The above results are all independent of the law of variation of the current and involve only the assumption $\mathsf{B}_1=\mathsf{B}_2$. They are due to Blakesley, but were first proved by methods similar to those used above, by Ayrton and Taylor in their paper above referred to.

35. Measurement of activity by current-meter only. In any practical case of measurement of power in which a wattmeter is inapplicable, if the actual resistance of the portion of the circuit considered is known and the mean square of the current can be measured with accuracy, the product of the two will, as shown in 17 above, be the true mean value of the activity if that is spent in heat. This of course will be given in watts, if the resistance is taken in ohms and the current in amperes.

As we have seen above, the proper mean value of the current, and of the difference of potential, and therefore also of the activity, can be found for any part of a circuit in the case of negligible self-induction, either by means of an electrodynamometer, or by means of an electrometer, when the resistance of one part of the circuit is known. When the resistance is unknown or uncertain, as for example in the case of incandescence lamps, the current and difference of potential may be measured for the lamp circuit in the following manner. A coil of german silver wire, having a resistance considerably greater than that of the lamps as arranged, constructed so as to have no self-inductance, is connected in series with a current-meter between the terminals of the machine so as to be a shunt on the lamps. The lamps are brought to their normal brilliancy, and the mean square γ'^2 of the current through the german silver wire measured. If R be the resistance of this wire, including, if appreciable, the resistances of the current-meter and its connections, and R be great in comparison with the self-inductance of the current-meter divided by T, we have for the mean square, V'^2, of the difference of potential between the terminals of the lamp system, the value γ'^2R^2. The current-meter is now employed to measure the whole current flowing to the lamps while their brilliancy is kept the same. Denoting the mean square of this current by $\gamma_1'^2$, we have for the value A_m of the mean activity spent in the lamp system

$$A_m = V'\gamma' = \gamma'\gamma_1'R. \qquad \text{.......................(89)}$$

36. Testing dynamos. Method of Messrs. Hopkinson. Messrs. J. and E. Hopkinson* have employed the following method of testing the efficiency of dynamo-machines, which obviates the difficulty of measuring accurately the mechanical power transmitted to the driving shaft of a dynamo by a steam engine or other motor. Two equal dynamos of the type to be tested are used, and one of these is run as a motor at the required speed and with the proper amount of electrical activity in the circuit. This can be adjusted by suitably varying the magnet resistances of one of the machines. The motor is made to spend the available activity which it gives out in driving the generator, and the difference in power required is supplied by a steam- or other engine, and measured by a Hefner-Alteneck dynamometer, or by any other similar method by which the difference of the pulls in the two parts of the belt is determined. This latter amount of power represents the losses in transmission, and added to the power returned to the generator by the motor gives the mechanical power required to drive the generator. The errors inherent in the determination of mechanical power transmitted to a driven shaft are thus made to affect only the comparatively small balance of power, and the efficiency is obtained to a much higher percentage of accuracy.

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* *Phil. Trans. R.S.* Part i. 1886.

The whole electrical power $E\gamma$ developed by the generator is then found by calculating that spent on each part of the circuit from the observed differences of potential between the terminals of the generator and motor, the current in the circuit, and the known resistances of the different parts of the machines. By adding to this the power w, in watts, wasted in the machine, the power spent in driving it is obtained, and hence at once the gross efficiency $E\gamma/(E\gamma+w)$.

Then the sum of the heats developed in the armature and magnets of each machine, and in the leads and other resistances in the circuit, subtracted from the power transmitted from the engine and measured by the dynamometer, gives a balance which represents the total loss in the circuit over and above those here enumerated. This is made up of power wasted in the iron cores of the armatures and in the pole pieces in consequence of hysteresis or eddy currents, in reversals of the currents in the sections of the armatures, in connexions, in sparking if any, and in the friction of the bearings and brushes. Half of this balance may be taken as spent in each machine. The whole power spent in driving the generator is therefore the sum of the whole electrical power $E\gamma$ given out in the circuit, and half the balance, w say. Thus the efficiency is

$$e=\frac{E\gamma}{E\gamma+w}=1-\frac{w}{E\gamma}, \quad \text{..........................} (90)$$

nearly.

37. Swinburne's method of testing dynamos. Swinburne measures electrically the loss of power w here described, and requires only one machine of the type to be tested. The magnets of the machine are excited separately, so that the armature is under the induction which would exist if the machine were working under the load specified for it. The machine is then driven by a small dynamo which furnishes current at the electromotive force of the machine just sufficient to drive it at the required speed, without any load beyond that involved in w, namely the losses in eddy currents, hysteresis, and friction in the machine which is being tested. The speed can be adjusted as in the tests above described by suitably varying the resistance of the magnet circuit. The power spent on the machine by the small dynamo is determined electrically in the ordinary way by measuring the number of volts difference of potential between the terminals and the current in amperes. The former will of course be approximately the full electromotive force of the machine when working under the prescribed load. The power thus determined, diminished by that spent in heat in the armature (which is generally negligible), is the waste power w required.

The efficiency can then be found by calculating the total electrical activity in the circuit when the machine is running under the prescribed load, by adding to the activity in the external circuit the electrical activities in the armature and magnets, found in watts by multiplying the resistance of each part in ohms by the square of the current in amperes. Call this electrical activity $E\gamma$, as in 17 above. Then the mechanical power spent in driving is $E\gamma + w$. The gross efficiency of the machine is thus $E\gamma/(E\gamma + w)$. The electrical efficiency of the arrangement is $E_1\gamma/E\gamma$, if E_1 be the difference of potential between the terminals of the external circuit. Finally the net efficiency is $E_1\gamma/(E\gamma + w)$.

38. Sumpner's method of testing transformers. On the analogy of the Hopkinson method of testing dynamos just described, Sumpner based the following method of testing power supplied to transformers. Two equal transformers have their primary coils c_1c_2 joined in parallel across the terminals of an alternating dynamo as shown in Fig. 75, and their secondaries C_1C_2 also joined in parallel between the points AB. Non-inductive resistances r and R are included in the primary and secondary circuits as shown.

Supposing the transformers to be alike, and the primary circuits to have the same resistance, the magnetizing currents will be the same in both, and there will be equal electromotive forces at any instant in the secondaries. Thus no current will flow in the secondary circuit whatever the resistance R. A non-inductive resistance r in the primary of either will cause the currents in the primaries to be different, and if r is in the circuit of c_2 a current will flow in the secondary which will load the transformer c_1C_1, and help to magnetize the core of c_2C_2, thus raising the electromotive force in the primary of that transformer.

If however the transformers be somewhat different, for example, so that (to take an example given by Sumpner) No. 1 converts from 100 to 2100 volts, and the other from 100 to 2000 volts, then there will be an electromotive force of 100 volts in the circuit of the secondaries which will produce any desired current if R be properly adjusted.

39. Determination of waste power. If then with two unequal transformers the current flowing through the secondaries be of the proper amount, each transformer will be fully loaded, but one, the more powerful, No. 1 say, will transform up, and the other down. That is the former will take energy from the mains, the other will return energy to the mains. The power-losses occurring in the double transformation are then, in the aggregate, the difference between the power taken by No. 1, and that given back by No. 2, diminished by the amount absorbed in the resistance R, and by the amount spent in heating the connecting wires and instruments applied.

It is only necessary therefore, in order to obtain w, to measure the balance of power supplied to the system at ab, and correct it as described. This may be done with a wattmeter, the fine wire coil of which is placed across the terminals ab, and the current coil in one of the mains, or by the electrometer-method described below. To calculate the efficiency we have then only to find the power W, say, supplied to No. 1 transformer. This can be done nearly enough by measuring the load on either transformer, say by placing a wattmeter with its fine wire coils across *ab*, and its current coil in *c*, or by measuring the difference of potential and current of any coil of either transformer. Then the efficiency of the double transformation, e^2 say, is given by

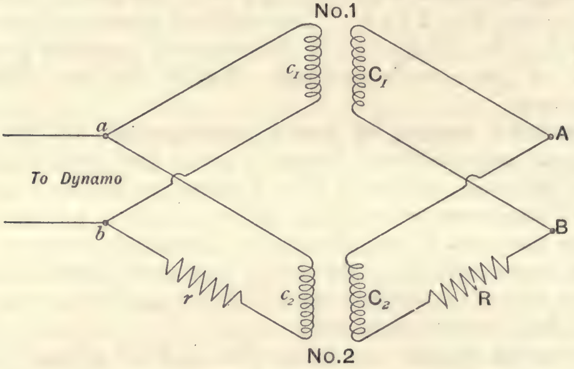

FIG. 75.

$$e^2 = \frac{W}{W+w} = 1 - \frac{w}{W},$$

nearly.

The efficiency of each transformer is approximately the square root of this, or

$$e = 1 - \frac{1}{2}\frac{w}{W} - \frac{1}{8}\frac{w^2}{W^2}, \quad \text{..........................} (91)$$

nearly.

40. Case of two equal transformers. One great advantage of this method lies in the fact that a considerable error in the estimation of w can only slightly affect that of e^2 or e, if e be not very different from unity. This method as it stands is only applicable to two transformers, the electromotive forces of the secondaries of which differ by at least twice the "drop" in difference of potential between the terminals of the secondary of either, when its load is raised from zero to the prescribed value. In the case of two similar transformers Sumpner used a small additional transformer which is able to supply the waste w for the two large transformers to be tested. The primary of this is connected in series with an adjustable non-inductive resistance x, across the main terminals *ab*, and the secondary is placed in, say, No. 2 transformer, in series with either c_2 or C_2, in place of the non-inductive resistance r or R.

This small transformer will supply an amount of energy, depending on the value to which x is adjusted, sufficient to cause any required current to flow in the secondaries of the large transformers. It is only necessary then to measure the energy given out by the small transformer by measuring the current and difference of potential on its primary and secondary, and further to measure as before the power supplied by the mains. The sum of these corrected as before will be w. Then W is measured as before for either of the large transformers, and the efficiency is determined by (91) above.

Different arrangements will suggest themselves to the engineer carrying out these tests as suitable in the varying circumstances in which he may be placed by his instruments, etc.*

41. Three-voltmeter method. Ayrton and Sumpner† gave the following method of measuring the power given out in any portion of a circuit. It will be seen that it is intimately related to the electrometer method described below. Three points on the circuit are taken, two (Fig. 76) AB, between which is the portion of the current in which the activity is to be found, while the portion BC consists of a non-inductive resistance of R ohms. Three alternate-current voltmeters of proper construction are used to give the mean squares of the differences of potential between A and B, B and C, and A and C. If D_1, D_2, D be the readings of these voltmeters each in volts, and A_m the mean activity in watts,

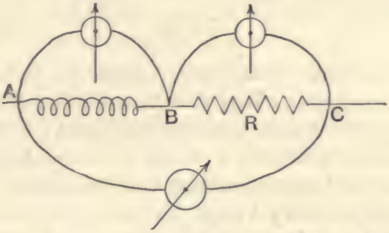

FIG. 76.

$$A_m = \frac{1}{2R}(D^2 - D_1^2 - D_2^2). \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(92)$$

For
$$A_m = \frac{1}{T}\int_0^T V_1 \gamma \, dt,$$

if V_1 be the difference of potential between A and B at any instant. But if V_2 be the difference of potential existing at the same instant between B and C we have $\gamma = V_2/R$. Hence

$$A_m = \frac{1}{RT}\int_0^T V_1 V_2 \, dt.$$

The difference of potential between A and C is at the same instant $V_1 + V_2$, and we have

$$V_1 V_2 = \tfrac{1}{2}\{(V_1 + V_2)^2 - V_1^2 - V_2^2\}.$$

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* See a paper by Ayrton and Sumpner, *Electrician*, Oct. 7, 1892.

† *Proc. R.S.* April 9, 1891, or *Electrician*, April 17, 1891.

Hence
$$A_m = \frac{1}{2RT}\left\{\int_0^T (V_1+V_2)^2\,dt - \int_0^T V_1^2\,dt - \int_0^T V_2^2\,dt\right\},$$

or
$$A_m = \frac{1}{2R}(D^2 - D_1^2 - D_2^2). \qquad\qquad (93)$$

It can be shown by the Theory of Errors of Observation that on the assumption of equal proportional errors in the quantities observed the best arrangement for this measurement is one in which the mean square of the difference of potential between *A* and *B* is equal to that between *B* and *C*. This however is an arrangement in which the power consumed in the non-inductive resistance is equal to the power measured.

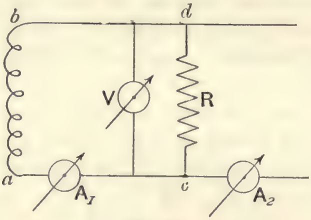

FIG. 77.

42. Method with two current-meters and voltmeter. A modification of this method was proposed by Ayrton and Sumpner, in which two current-meters A_1, A_2, and a voltmeter V are arranged as shown in Fig. 77. *ab* is the portion of the circuit in which the power is to be measured, *cd* a non-inductive resistance placed across its terminals, V is a voltmeter placed parallel to *ab*, and *cd*, and measuring the mean square of the difference of potential between *ac* and *bd*. If V be the difference of potential between a and b at any instant, and γ the current at that instant, the activity is

$$A_m = \frac{1}{T}\int_0^T V\gamma\,dt,$$

or if γ' be the current in *cd* at the same instant,

$$A_m = \frac{R}{T}\int_0^T \gamma\gamma'\,dt, \qquad\qquad (94)$$

since $\gamma' = V/R$. But if D_1 be the reading of the current-meter, A_1, D_2 that of A_2 (each giving the mean square of the current in amperes), we have

$$D_1 = \frac{1}{T}\int_0^T \gamma^2\,dt,$$

and
$$D_2 = \frac{1}{T}\int_0^T (\gamma+\gamma')^2\,dt = \frac{1}{T}\int_0^T (\gamma^2+\gamma'^2+2\gamma\gamma')\,dt$$
$$= D_1 + \frac{D^2}{R^2} + \frac{2}{T}\int_0^T \gamma\gamma'\,dt,$$

if D be the reading of the voltmeter expressed in volts. Hence by (94)

$$A_m = \frac{R}{T}\int_0^T \gamma\gamma' dt = \frac{R}{2}\left(D_2 - D_1 + \frac{D^2}{R^2}\right). \quad \text{..............(95)}$$

This was given* as an improvement upon a method proposed by Dr. J. A. Fleming, in which a current-meter is placed in *cd*, and A_m is given by (95), with D^2 the reading of this current-meter used instead of the term D^2/R^2. The current-meter introduces a certain amount of inductance into *cd*, although this might be made negligible by taking *cd* large enough.

43. Electrometer-method measurement of mean squares of current and potential. An electrometer may be used in the following manner to give the mean square of the current, and of the difference of potential for any part of a circuit, whether containing motors or arc lamps or any arrangement with or without counter-electromotive force or self-inductance. A coil of thick german silver wire (or to prevent sensible heating a set of two or more coils arranged in parallel) having no self-inductance is included in the part of the circuit considered, so that the current to be measured also flows through the wire. The mean square of the difference of potential between the ends of this resistance is measured as described in 14 above by connecting one pair of quadrants of the electrometer to one end, and the needle and the other pair of quadrants to the other end, and the mean square γ'^2 of the current obtained by dividing by the square of the resistance of the wire. The mean square of the difference of potential between the terminals of the part of the circuit considered is then found in the same manner. A multicellular electrostatic voltmeter, or any electrostatic voltmeter of large range of sensibility, is very convenient for such measurements.

The product is not generally to be taken as the mean square of the activity in the part of the circuit considered, for it is evident that in this case what is obtained is the value of

$$\frac{1}{T}\int_0^T V^2 dt \times \int_0^T \gamma^2 dt,$$

where V and γ are the difference of potential and the current at any instant. The square root of this quantity is not generally the same thing as

$$\frac{1}{T}\int_0^T V\gamma\, dt,$$

the true mean value of the activity. This is, however, given indirectly by the following method.†

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* "Alternate Current and Potential Difference Analogies," *Phil. Mag.* Aug. 1891.

† This method is described by A. Potier, *Journal de Physique*, t. ix. p. 227, 1881, but was independently invented also by Prof. W. E. Ayrton and Prof. G. F. Fitzgerald (see Prof. Ayrton on "Testing the Power and Efficiency of Transformers," *Proc. Soc. Tel. Engs. and Els.* Feb. 1888).

44. Electrometer-method of determining activity. Let the two ends of the resistance coil of zero self-inductance and known resistance R be called A and B, and let the extremities of the portion of the circuit for which the measurements are to be made be called C and D. One of the pairs of quadrants is connected to A, the other pair to B, and the needle to C, and the reading, d say, taken. The quadrants remaining as they were, the needle is connected to D, and the reading d' taken. Now, if at any instant V_1 be the potential of A, V_2 of B, V_1' of C, and V_2' of D, we get, if (17) above is applicable to the instrument (see 12 above),

$$\left.\begin{aligned} d &= \frac{k}{T}\int_0^T (V_1 - V_2)\left(V_1' - \frac{V_1 + V_2}{2}\right)dt, \\ d' &= \frac{k}{T}\int_0^T (V_1 - V_2)\left(V_2' - \frac{V_1 + V_2}{2}\right)dt, \end{aligned}\right\} \quad \text{.............} (96)$$

and by subtraction and division by kR,

$$\frac{d - d'}{kR} = \frac{1}{RT}\int_0^T (V_1 - V_2)(V_1' - V_2')dt. \quad \text{...............} (97)$$

But it is clear that the expression on the right-hand side of (97) is the true mean value of the activity required.

If $V_1' - V_1$ be great in comparison with $V_1 - V_2$ and A, say, be connected with the case of the instrument, the first of (96) becomes

$$d = \frac{k}{T}\int_0^T (V_1 - V_2) V_1' dt. \quad \text{.........................} (98)$$

If A and D coincide $V_2' = V_1$, and the activity in the part of the circuit between C and D is given by the first of (96) alone when put in the form

$$\frac{d}{kR} = \frac{1}{RT}\int_0^T (V_1 - V_2) V_1' dt. \quad \text{.......................} (99)$$

This observation is due to Sayers. It is thus possible in the case supposed to use an electrometer as a direct-reading wattmeter.

If a quadrant electrometer is used as here explained, care must be taken to see that the equation (17) holds for the instrument (see p. 295 above). Dr. Hopkinson found (*Phil. Mag.* Ap. 1885) that the indications of his instrument were very exactly expressed by the equation

$$D = \frac{1}{1 + mV^2}(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right), \quad \text{............} (100)$$

where m is a small constant. Hence for high values of V it is necessary to know and use this second constant if its value is sensible. The deviation from fulfilment of the ordinary equation here shown was found to be in great part due to downward electrical force on the needle caused by its hanging a little too low in the quadrants.

CHAPTER XI.

THE COMPARISON OF RESISTANCES.

1. Comparison of resistances to steady currents. Galvanometers. We give here an account of methods for the comparison of the resistances of conductors in which steady currents are kept flowing. In most cases the conductor to be compared is arranged in a particular way in connection with other conductors, which are then adjusted so as to render the current through a certain conductor of the system zero. From the known relation of the resistances of the other conductors the required comparison is deduced.

The form of galvanometer generally employed in the measurement of resistances is the well-known reflecting galvanometer, one arrangement of which is shown in Fig. 78. For most ordinary purposes the form of the instrument here described is convenient. A mirror of silvered glass to which the needle-magnets are cemented at the back is hung within a cylindrical cell about half a centimetre in diameter. The ends of the cylinder are closed by glass plates from four to five

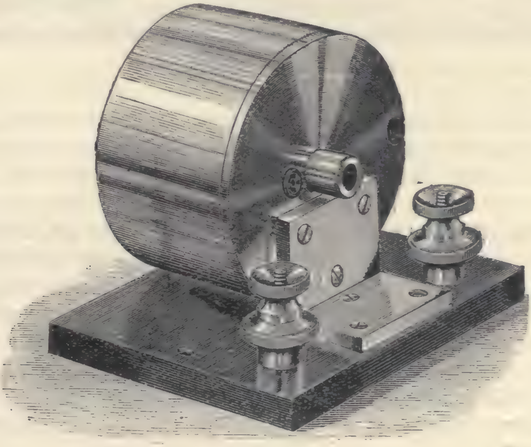

FIG. 78.

millimetres apart, held in brass rings which can be screwed out or in so as to increase or diminish the length of the cell. The mirror is hung by a piece of a single silk fibre passed through a small hole in the cylindrical surface of the chamber and fixed there with a little shellac. The mirror is only of slightly smaller diameter than the cylinder in which it hangs, so that in this arrangement the fibre is very short, rendering it necessary in cases in which deflections have to be read off to allow in one way or another for the effects of torsion. The cylindrical chamber is screwed into one end of a cylinder of slightly greater diameter which fits the hollow core of the coil, and is called the galvanometer-plug.

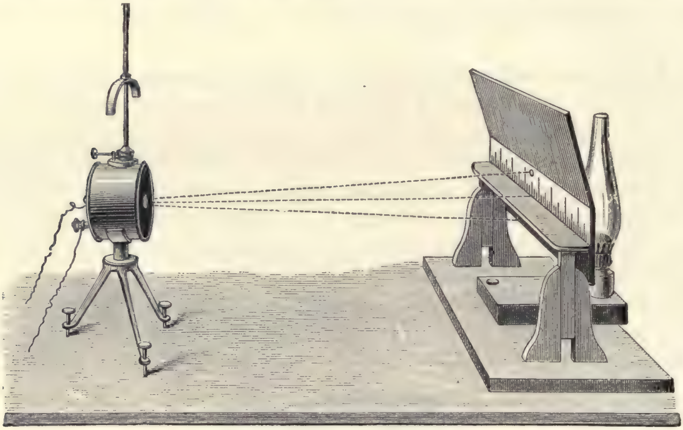

FIG. 79.

When the plug is in position the mirror hangs freely within its cell, with therefore the point of suspension on the highest generating line of the cylinder. Deflections of the needle are observed either by the Poggendorff telescope method (Fig. 15), or, and much more frequently, by using a ray of light reflected from the mirror as an index (Fig. 79). The mirror being made in this case concave, the reflected light is brought to a focus on a fixed graduated scale, and the displacement of the spot of illumination gives the deflection. If the scale is straight and horizontal, and is set, as it usually approximately is, at right angles to the undeflected direction of the reflected ray, supposed also horizontal, the ratio s/r, of the displacement s to the distance r of the scale from the mirror, is $\tan 2\theta$, if θ be the angle turned through by the mirror.

The weight of the needle and mirror is small, generally under one grain, and the period of free vibration of the suspended system about any position of equilibrium is short. The needle is also made to come quickly to rest by the smallness of the chamber in which it hangs. Since the mirror nearly fills the whole cross-section of the cell, the air damps the motion of the mirror to a very great extent even when the cell has its largest volume. The mirror may be made quite "dead-beat," that is, to come to rest without oscillation, by screwing in the front and back of the cell until the space is sufficiently limited.

In instruments in which it is desirable to avoid effects of torsion the galvanometer coil is made in two halves which are fixed coaxially, with a narrow space between them to receive the suspension piece. This piece forms a chamber in which the needle hangs between the two halves of the coil, and gives a length of fibre which at shortest is equal to the radius of the outer case of the coil, and which can obviously be made as long as is desired. The part of the hollow core at the needle is closed in front and at back by glass plates carried by brass rings. These can be screwed in or out by a key from without so as to diminish or increase the size of the chamber, and thus render the needle system more or less nearly "dead-beat."

We shall suppose the galvanometer set up so that the deflections are read by the ordinary deflection method. It is only necessary to arrange that when no current is flowing in the wires the mean plane of the coils shall be parallel to the magnetic axis of the needle-system. This is done as follows. A straight thin wire of steel (a knitting needle) is magnetized and hung by a single silk fibre of a foot or so in length. This can easily be done by taking a sufficiently long single fibre of silk and forming a double loop on one end by doubling twice and knotting. In this double loop, made widely divergent, the steel wire is laid horizontally, and the single end of the fibre is attached to a support carried by a convenient stand, which is then placed so that the wire takes up a position in the direction of the horizontal component of the magnetic field where the needle is to be placed. A line can now be drawn parallel to the wire on the table beneath it. All that is necessary then is to place the galvanometer so that the front and back planes of the coil are vertical and parallel to this line, and adjust the lamp and scale as described above.

It is sufficient for our present purpose to state that if the needles be so small as in the Thomson reflecting galvanometer, and torsion can be neglected, the current in the coil may be taken as proportional to the tangent of the deflection angle, and therefore if that angle be not greater than three or four degrees the current may, with an error not greater than $\frac{1}{7}$ per cent., be taken as proportional to the deflection simply.

2. Sensitiveness of a galvanometer. The galvanometer should be made as sensitive as possible by diminishing the directive force on the needle as far as is practicable without rendering the needle unstable. This is easily done by placing magnets near the coil so that the needle hangs, when the current in the coil is zero, in a very weak magnetic field. That the field has been weakened by any change in disposition of the magnets, made in the course of the adjustment, will be shown by a lengthening of the period of free vibration of the needle when deflected for an instant by a magnet and allowed to return to zero. The limit of instability has been reached when the position of the spot of light for zero current changes from place to place on the scale, and the intensity of the field must then be slightly raised to make the zero position of the needle one of stable equilibrium. For ordinary testing, attention to this matter of sensitiveness is important. Very frequently, especially in laboratories for students, the magnetic fields at the galvanometer needles might be weakened with advantage. The less sensitive arrangement is more easily made by the student.

Although not absolutely essential, except when accurate readings of deflections are required, it is always well when the field is produced by magnets, to arrange them so that the field at the needle is nearly uniform. It may therefore be produced by two or more long magnets placed parallel to one another at a little distance apart symmetrically with respect to the centre of the needle above or below it, and with their like poles turned in the same directions; or a long magnet placed horizontally with its centre over the needle, and mounted on a vertical rod so that it can be slided up or down to give the required sensibility, may be used. The earth's horizontal field must of course always be taken account of in such adjustments. Also when the directive force on the needle is much reduced and deflections have to be measured and compared, it is to be remembered that the couple due to the suspension may be of very sensible amount.

Sensibility is sometimes obtained by the use of astatic galvanometers, but these are rarely necessary and, except in the hands of people with some skill in electrical work, are more troublesome to use than the ordinary non-astatic instrument.

3. Resistance coils and resistance boxes. For the comparison of the resistances of conductors other resistances the relations of which are known are employed. These are generally coils of insulated wire wound

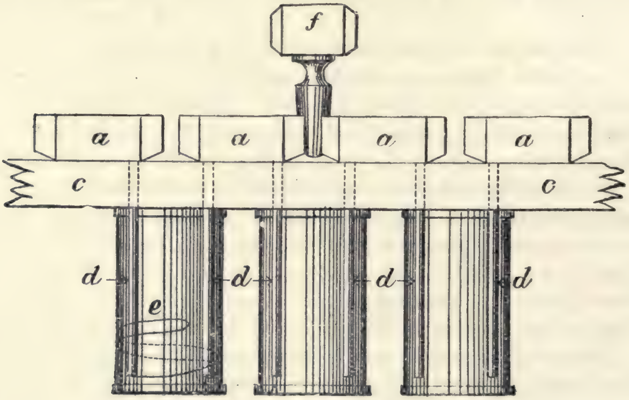

FIG. 80.

on bobbins (Fig. 80) which are arranged so that the coils can be used conveniently in any desired combination. Such an arrangement of coils is called a resistance box. Figs. 81 and 82 show resistance boxes of different forms.

In a resistance box each coil has a separate core, which ought to be a brass or copper cylinder split longitudinally to prevent induction currents, and covered with thin rubber or varnished paper for insulation. These cores are shown in Fig. 80. The metallic core facilitates the cooling of the coil if an appreciable rise of temperature is produced by the passage of a current through it. After each layer of the coil has been wound it is dipped in melted paraffin wax, so as to fix the spires relatively to one another, preserve them from damp, and insure better insulation. It is of great importance to use perfectly pure paraffin wax, and especially to make sure that no sulphuric acid is present in it. Unless this precaution is observed trouble may be caused not only by the action of the acid on the metal of the conductor, but by the polarization effects due to electrolytic action in the acid paraffin. Paraffin

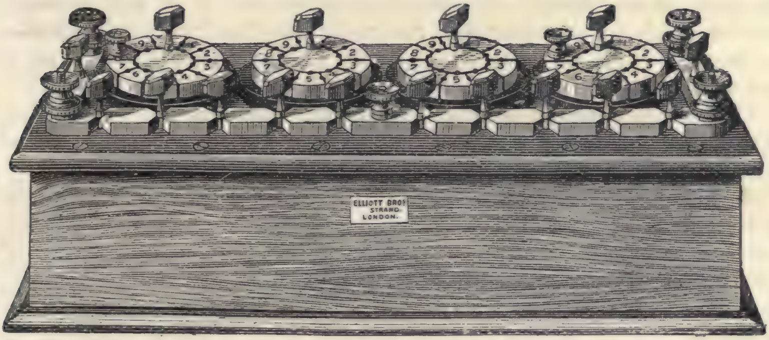

FIG. 81.

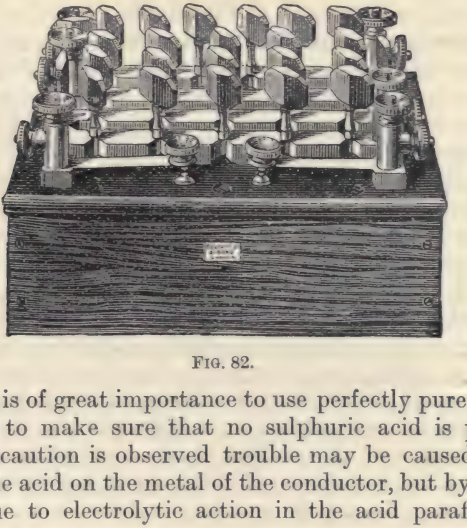

FIG. 82.

which is at all doubtful should be melted and well shaked up with hot water to remove the acid.

The wire chosen for the higher resistances is generally an alloy of one part platinum to two parts silver. This has a high specific resistance (see 37 below) combined with a small variation of resistance with temperature. Standard coils are made of various metals and alloys (see 10 and 40 below). For the lower resistances wire of greater thickness is employed on account of its greater conductivity, which enables a greater length of wire to be used, and this facilitates accurate adjustment.

Coils are sometimes made of "platinoid," a species of german silver which does not tarnish seriously with exposure to the air and has a low variation of resistance with temperature (see Table V.).

4. Construction of resistance coils. When a coil of given resistance is to be wound, a length of well-insulated wire of slightly greater resistance (determined by comparison at ordinary temperature by one of the processes to be described) is cut, doubled on itself at its middle point, and wound thus double on its core. This is done to avoid the effects of induction (see 16 below) when the current is in a state of variation, as when starting or stopping. After the coil has been wound its resistance is again measured, and if good insulation has been obtained, it ought now to show a slightly increased resistance, on account of the change produced in the wire by bending. The coil is fixed in position by two long brass or copper screws *d*, *d*, Fig. 80, passing through ebonite discs in the ends of its core, which fasten it to the cover of the box. These screws should be sufficiently massive to give no appreciable resistance. They are attached to two adjacent brass pieces, *a*, *a*, on the outside of the cover, and have the ends of the wire of the coil soldered to them so that the coil bridges across the gap shown in the figure between every adjacent pair of brass pieces. The coil is now brought to the temperature at which it is to be accurate and finally adjusted so that its resistance taken between the brass pieces is the required resistance. The method of adjustment of the resistance of a coil by shunting it by a wire of sufficiently high resistance will be understood from the examples of its use in 30 below and elsewhere.

5. Legal and International ohm. Coils are made in multiples of the "Ohm" or practical unit of resistance. The ohm is defined absolutely in Chap. I. 48. It was agreed at an International Conference on Electrical Standards and Units, held in London in October 1908, that the resistance offered to an unvarying electric current by a column of mercury, at the temperature of melting ice, 14·4521 grammes in mass, of a constant cross-sectional area, and of a length of 106·300 centimetres should be the International Ohm. This choice was legalized by an Order in Council issued on January 10, 1910. The Order in Council is given in an Appendix and is quoted in I. 49. Different forms in which copies of such a standard are made are also described below, pp. 368, 369.

6. Different forms of resistance boxes. A series of coils are arranged in a resistance box in some convenient order either in series or in parallel. Fig. 83 shows a series arrangement suitable for many purposes. Each number indicates the number of ohms in the corresponding coil. The space between each pair of blocks is narrow above and widens out below, as shown in Fig. 80, to increase the effective distance along the vulcanite from block to block. In the adjacent ends of the brass pieces, between which is the narrow gap, are cut two narrow opposite grooves, so as to form a slightly conical vertical socket. This fits a slightly conical plug, *f* in Fig. 80, which when inserted bridges over the gap by making direct contact between the blocks, and when not thus in use is held in a hole drilled in the middle of the upper surface of the block. The coil is short-circuited when the plug is inserted, that is a current sent from one block to the other passes almost entirely across the plug on account of the much greater resistance of the coil. The handle, *f*, of the plug is generally made of ebonite.

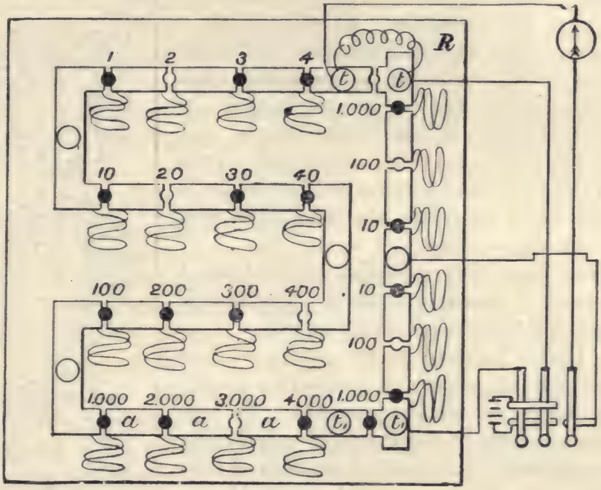

FIG. 83.

The plan of arranging a series resistance box which is most economical of coils is a geometrical progression with common ratio 2, though this is not much used. It was, however, employed in some of the earlier boxes used at Glasgow when methods of testing were being worked out. In such a box two unit coils are generally provided to enable the box to be conveniently tested. The inconvenience of the arrangement is in the reduction of any resistance which it is proposed to unplug in the box to its expression in the binary scale of notation. For example if the resistance 370 is to be found on the box, this is expressed as $2^8 + 2^6 + 2^5 + 2^4 + 2$ or 101110010, and the corresponding plugs inserted, namely the first, fourth, fifth, sixth, and eighth of the plugs beyond the units. The process of reduction is performed as follows by dividing successively by 2, and writing the remainders as successive figures of the number from right to left in the order in which they are obtained, ending with the last quotient, which is of course 1.

2	370	
	185	0
	92	1
	46	0
	23	0
	11	1
	5	1
	2	1
	1	0

Hence 370=101110010 in the binary scale. It is not however always necessary to go through this process. Practice with a box on this principle leads soon to readiness in deciding what coils are to be unplugged, or what is the resistance of any set of coils which may be unplugged. It is well to remember that any coil of the series is greater by unity than the sum of all the preceding coils of the series.

The coils form a geometrical series from 1 to 4096 with a common ratio 2. The unit is duplicated for the reason stated above.

The "Dial" form of series resistance box shown in Fig. 82 above, is preferable to the ordinary forms for many purposes. It contains three or four or more sets of equal coils, each nine in number. One set consists of nine units, the next of nine tens, the next of nine hundreds, and so on. Besides these the box sometimes contains a set of nine coils each a tenth of a unit. Fig. 84 is a plan of a five-dial box. The sets of coils are arranged along the box in order of magnitude. Each set is arranged in series, and the blocks to which the extremities of the coils are attached are arranged in circular order round a central block, which can be connected to any one of the ten blocks of the set surrounding it, by a plug inserted in a socket provided for the purpose. Each central block, except the first and last, is connected by a thick copper bar inside to the initial block of the succeeding series of nine coils, as shown in Fig. 84 by the dotted lines. The ten blocks of each set of coils are numbered 0, 1, 2, ...9, as shown. Thus a current passing to one of the central blocks passes across through the bar to the next series of coils, then through the coils

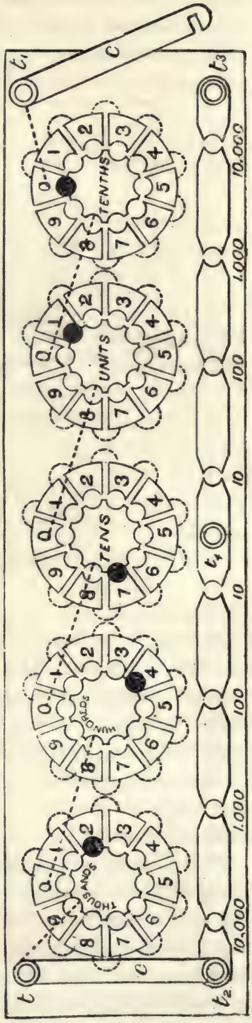

FIG. 84.

until it reaches a block connected to the central piece by a plug, when it passes across to the centre and then to the next series of coils. If no coil of a series is to be put in circuit, the plug joins the central block to the coil marked zero.

In a five-dial box the central blocks are marked respectively tenths, units, tens, hundreds, thousands, and the resistances are read off decimally at once. Thus supposing the centre in the first dial to be connected to the block marked 5, in the second dial to the block marked 7, in the third to that marked 6, the resistance put in circuit is 67·5 units.

The advantage of the arrangement consists in the fact that only one plug is required in each dial whatever the resistance may be, and since the plugs when no coils are included complete the circuit through the zeros, there is always the same number of plug contacts in circuit, instead of a variable number as in the ordinary arrangement. Moreover a disadvantage of a series of plugs in line, joining metal blocks on a single base plate, is avoided. When one of these is forcibly tightened the contacts of the others may be altered. For this reason high precision resistances are placed in blocks detached in position, with the metal pieces joined by heavy flexible conductors.

Besides the dial resistances there is generally in each box a set of resistances arranged in the ordinary way, and comprising two tens, two hundreds, two thousands, and sometimes two ten-thousands, fitted with terminals to allow the box to be conveniently used as a Wheatstone Bridge, as described below. The extremities of this series of resistances can be connected by means of thick copper straps with the series of dial resistances. Each pair of equal coils are sometimes wound on one bobbin to ensure equality of temperature.

7. Resistance slides. It is sometimes desirable to have a ready means of varying the ratio of two resistances, or of increasing a single resistance by steps of any required amount. For this purpose a resistance slide is a convenient arrangement. One form is shown at CD in Fig. 85. Along a metallic bar r in front of a series of equal resistance coils slides a contact piece s by which r is put in conducting contact with any one of the series of brass or copper blocks by which the coils are connected. The figure shows a combination of two slides used by Thomson and Varley for cable testing. Each resistance in AB is five times that of each coil in CD, and there is the same number in each, so that the whole resistance of CD is twice that of each coil in AB. The slider, S, of AB consists of two contact pieces insulated from one another on the slider, and at such a distance apart as to embrace two coils. The terminals of CD are connected to CC as shown in the figure, and therefore in whatever ratio the resistance CD is divided by the contact piece s, in that ratio is the joint resistance of the two coils CC divided. CD thus forms a vernier for AB. In the arrangement figured the resistance CD is divided into the two parts 12 and 8, and therefore the sixth and seventh coils of AB which are between the terminals of S are divided into two similarly situated parts 12 and 8. Hence the whole resistance between *A* and *B* is divided into the two parts 56 and 44.

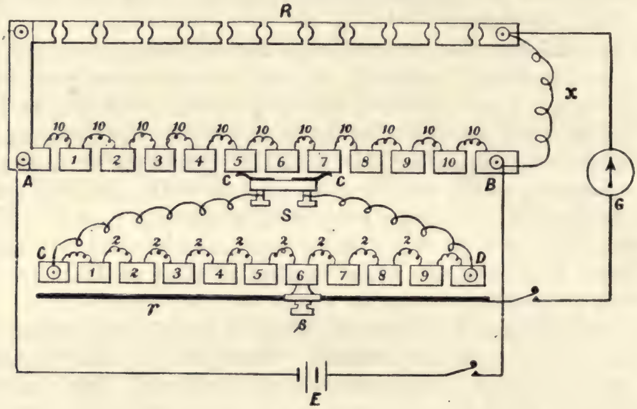

FIG. 85.

Dial forms of the double resistance slide are also used, and are very convenient.

8. Conductivity box. Boxes in which the coils in circuit are in parallel seem to have been first made at the suggestion of Lord Kelvin, and called Conductivity Boxes, because the conductance* (the reciprocal of the resistance) in circuit is obtained by adding the conductances of the coils. Fig. 86 shows the arrangement. Each coil is a resistance coil wound on a bobbin as described above and has one extremity connected to a massive bar *a*, the other to a brass block *c*, outside the box, which can be connected by a plug to the massive bar *b*. The resistance in circuit is obtained at once by adding the conductances of

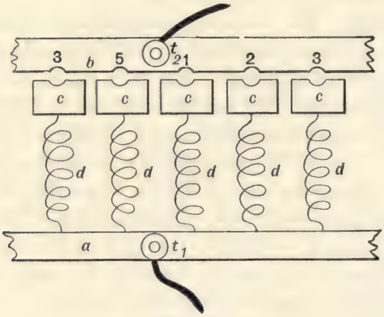

FIG. 86.

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* The word "Conductance" is now widely used instead of "Conductivity," and is also adopted in this book.

the coils thus in circuit, and taking the reciprocal of their sum. The conductances of the coils are marked on the corresponding blocks outside the cover.

This arrangement is very convenient for the measurement of low resistances such as one ohm and under, as it gives a long graduation of fractions by combination of the coils.

Lord Kelvin proposed to call a box arranged thus a Mho-Box, where " Mho " is the word " Ohm " read backwards to indicate that the box gives conductances, that is reciprocals of resistances. "*Perversion*" of spelling of the word Ohm thus indicated *inversion* of numerical value.

9. Temperature variation of resistance. The resistance of almost all wires increases with rise of temperature, and the box is generally adjusted to be correct at a convenient mean temperature which is marked on the cover. The value of the resistance shown by the box at any other temperature is obtained when the change of temperature can be ascertained from the known variation of resistance with temperature. A table of the variation of the resistances of different substances with temperature is given at the end of this volume.

The general internal temperature can be observed by means of a thermometer passed through one of the orifices which should be left in the side of the box to allow free circulation of air. Local changes of temperature may sometimes be produced in the coils without affecting appreciably the general internal temperature. These changes cannot be accounted for, as it is impossible to observe them with any accuracy, but can be avoided by using only the very feeblest currents, and continuing these for the shortest possible time.

The general internal temperature can also be measured by means of an auxiliary coil provided for the purpose. This is constructed of thick copper wire wound on ebonite, and extends along the whole length of the box. Since the variation of resistance of copper relatively to that of the wire of which the coils are constructed is known, we can by measuring the resistance of this auxiliary unit by the box itself obtain a closely approximate estimate of the internal temperature.

The temperature variation may be made for all the coils the same as the highest variation for any one, by introducing into each a piece of copper (conveniently at the bight after the coil is wound) just sufficient for the purpose.

10. Testing a resistance box. In every case the blocks to which the coils are attached should be pierced with a socket for special plugs with binding terminals attached, by means of which any coil in the box may be brought into circuit itself. This is necessary for the testing of the box, which is done as follows. In the case of the ordinary arrangement of coils (Figs. 81, 83), each of the units is compared with a standard unit, then the two units together are tested against each of the 2s, then the 2s and a 1 are attested against the 5 and so on, until the 100s are reached. All the preceding coils put together give 100, which can be tested against each of the 100s, and this process is continued until the box is completely tested. The process can be checked by other possible combinations, and the whole of the results, if necessary, put together by the ordinary methods of combination.

If a dial box is to be tested the auxiliary unit, if it has one, suffices for the comparison of each of the units, then the nine units and the auxiliary unit give 10 for the comparison of each of the nine tens. These when compared give with the ten units 100 for the comparison of each of the hundreds, and so on.

In the case of a box arranged in geometrical progression with common ratio 2, and first term 1, the unit is duplicated for the sake of comparison. Each unit having been compared with a standard, they give together a comparison of the next coil, which is 2, then that with the two units give 4, with which the coil of 4 units can be compared, and so on.

The actual methods of comparing coils are described below (p. 352 *et seq.*). It is to be remembered that in the comparison of the coils of low resistance the connecting wires (which should be in all cases short and thick) must be taken into account.

In the use of a set of resistance coils it is important that the plugs be kept clean, and the ebonite top of the box, especially between the blocks of brass, kept free from dust and dirt. The ebonite may be freed from grease by washing it with benzole applied sparingly by means of a brush, and a film of paraffin oil should then be spread over its surface. The plugs and their sockets may also be freed from adhering greasy films by washing in the same way with benzole or very dilute caustic potash. The latter should not however be allowed to wet the ebonite surface. If necessary the sockets may be scraped with a round-pointed scraper. On no account should the plugs or sockets be cleaned with emery or sand paper.

11. Rheostats. It is frequently necessary to adjust a current to a convenient strength by varying the amount of resistance in circuit. When the amount of resistance in circuit need not be known, this can be done most readily by means of a rheostat, or resistance coils in series with a rheostat, an arrangement which has the advantage of giving a continuous variation of the resistance. Rheostats of convenient design consist of resistance wires of constantan alloy, which has a temperature coefficient very nearly zero. These wires are bare and are wound " non-inductively " on slate blocks, and a slider making contact with the wires brings into the circuit more or less of the wire. This alloy consists of 50 per cent. copper and 50 per cent. nickel, and has a temperature coefficient of only about 0·003 per cent. per degree centigrade.

It has the disadvantage of a considerable thermoelectric power against copper.

12. Christie's, or Wheatstone's bridge. The method of comparing resistances of most general use is that usually referred to as Wheatstone's Bridge, though as a matter of fact it was invented by Mr. S. Hunter

Christie [*Phil. Trans.* 1833]. It was brought into general use by Sir Charles Wheatstone. The arrangement of conductors employed is that shown in Fig. 86, with a battery, generally a single Daniell's or Menotti's cell, included in r_6, and a galvanometer in r_5. A much higher battery power is however sometimes required, especially in cable and other testing. The three conductors whose resistances are r_1, r_2, r_3 are coils of a resistance box provided with terminals so arranged that connections can be made at the proper places to form the bridge, for example as in Fig. 88, which shows diagrammatically how a resistance box is joined up as a Wheatstone bridge. It will be easy to make out in Fig. 83 the terminals corresponding to *A*, *B*, *C*, *D* respectively of Fig. 87. Fig. 83 above shows how in a so-called "Post-Office Resistance Box," the battery and galvanometer keys are mounted on the cover, and permanently connected to the proper points inside the box.

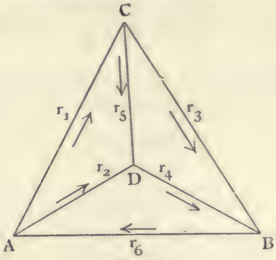

FIG. 87.

The resistance to be compared is placed in the position *BD* (Fig. 88), and convenient values of r_1 and r_2 are chosen, while r_3 is varied until no current flows through the galvanometer. The value of r_4 is then found by (24) of Chapter IV. with $c = 1$, which, since γ_5 is zero, may be written

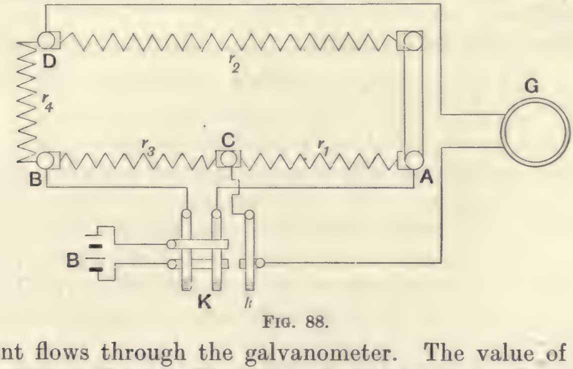

FIG. 88.

$$r_4 = \frac{r_2}{r_1} r_3. \quad \text{.................................}(1)$$

If r_1 and r_2 are equal, r_4 is equal to r_3, and is read off at once from the resistance box.

13. Arrangement of bridge for greatest sensibility. In the practical use of Wheatstone's bridge we have generally to employ a certain battery and a certain galvanometer for the measurement of a wide range of resistances; and it is possible if great accuracy is required so to choose the resistances of the bridge as to make the arrangement have maximum sensibility. An approximate determination is first made of the resistance to be measured. Call this r_4. It has been shown in IV. 13 above that if the battery and galvanometer are invariable we should make

$$r_1 = \sqrt{r_5 r_6}, \quad r_3 = \sqrt{r_4 r_6 \frac{r_4 + r_5}{r_4 + r_6}}, \quad r_2 = \sqrt{r_4 r_5 \frac{r_4 + r_6}{r_4 + r_5}}. \quad \text{.........(2)}$$

When the other resistances r_1, r_2, r_3, r_4 are fixed the coil of the galvanometer, supposed wound to fill a given bobbin, should have the resistance g given by

$$g = \frac{r_1(r_3 + r_4)}{r_1 + r_3}.$$

If the resistances of the given battery (see p. 376 below) and galvanometer are at the disposal of the experimenter, then on the supposition that the resistances of the connections are so slight that the resistance of the galvanometer may be taken as equal to r_5, the most sensitive arrangement is that in which each of the resistances is equal to r_4.

Unless in particular cases in which great accuracy is necessary, any convenient values of r_1, r_2 will give results sufficiently accurate for all practical purposes, but in arranging the bridge with these the following rule should be observed: of the resistances r_5, r_6 of the galvanometer and battery respectively, connect the greater so as to join the junction of the two consecutively greatest of the four other resistances to the junction of the two consecutively least.

14. Practical rule for sensibility. This rule follows easily from IV. (14) above. For interchanging r_5 and r_6 we alter only the value of D, and calling the new value D' we get

$$D' - D = (r_5 - r_6)(r_1 - r_4)(r_3 - r_2). \quad \text{..................... (3)}$$

The expression on the right will be negative if $r_6 > r_5$ and r_1, r_3 be the two greatest or the two least of the other resistances. Hence on this supposition the value of D has been diminished, and therefore the current through the galvanometer for any small value of $r_2 r_3 - r_1 r_4$ increased by making r_6 join the junction of r_1, r_3 to that of r_2, r_4. In cases in which the resistances in the bridge are large, a galvanometer of high resistance should also be used.

15. Further discussion of sensibility. The rule, however, that the given battery should be so arranged as to make the internal resistance equal to the resistance to be worked through in the bridge is utterly impractical, as it cannot be carried out. But one or two storage cells of negligible resistance are usually available, and the e.m.f. applicable can be made as large as the resistances will bear without overheating, so that the conditions to be fulfilled are quite different from those which contemplate the use of a certain battery of cells, to be joined in some combination of series and parallel arrays.

Using then the battery of negligible resistance, we see that, when balance is nearly obtained, the e.m.f. E applied is $\gamma(r_2+r_4)$, where γ is the current in AD and DB [Fig. 87]. The best value, g, of the galvanometer resistance, *if a coil of given size is to be made for the test*, is [IV. 13 (26)] $r_1(r_3+r_4)/(r_1+r_3)$. When this value is used the deflection D is, if k be a constant, given by

$$kD=\tfrac{1}{2}\gamma\frac{dr_4}{\sqrt{r_4}}\bigg/\left\{\left(1+\frac{r_3}{r_1}\right)\left(1+\frac{r_3}{r_4}\right)\right\}^{\frac{1}{2}}. \qquad\text{.......... ......(4)}$$

In order that the deflection should be as great as possible it is clearly required by this formula that r_3 should be small in comparison with r_1 and with r_4. Since $r_3/r_4=r_1/r_2$ it is clear that r_2 must be large. Thus we get again the rule that the galvanometer should connect the junction of the consecutively largest pair of resistances with the junction of the two which are consecutively least.

The actual resistance r_5 of the available galvanometer may be different from the best resistance g. As we have seen [IV. 13 (26′)] the current through the galvanometer is then

$$\frac{\gamma dr_4}{r_5+g}\,\frac{r_1+r_2}{r_1+r_2+r_3+r_4},$$

and the deflection may be compared with that for the coil of best resistance, if the bobbins are similar, by multiplying this value of the current by $\sqrt{r_5}$. Thus for the resistances chosen for the bridge the deflection is proportional to

$$\frac{\sqrt{r_5}}{r_5+g}=\frac{\sqrt{n}}{n+1}, \qquad\text{..............................(5)}$$

if $r_5=ng$. The maximum deflection is obtained for $n=1$, and hence the ratio of the actual deflection to the maximum is $2\sqrt{n}/(n+1)$. This result was given by Schuster [*Phil. Mag.* 39 (1895)], who also points out in the same paper that, obviously, as sensitiveness is always increased by an increase in electromotive force, the limit is only reached when the current is so strong that there is danger of overheating one or other of the resistances.

If the current through r_4 be nearly the maximum, γ, which the conductor will bear, the galvanometer current will, if $r_3=r_4$, be $k\gamma\,dr_4/r_4$ [see (27′), p. 141], where k is a constant. If this be the smallest current, $d\gamma_5$, which can be measured by the galvanometer, we have $d\gamma_5/\gamma=k\,dr_4/r_4$. Thus the percentage accuracy of measurement of r_4 is proportional to the current, γ, which can be safely passed through that resistance.

16. Operations in testing with a bridge. In the practical use of the method the electrodes of the battery should be carried to the terminals of a reversing key, so that the testing current may be sent in opposite directions if desired through the resistances of the bridge. Also a single spring contact-key, which makes contact only when depressed, should be placed in r_5. These keys are convenient when arranged side by side, so that the operator placing a finger on each can depress one after the other. A convenient form of wire rocker with mercury cups, combining the two keys, may be easily made by the operator. When the bridge has been set up and a test is about to be made, the single key in r_5 is first depressed to test whether any deflection of the galvanometer needle is produced without closing the battery circuit. If there is a deflection, this must be due either to thermoelectric action in the galvanometer circuit, or to leakage from the battery to the galvanometer wires. The procedure in this case will be stated presently. If there is no deflection, the operator then opens the galvanometer circuit, depresses the key which completes the battery circuit, and immediately after, while the former key is kept down, depresses also the galvanometer key. After the circuits have been completed just long enough to enable the operator to see whether there is any deflection of the needle, the keys are released so as to break the contact in the reverse order to that in which they were made. This order of opening the circuits enables him to make a second observation of deflection without its being necessary again to send a current. It is easy to imagine and construct a form of contact-making key, which being depressed a certain distance completes the battery circuit, and on being depressed a little further completes the galvanometer circuit, and therefore on being released interrupts these circuits in the reverse order. This form of key is of use in the testing of resistance coils in which there is considerable self-induction. For general work, however, it is inconvenient, as the reverse order of making the contacts may have to be adopted for certain other tests. Again, in many practical operations, such as cable testing, etc., the contacts have to be made after different intervals of time in different cases.

17. Effect of self-inductance in a bridge network. The object of thus completing and interrupting the battery circuit before that of the galvanometer is partly to avoid error from the effects of *self-inductance*. When a current in a conducting wire is being increased or diminished, an electromotive force, the amount of which depends on the arrangement of the conductor, is called into play, so as to oppose the increase or diminution of the current [see p. 240]. The effect of this electromotive force is to produce, therefore, a weakening of the electromotive force of a battery for a very short time after the circuit is completed, and a strengthening during the very short interval in which the current falls from its actual value to zero at the interruption of the circuit. Its value is small, though not zero, when the wire is doubled on itself so that the two parts lie along side by side, the current flowing out in one and back in the other; but is very considerable if the wire is wound in a helix, and still greater if the helix contains an iron core. It is shown in the discussion at p. 240 that the electromotive force of self-inductance is directly proportional to the rate of variation of the current in the circuit, and is greater the larger the magnetic induction through the circuit, and thus is explained the bright spark seen when the circuit of a powerful electro-magnet is *broken*.

If, then, one or more of the coils of a bridge arrangement were wound so as to have self-inductance, the electromotive force thus called into play would, if the galvanometer circuit were completed before that of the battery, produce a sudden deflection of the galvanometer needle when the battery circuit is closed. All properly constructed resistance coils are, as has been stated, made of wires which have been first doubled on themselves and then wound double on their bobbins, and have therefore no self-inductance. The wire tested, however, and the connections of the bridge have generally more or less self-inductance, the effect of which, unless the contacts were made as described above, might be mistaken for those of unbalanced resistance. This mode of winding the coils also avoids direct electromagnetic effects of the coils on the galvanometer needle when the coils are placed near it.

If on depressing the galvanometer key at first as described above a current is found to be produced by thermoelectric or leakage disturbance, and the spot of light is therefore displaced, the operator keeping down the galvanometer key depresses the battery key, and observes if there is any permanent deflection of the spot of light from its displaced position during the time that the battery key is kept down. This is easily distinguished from the sudden deflection due to self-inductance, as that immediately dies away to zero as the current rises to its permanent value.

If the coil which is under test for resistance has an iron core the battery key must be kept down for a little time before the galvanometer key is depressed, to allow inductive action, due to the growth of magnetism in the iron, to have ceased.

18. Testing with a known ratio of arms of bridge. Interpolation. When comparing a resistance the operator first observes the direction in which the mirror or needle is deflected when a value of r_3 (Fig. 88) obviously too great is used, and again when a much smaller value of r_3 is used. If the deflections are in opposite directions, the value of r_3, which would produce no deflection of the needle, lies between these two values, and the operator simply narrows the limits of r_3, until on depressing the galvanometer key no motion, or only a very small motion, of the needle is produced. It may happen, however, that the value of the resistance which is being compared may be between two resistances which have the smallest difference which the box allows. Thus with a resistance box by which with equal values of r_1 and r_2 he cannot measure less than $\frac{1}{10}$ of an ohm, he may either by making the ratio of r_1 to r_2, 10 to 1, or 100 to 1, obtain the values of r_4 to one or two places of decimals. Any inaccuracy in the relation of the arms of the bridge may be eliminated by reversing the arrangement, that is, interchanging r_1 and r_2, and r_3 and r_4, and taking the mean of the results.

Whatever be the ratio of r_1 to r_2, if he can read the deflections when first one and then the other value of r_3 (between which r_4 lies, and which differs by only $\frac{1}{10}$ of an ohm) is used, he can find r_4 to another place of decimals by interpolation by proportional parts. For example, let the value 120·6 of r_3 produce a deflection of the spot of light of 6 divisions to the left, and 120·5 a deflection of 14 divisions to the right: the value of r_3 which would produce balance is equal to

$$120{\cdot}5 + {\cdot}1 \times 14/(14+6) = 120{\cdot}57.$$

19. Slide-wire bridges. A convenient and accurate form of bridge is that introduced by Kirchhoff. In this an exact balance is obtained by moving a sliding contact, *D*, say, along a graduated wire which joins the two points *A*, *B* of Fig. 87. A diagrammatic sketch of the arrangement is shown in Fig. 89. *S* is the sliding-piece, *A*, *B* the wire along which it slides. *A*, *B* is stretched in front of a scale a metre in length graduated to half-millimetres and doubly numbered, from left to right and from right to left. The coils *a*, *c*, *d*, *b* of the diagram have the respective resistances r_1, r_2, r_3, r_4. Fig. 90 shows a form of the instrument manufactured by Messrs. Elliott Bros.

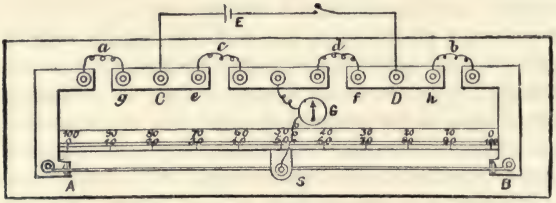

FIG. 89.

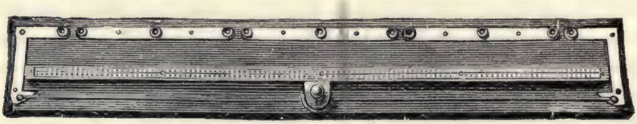

FIG. 90.

Fig. 91 shows an easy-made and cheap form of wire bridge devised by Prof. T. Gray. *w*, *w* is the wire, made of platinoid or german silver, which is stretched above, but not in contact with, a base-board, passing round the insulating and supporting vulcanite block *B* from the mercury cup c_1 to the other c_5. A vulcanite crossbar *A* clamps the wire in position near the cups. If the wire be long several such crossbars may be used. Each end of the wire is soldered to a stout bar of copper, bent, as shown in Fig. 91, so as to dip into a mercury cup without any risk of contact of the mercury with the soldered junction. The cups should be of copper, and may conveniently be made of the form shown in the figure, and fixed in holes in the wooden or ebonite supporting-block. The ends of the copper pieces dipping into them should be carefully squared and bear against the copper bottoms. They should be freshly amalgamated with mercury.

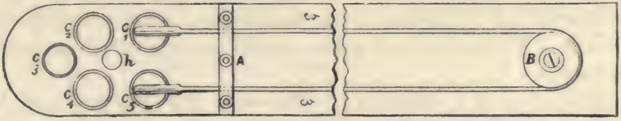

FIG. 91.

The wire is divided into parts of equal resistance by a process of calibration (p. 342 *et seq.*, below), and marks indicating these parts are made on a rule attached to the base-board, along which the contact-piece slides. A movable scale subdivides the space between two divisions.

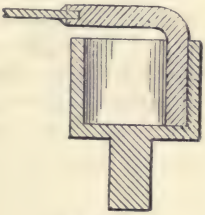

FIG. 92.

On a plate of ebonite or well-paraffined hard wood are fixed mercury cups c_2, c_3, c_4, made as just described. The auxiliary resistances r_1, r_4 of the bridge when required are placed between c_1 and c_2, c_5 and c_4, while the wires to be compared connect c_2 and c_3, c_3 and c_4. Since the wire w, w can be made long, the auxiliary resistances are not frequently required. When they can be dispensed with c_2 and c_4 are removed and c_3 placed in the socket h, and the wires to be compared are then placed between c_3 and c_1, c_3 and c_5.

A form of slide-wire bridge was used by Matthiessen and Hockin in the comparisons of resistance made by them in their work as members of the British Association Committee on Electrical Standards; and it was found by these experimenters that an alloy of 85 parts of platinum with 15 parts of iridium formed an excellent material for the graduated wire. This alloy, they found, did not readily become oxidized. Platinum-silver alloy is however frequently employed.

The contact piece is generally a well-rounded edge of steel with a slight notch to receive the wire. The knob pressed by the operator bends a spring which presses the contact piece with just sufficient pressure against the wire. A turning bar can be put into position to keep down the contact when desired. The sliding piece carries a vernier which enables fractions of a division to be read on the scale.

The method of testing by a slide-wire bridge is precisely the same as by the ordinary bridge, except that when balance has been nearly obtained in the usual way, by varying the relation of the resistances r_1, r_2, r_3, for a particular position of the sliding piece, an exact balance is obtained by shifting the sliding piece in the proper direction along the wire. Supposing that the resistance of the wire per unit of length has been determined for different parts of the wire, and that the resistances of contacts have been determined (32, 33) and allowed for, the value of r_4 is at once found by taking into account the resistances of the segments of the wire AB, on the two sides of the point contact at which gives zero deflection.

20. Calibration of a slide-wire. The wire AB (Fig. 89) may be "calibrated" by one of the following methods. The first is that which was employed by Matthiessen and Hockin.* Let r_1 and r_4 (a, b in Fig. 89) be such resistances that balance is obtained at some point P in AB, with two coils r_2, r_3 (c, d in Fig. 89) differing in resistance by say $\frac{1}{10}$ per cent. Let $r_1+\alpha$ be the total resistance, including contacts, between C and P, and $r_4+\beta$ that between D and P. Now alter r_1 by inserting a short piece of wire. This will shift the zero point along the wire through a certain distance to the left. Balance so as to find this point, which call P_1; then interchange r_2 and r_3, and balance again, and call the second point thus found P_2. Let z denote the resistance between P and P_1, z' the resistance between P and P_2, x the resistance of the short piece of wire added to r_1, and l the length of wire between P_1 and P_2. We have, neglecting connections of r_2, r_3,

$$\left.\begin{aligned}\frac{r_1+\alpha+x-z}{r_2}&=\frac{r_4+\beta+z}{r_3},\\ \frac{r_1+\alpha+x-z'}{r_3}&=\frac{r_4+\beta+z'}{r_2},\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

from which we obtain for the resistance per unit of length between P_1 and P_2,

$$\frac{z-z'}{l}=\frac{r_2-r_3}{l(r_2+r_3)}(r_1+r_4+\alpha+\beta+x)\ .\ \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

The value of x is easily obtained with sufficient accuracy from either of equations (6), as z is approximately known from the known resistance of the whole wire. In this way the resistance per unit of length at different parts of the wire can be easily found, and, if necessary, a table of corrections formed for the different divisions of the scale.

21. Carey Foster's method of calibrating a slide-wire. Professor Carey Foster has given the following method for the calibration of the bridge wire. The arrangement is shown diagrammatically in Fig. 93. The battery shown in Fig. 89 is removed, and two equal copper bars are attached at C, D (Fig. 93), at right angles to the bars of the bridge at those points. Between the extremities of these is stretched a second slide wire. Or the slide wire of a second bridge, from which all other connections have been removed, may be connected to C and D by wires from the end bars to which it is attached. In place of the coils c, d of Fig. 89, and the middle bar of the bridge, is substituted a single Daniell's or other cell. One terminal of the galvanometer is connected to a sliding piece on the wire W, the other to a sliding piece on the other wire, W'. In place of r_1 and r_4 are substituted two small resistances,

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* *Reports on Electrical Standards*, p. 171 (1912 edition).

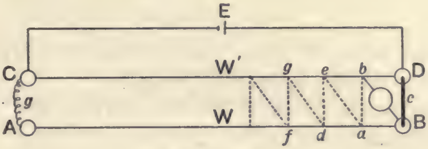

FIG. 93.

one simply a piece of thick wire c, the other a resistance g, equal to that of a convenient portion, say from 80 to 100 millimetres of the bridge wire. The former of these has been called the connector, the latter the gauge. They are connected to the bridge by mercury cups in the manner described in 19 above, and some form of switchboard is usually employed to effect the interchanges described below.

The connection by massive contacts of copper in mercury-filled copper cups, or at least cups with copper bottoms, on which the massive contact pieces rest in the mercury, is of the utmost importance. It is the only method of connection which admits of interchange of coils without error from resistance of contact.

Supposing the gauge placed first on the left and the connector on the right, the slide on W is moved close up to the extremity B, and balance is obtained by placing the slider on W' at some point near D. The gauge and connector are then interchanged, and balance is again obtained by shifting the slider on W' towards the left to some point b.

The gauge and connector are again interchanged, and balance obtained by shifting the slide on W to the left, and so on until both wires have been traversed almost completely from end to end. The distance through which the slider is moved at each interchange of the resistance is read off, and gives, as we shall now show, a determination of the average resistance per unit of length over that portion of the wire. Let P and P' be points of contact on W and W' when balance is obtained, let the permanent resistances included with W, W' at the left-hand ends be denoted by a, a', and at the other ends by b, b' respectively, the resistance of the connector by c, of the gauge by g, of the wire from A to P by z, of the whole wire by w, of the wire W' from C to P' by z', and of the whole wire by w'. If the connector be on the left and the gauge on the right, we have

$$\frac{c+a+z}{a'+z'}=\frac{g+b+w-z}{b'+w'-z'}, \quad \text{.........................(8)}$$

and if the gauge and connector be interchanged so that z receives a new value z_1,

$$\frac{g+a+z_1}{a'+z'} = \frac{c+b+w-z_1}{b'+w'-z'}. \quad \text{.........................(9)}$$

From these equations we get at once

$$g-c=z_1-z, \quad \text{..............................(10)}$$

that is, the steps along W have each a total resistance equal to $g-c$, a result evident without calculation at all.

Again, supposing the gauge at first on the left, and next on the right, the slider on W' is shifted, and we get the equations

$$\frac{a'+z'}{g+a+z} = \frac{b'+w'-z'}{b+c+w+z},$$

$$\frac{a'+z_1'}{c+a+z} = \frac{b'+w'-z_1'}{b+g+w-z}.$$

These give $$z'-z_1' = (g-c)\frac{a'+b'+w'}{a+b+c+g+w}. \quad \text{..................(11)}$$

The quantities on the right-hand side are all constants, and therefore the wire W' is thus divided into parts of equal resistance. From the known resistance of the whole wire, which can be found as shown in 23, p. 347 below, the resistance of each part can be found. The steps on each wire are thus steps of equal resistance.

The following are the actual results obtained in the calibration of the slide-wire of a bridge performed by the method just described.

Parts of the wire of equal resistance ($=r$).		Resistance of the parts included between the corresponding readings.	
Readings (zero taken at right-hand end).	Lengths l.	Readings.	Resistance$=\frac{10r}{l}$.
0 ... 10·59	10·59	0 ... 10	·94429 r
9·79 ... 20·35	10·56	10 ... 20	·94697 ,,
19·70 ... 30·26	10·56	20 ... 30	·94697 ,,
29·84 ... 40·41	10·57	30 ... 40	·94607 ,,
39·69 ... 50·22	10·53	40 ... 50	·94967 ,,
49·71 ... 60·27	10·56	50 ... 60	·94697 ,,
59·80 ... 70·35	10·55	60 ... 70	·94787 ,,
69·82 ... 80·32	10·50	70 ... 80	·95238 ,,
79·86 ... 90·38	10·52	80 ... 90	·95057 ,,
89·41 ... 99·97	10·56	90 ... 100	·94697 ,,
		0 ... 100	9·47873 r

The numbers in the right-hand column are taken from tables. The results are of course not correct to the number of decimals given.

It will be noticed that the second reading in any line of the first column is not exactly the same as the first reading in the next line. This was caused through its being difficult to balance by adjusting the contact on the auxiliary wire. Balance was therefore obtained after a step was taken along the auxiliary wire by moving the slider through a short distance on the wire which was being calibrated.

The value of r found as described below, p. 347, was ·0452 ohm. From this the resistance of the part of the wire between two readings of the scale is found as shown in the table.

22. T. Gray's method of calibrating a slide-wire. A modification of this method, which works well in practice and avoids some difficulties, has been made by Prof. T. Gray. The two wires W, W', are arranged parallel to one another as in Fig. 94, and are connected at the ends A, C and B, D by two equal small resistances of suitable amount g, the terminals of which rest in mercury cups as described above (p. 343). The equality of these resistances can be tested with great ease and delicacy by connecting the battery at A, B, and balancing with the galvanometer between a point on W and another on W', then interchanging the small resistances g, g, and observing if the balance is disturbed. If it is not the resistances are equal. When the resistances have been adjusted to equality, the battery is brought into contact at A and D and balance is obtained by placing one galvanometer terminal close to B on W, and the other at b on W'. The battery contacts are then transferred to B and C, and balance is obtained by shifting the terminal of the galvanometer on W to some point a, while that on W' is kept at b. The battery contact is then transferred to A, D, and balance obtained by moving the terminal on W' so that the points of contact are a, e, and so on.

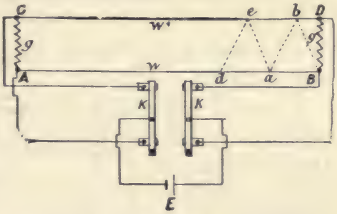

FIG. 94.

The readings on the graduated scales are taken for the successive points of contact, and divide each wire, as will be shown presently, into steps each of resistance g.

The contact of the battery at A, D or B, C can be made by means of two simple rockers K, K, working between mercury cups or ordinary metal contacts, or by means of any simple key. This renders unnecessary any mercury-cup switchboard arrangement for transferring coils.

Thus the method has the very great advantage that the contacts are all permanent except those of the battery and the sliders, no one of which of course introduces any error.

Let contact be made by the battery at A and D, and balance be obtained with the galvanometer at points a and e on the wires W and W', then calling as before z, z' the resistances of the wires between A

and a, C and e respectively, and w, w' the resistances of the whole wires, we have, neglecting (which will not affect the result) constant resistances of connecting bars, etc.,

$$\frac{w-z+g}{z}=\frac{w'-z'}{z'+g}. \quad \text{.........................(12)}$$

Let the battery now be transferred to B and C and balance be obtained at d and e. Denoting the resistance between A and d by z_1, we again have

$$\frac{w-z_1}{z_1+g}=\frac{w'-z'+g}{z'}. \quad \text{.........................(13)}$$

Equations (12) and (13) give

$$z-z_1=g\left(1+\frac{w+g}{w'+g}\right), \quad \text{.......................(14)}$$

or the steps along W are steps of equal resistance. The same can of course be proved for W'.

To avoid thermoelectric effects in such processes, the mean of the two positions of balance for opposite currents should always be adopted as the true position.

The slide-wire bridge may be used for the accurate comparison of resistance coils with a standard, say for the adjustment of single ohms with a standard ohm. Fig. 89 (p. 340 above) shows the arrangement adopted. r_1 and r_4 are the resistances of the coils a, b, to be compared, and are nearly equal. r_2 and r_3 are the resistances of the two coils c, d, and are each nearly equal to r_1 or r_4. The connections are made by mercury cups as already described. Balance is obtained with the contact-piece somewhere near the middle of the slide-wire. The coils r_1, r_4, are then interchanged and balance again obtained. By (10) above we have

$$r_1-r_4=z_1-z_2, \quad \text{.............................(15)}$$

where z_1, z_2 are the resistances of the wire from A to the points of contact in the two cases. If ρ be the resistance per unit of length for the whole wire, s_1, s_2 the distances (reduced, if necessary, by calibration, as shown above, to distances along a wire of uniform resistance ρ per unit of length) measured along the wire from A, we have

$$r_1-r_4=\rho(s_1-s_2). \quad \text{..............................(16)}$$

These results are evidently free from any uncertainty as to the resistance of the junctions of the slide-wire to the copper bars at its ends, and from any error due to want of correspondence between the index mark on the sliding piece and the point of contact.

It is to be observed in this connection that the resistance of a coil may be accurately adjusted to any required value by first making it slightly too great, and then joining it in parallel with a thin wire cut so as to give as nearly as possible the required correction. If the observed resistance be r_4, and that required r_1, the resistance of the correcting wire is $r_1 r_4/(r_4 - r_1)$.

If a separate experiment be made with a coil of accurately known resistance r_1, just a very little less than that of the whole wire, and a second conductor of resistance r_4 so small that it may be neglected, the value of ρ may be obtained from the equation

$$\rho = \frac{r_1}{s_1 - s_2}. \qquad \text{...............................(17)}$$

If the coils compared are too unequal to allow balance to be made on the wire, a series of intermediary coils may be obtained, so as to give a gradual descent from one coil to the other.

23. Resistance of the slide-wire between two readings. The resistance of the wire between any two readings may also be determined by the following method, due to Mr. D. M. Lewis. The total resistance of the wire is approximately found by measuring it with an ordinary bridge consisting of a post-office set of coils, or other available form of a resistance box. Two coils are then made, the resistance of each of which is less than unity by a quantity which is nearly equal to, but not greater than, the total resistance of the wire. These can be also made by means of an ordinary resistance box. Let R_1, R_2 be the as yet not accurately known resistances of these coils. Each is tested as follows in the slide-wire bridge against a unit coil, a standard ohm for example. The unit coil is first placed in the position a of Fig. 89 and one of the two resistances, R_1 say, is placed in the position b. The connections should be made by mercury cups as already described. In the positions marked c, d are placed permanently two coils of nearly equal resistance. The magnitudes of these need not be known, but should not be greater than one or two units. Balance is obtained with the slide S at a point near the end B of the slide-wire, and the reading on the slide-scale is taken. The coil R_1 and the unit are then interchanged, and balance obtained with the slide near A. The difference of the two readings gives the length of wire intercepted between them, and this must be equal in resistance to $1 - R_1$.

The other coil R_2 is now substituted for R_1 and two readings for which balance is obtained taken in the same way. These give a length of the wire the resistance of which is $1 - R_2$.

The two resistances are now put together in series and tested against the unit in precisely the same way, and give between the two readings taken a length of wire of resistance $R_1 + R_2 - 1$.

Now from a previously made calibration of the wire the resistances of the three portions of the wire thus observed can be obtained in terms of the resistance of the calibration-step, and three equations are thus available for the determination of the three unknown quantities R_1, R_2, and r, the resistance of the step used in calibration, as in 21 above.

The following table gives the results of this process applied to the slide-wire the calibration of which is given above :

Positions of the Resistances.		Readings on Slide-wire.	Resistances between these readings in terms of r. Obtained from Table, p. 344 above.
Left.	Right.		
R_1	1	1·40	9·131
1	R_1	97·72	$[9{\cdot}47875 - {\cdot}13220 - {\cdot}21590 = 9{\cdot}13065r]$
R_2	1	0·14	9·368
1	R_2	98·97	$[9{\cdot}47873 - {\cdot}01322 - {\cdot}09754 = 9{\cdot}36797r]$
$R_1 + R_2$	1	69·70	3·625
1	$R_1 + R_2$	31·45	$[3{\cdot}79058 - {\cdot}02843 - {\cdot}13717 = 3{\cdot}62498r]$

Here

$$1 - R_1 = 9{\cdot}131r, \quad 1 - R_2 = 9{\cdot}368r,$$

$$R_1 + R_2 - 1 = 3{\cdot}625r,$$

and therefore

$$r = \frac{1 - R_1}{9{\cdot}131} = \frac{1 - R_2}{9{\cdot}368} = \frac{R_1 + R_2 - 1}{3{\cdot}625} = \frac{1}{22{\cdot}124} = {\cdot}0452.$$

Substituting this value of r in the first two equations we find R_1 and R_2. This can be used to find the resistance of the portion of the wire between any two readings of the scale.

24. Comparison of two standards. An accurate comparison of two nearly equal resistances, for example a unit with its copy, can be obtained by making r_2 and r_3 to be compared occupy the positions c, d, of Fig. 89. Balance is first obtained with r_2 and r_3 in one pair of positions, then they are interchanged and balance again obtained. Assuming that the permanent resistances are included in r_1, r_4, r_2, r_3, and giving z_1, z_2 the same meanings as at p. 346 above, we have

$$\frac{r_2}{r_3} = \frac{r_1 + z_1}{r_4 + w - z_1} = \frac{r_4 + w - z_2}{r_1 + z_2} = \frac{r_1 + r_4 + w + z_1 - z_2}{r_1 + r_4 + w - (z_1 - z_2)},$$

and therefore

$$\frac{r_2 - r_3}{r_3} = \frac{2(z_1 - z_2)}{r_1 + r_4 + w - (z_1 - z_2)}. \quad \text{..................(18)}$$

Hence the greater $r_1 + r_4$ the greater $z_1 - z_2$. Thus, by choosing a pair of resistances as nearly equal as possible, and sufficiently great, r_2 and r_3 may be compared to any needful degree of accuracy.

The permanent resistances, α, β say, corresponding to the coils a, b of Fig. 89, may be estimated by the following method, by which two low resistances can be measured when the ratio of two others is accurately known. Let the resistances r_2, r_3 of c, d in Fig. 89 have the known ratio

μ. We shall suppose r_1 and r_4 to be so low resistances that, with a value of μ differing considerably from unity, balance can be found on the wire. Balance is obtained with the coils in the positions c, d, shown in Fig. 89; then r_2 and r_3 are interchanged, and balance is again obtained. We have

$$\mu = \frac{r_1 + z_1}{r_4 + w - z_1} = \frac{r_4 + w - z_2}{r_1 + z_2}.$$

From these equations we obtain

$$r_1 = \frac{z_1 - \mu z_2}{\mu - 1}, \quad r_4 = -w + \frac{\mu z_1 - z_2}{\mu - 1}. \qquad\qquad(19)$$

If thick copper pieces be substituted for the coils a, b of Fig. 89, their resistances, if the connections as is understood are made with proper mercury cups, may be taken as zero, and α and β are approximately given by (19). The values of α, β thus obtained may be used for the correction of the values of r_1, r_4 found as just described. This correction will not be appreciably affected by the unknown permanent resistances corresponding to the coils c, d, if r_2, r_3 are taken moderately large so that the actual ratio may be taken as equal to their known ratio.

Neither of the arrangements of Wheatstone's bridge described above is at all suitable for the comparison of the resistances of short pieces of thick wire or rod, for example, specimens of the main conductors of a low-resistance electric-light installation, the resistances of which are so small as to be comparable with, if not less than, the resistances of the contacts of the different wires by which they are joined for measurement. To obtain an accurate result in such a case, we must compare, directly or indirectly, the difference of potential between two cross-sections in the rod which is being tested, with the difference of potential between two cross-sections in a standard rod, while the same current flows in both rods, in a direction parallel to the axis at and everywhere between each pair of cross-sections.

25. Thomson's double bridge. Thomson modified Wheatstone's bridge, by adding to it *secondary conductors*, to enable it to be used with the convenience of the ordinary arrangement, for the accurate comparison of low resistances. The arrangement is shown in Fig. 95, as applied to the comparison of the resistance of a certain length of a rod of metal with that of a similar length of a standard rod. CD are two cross-sections, at a little distance from the ends of the conductor to be tested, and AB are two similar cross-sections of the standard conductor. These rods are connected by a thick piece of metal, so that the resistance between B and C is very small, and the terminals of a battery of low resistance are applied at the other extremities of the rods as shown. The sections B, C are connected also by a wire BLC, and the sections A, D by a wire AMD, in each case by as good metallic contacts as possible. BLC and AMD may very conveniently be wires, along which sliding contact-pieces L and M can be moved, with resistances R, R, R, R

of half an ohm or an ohm each, inserted as shown in the figure. The sections A, D are so far from the ends of the rods, and the wires AMD, BLC are made of so great resistance (one or two ohms is enough in most cases), that the current throughout the portions of the conductors compared is parallel to the axis, and the effect of any small resistance of contact there may be at A, B, C, D is simply to increase the effective resistance of BL and LC and AM and MD by a small fraction of the actual resistance of the wire in each case. The terminals of the galvanometer G are applied at L and M, and the circuits of the galvanometer and battery are completed through a double key as in the ordinary bridge. A reversing key is inserted in the battery circuit as in other cases, to enable the comparison to be made with both directions of current.

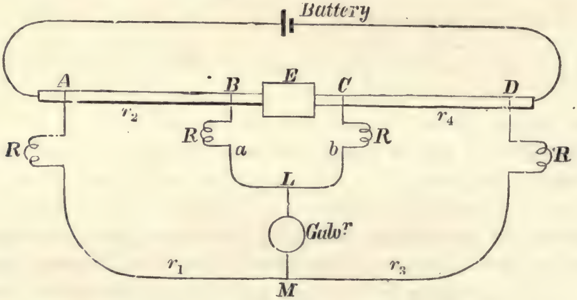

FIG. 95.

Let the resistances AM, DM be denoted by r_1, r_3; BL, CL by a, b; AB, CD by r_2, r_4; and BC by s. Suppose r_1 and r_3 to be varied by moving the sliding piece at M till no current flows through the galvanometer. To find the relation which must hold among the resistances when this is the case, we may suppose the point L connected by a bar of zero resistance, with the cross-section of E which is at the same potential as L. Call this cross-section K. The resistance of the portion of BC to the left of K is $as/(a+b)$, and the portion to the right $bs/(a+b)$. The resistance between B and KL is therefore

$$\{a^2s/(a+b)\}/\{a+as/(a+b)\} \quad \text{or} \quad as/(a+b+s),$$

and similarly that between C and KL is $bs/(a+b+s)$. Hence by (1) we have

$$r_3\left(r_2+\frac{as}{a+b+s}\right)=r_1\left(r_4+\frac{bs}{a+b+s}\right)$$

or

$$r_1r_4-r_3r_2=\frac{s}{a+b+s}(ar_3-br_1)\,. \quad \ldots\ldots\ldots\ldots\ldots\ldots(20)$$

Now s has been supposed very small in comparison with $a+b$, and

a and b can be easily chosen so as to make $ar_3 - br_1$ approximately equal to zero. Hence equation (7) reduces to

$$r_1 = \frac{r_3}{r_4} r_2 , \qquad \text{.........................(21)}$$

the formula found above for the ordinary Wheatstone bridge.

26. Theory of Thomson double bridge. If we go back to the description of the Wheatstone bridge network and the discussion of the sensitiveness of its arrangement, it will be clear that we have to add to the resistance G of the galvanometer the term $ab/(a+b)$ on account of the two conductors LB and LC. Thus the current through the galvanometer is here

$$\gamma_5 = \frac{\gamma\, dr_4}{G + \dfrac{ab}{a+b} + \dfrac{(r_1+r_2)(r_3+r_4)}{r_1+r_2+r_3+r_4}} \, \frac{r_1+r_2}{r_1+r_2+r_3+r_4} . \qquad \text{.........(22)}$$

The deflection is assumed above to be proportional to the product of this by $\sqrt{G}$, so that the maximum deflection is obtained when

$$G = \frac{ab}{a+b} + \frac{(r_1+r_2)(r_3+r_4)}{r_1+r_2+r_3+r_4} , \qquad \text{.....................(23)}$$

that is when

$$G = \frac{ab}{a+b} + \frac{r_1(r_3+r_4)}{r_1+r_3} . \qquad \text{..........................(23')}$$

Hence the maximum deflection D_m is proportional to

$$\tfrac{1}{2}\gamma\, dr_4 \frac{1}{\sqrt{G}} \frac{r_1+r_2}{r_1+r_2+r_3+r_4} ,$$

where γ is the current in r_2 and r_4, and G has the special value just stated. Inserting this value, and noticing that

$$(r_1+r_2)/(r_1+r_2+r_1+r_4) = r_1/(r_1+r_3),$$

and that $a/(a+b) = r_1/(r_1+r_3)$, we get

$$kD_m = \tfrac{1}{2}\gamma\, dr_4 \frac{1}{\left\{(b+r_3+r_4)\dfrac{r_1+r_3}{r_1}\right\}^{\frac{1}{2}}} , \qquad \text{..............(24)}$$

where k is a constant. Thus we have

$$kD_m = \tfrac{1}{2}\gamma \frac{dr_4}{\sqrt{r_4}} \left\{\frac{r_1 r_4}{(b+r_3+r_4)(r_1+r_3)}\right\}^{\frac{1}{2}} , \qquad \text{..............(25)}$$

which is the form that the result stated at 15 (4) above for the Wheatstone bridge takes for the Thomson double bridge.

For $r_4 = 0{\cdot}0001$, $r_2 = 0{\cdot}001$, $r_3 = 1{\cdot}0$, $r_1 = 10$, $b = 1$, $a = 10$, the sensitiveness would be proportional to $0{\cdot}0034\, \gamma\, dr_4/\sqrt{r_4}$. With $a = 10$, $b = 1$, the best resistance of the galvanometer would be about 20/11.

27. Example of a test by double bridge. The following example taken from a paper, "On methods of high precision for the comparison of resistances" by Mr. F. E. Smith,* gives the details of an exact determination made at the National Physical Laboratory. As indicated, we alter the notation of the foregoing theory to agree with the example as presented by Mr. Smith. The resistances, in ohms, were as follows:

$r_4=P$ = 0·1, with potential leads. Value to be found.
$r_2=Q$ = 1·0, ,, ,, ,, Value, $1{\cdot}00000_5$ at 17° C.
$r_3=R$ = 1·0, ,, ,, ,, Value, $1{\cdot}0000_2$,, ,,
$r_1=S$ = 10·0, no potential leads. Value, $10{\cdot}0001_8$,, ,,

$$a=10, \quad b=1, \quad t=17^\circ \text{ C}.$$

The diagram shows the arrangement of the coils. The procedure is as follows. Balance is first obtained by shunting R or S; R', S' is put for the shunted values of $R+L$ and $S+L'+L''$. The value of L is obtained by disconnecting the battery lead from $P.L$ and joining it to $L.R$ and balancing again. Similarly $L'+L''$ is evaluated. To find the connector resistance d, a and b are disconnected, and the galvanometer terminal connected to the junction of Q and d. a is a resistance coil *plus* a potential lead of Q, b is another coil *plus* a potential lead of P, and the ratio b/a is determined with a and b in position, in the following way. The bridge is balanced by shunting R or S. The connector which joins P to Q at d is removed and balance restored by shunting a or b. Then the original arrangement is restored, and balance again obtained. Thus we get successively

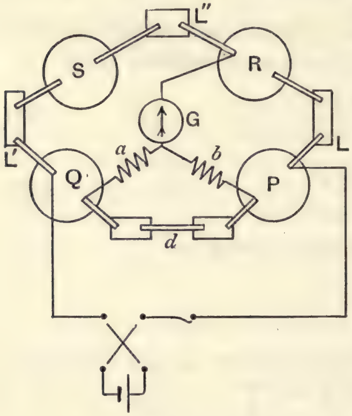

FIG. 96.

$$\frac{P}{Q}=\frac{P+b}{Q+a}=\frac{b}{a}=\frac{R'}{S'}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(26)$$

where R', S' are the shunted values of $R.S$.

The value of P is given by

$$P=Q\frac{R'}{S'}+\frac{ad}{a+b+d}\left(\frac{R'}{S'}-\frac{b}{a}\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots(27)$$

If that of $ad/(a+b+d)$ is not too great the second term of this expression will be negligible; $ad/(a+b+d)$ should not be greater than P. As Mr. Smith remarks, if this quantity be NP and the probable error of an observation be 1×10^{-n}, the error of the final result is not less than $N \times 10^{-n}$. It follows that the resistance of the current leads of standard resistances in the double bridge should not be greater than that of the standard. If the contrary is the case the potentiometer method is to be preferred ; its sensitiveness is higher, but the heating is greater and the tests take longer.

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* *B.A. Reports on Electrical Standards* (1913), p. 674.

Now taking the example for which numbers have been stated above, we have first balance obtained by shunting R with a resistance of 122,000 ohms. The connector at d was removed and balance restored by shunting b with 6500 ohms. This balance remained when the connector d was restored. The value of d was found (see below) to be 0·00012, which is less than P. Thus

$$P = Q\frac{R' + L}{S + L' + L''} = \frac{1\cdot00000_5 \times (1\cdot00001_6 + 0\cdot00011_9)}{10\cdot0001_8 + 0\cdot0001_9} = 0\cdot100011_0, \text{ at } 17^\circ \text{ C.}$$

The evaluation of d, L, $L' + L''$ is shown in the following table :

	Position, galvr. leads.		Position, battery leads.		Balancing condition.		Ohm.
(1)	$L''.R$	$a.b$	$P.L$	$Q.L'$	Shunt on $R = 122000$	Equivt. change	$0\cdot00000_{82}$
(2)	,,	,,	$L.R$	$Q.L'$	,, $S = 8150$	,, ,,	$0\cdot00122_6$
(3)	$L''.S$	,,	$P.L$	$S.L'$	,, $R = 7100$	,, ,,	$0\cdot00014_1$
(4)	$L''.R$	$Q.d$	$P.L$	$Q.L'$	,, $S = 8370$	,, ,,	$0\cdot00119_5$

From (1) and (2) $L = 10(0\cdot000130_8)/11 = 0\cdot00011_9$

,, (1) ,, (3) $L' + L'' = 10(0\cdot000133)\ /11 = 0\cdot00012_1$

,, (1) ,, (4) $d = 0\cdot00012_8$.

28. Thomson's apparatus for testing rods by a double bridge. The apparatus illustrated in Fig. 97 is interesting as that which Thomson constructed for the application of the method to thick rods of conducting material. It is not now used in the practice of the method. The description and cut, however, which appeared in the first edition of this book, are here allowed to stand ; the apparatus is in the historical collection in the Natural Philosophy Institute of the University of Glasgow. [On a massive sole plate of iron, P, are mounted two vertical guide-rods of copper, A, A, and parallel to these the rods to be compared, viz., a standard rod C, and the rod to be tested C_1. C, C_1 are supported with their lower ends in two mercury cups cut in a single block of copper. This block corresponds to the piece E in Fig. 95. The upper ends of C, C_1 are fixed in screw blocks of copper, t, t, to which also are attached the terminals of a constant battery B of low resistance. A reversing key K is interposed between t, t and the battery. A scale D graduated along its two edges nearly fills the space between the rods C, C_1.

A pair of resistance coils r, r are fixed to the sole plate, and have one terminal of each connected by a strip of copper, which also carries the terminal screw T. The other terminals of these coils are fixed to two copper slides, S_2, S_3, which move along, but are insulated from, the guide-rods, and carry contact pieces *c*, *c*, each of which is bevelled off to a knife-edge on a level with its upper side. This knife-edge is pressed against the corresponding rod by springs *s*, *s*, which are insulated so as not to touch the rods. The coils *r*, *r* are attached directly to the contact pieces *c*, *c*. Thus S_2rTrS_3 corresponds to the partial circuit *BRLRC* of Fig. 95.

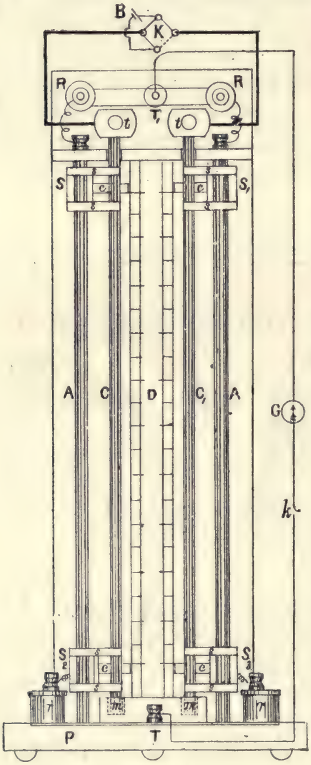

FIG. 97.

Near the upper ends of C, C_1 is a similar arrangement of sliders S, S_1, with spring contacts and attached coils *R*, *R*. These coils are connected by a copper strip which carries the terminal T_1. The coils *R*, *R* are attached to the upper ends of the guide-rods *A*, *A*, and through these to the sliders S, S_1. The guide-rods are so thick that no appreciable change is made in the ratio of the resistances of the parts of the partial circuit SRT_1RS_1 on the two sides of T_1 by varying the positions of the sliders. This partial circuit corresponds to *ARMRD* of Fig. 95.

Each pair of coils, *r*, *r* and *R*, *R*, may be wound on a single bobbin with advantage. The arrangement is thereby rendered more compact, and there is less risk of error from difference of temperature between the bobbins, or of thermoelectric disturbance between their terminals.

Between T and T_1 is placed the galvanometer *G*, which is provided with a simple key *k*, placed for convenience in the actual arrangement beside the reversing key *K*.

In the use of the instrument the rods to be compared are placed in position, and the sliders on the rod of lower resistance are placed so that their upper edges, and therefore their knife-edges, are opposite the lowest and uppermost divisions of the scale. The lower contact piece on the other rod is placed with its upper edge opposite the lowest division of the scale on that side. The upper contact piece on the same rod is then shifted until no current flows through the galvanometer. Balance is obtained for both directions of the current, and the mean position of the slider taken, to eliminate error from thermoelectric disturbance.

A number of standard rods of different thicknesses are provided with the instrument in order that nearly equal ratios may be obtained over a wide range of low resistances.]

29. Matthiessen and Hockin's method for low resistances. The following method was used for the same purpose by Messrs. Matthiessen and Hockin in their researches on alloys. *AB*, *CD*, Fig. 98, are the two rods to be compared. They are connected in circuit with two coils of resistances r, s, which have between them a graduated wire WW', as in Kirchhoff's bridge. SS' are two sharp knife-edges, the distance of which apart can be accurately measured, fixed in a piece of dry hard wood or vulcanite, and connected with mercury cups on its upper side. This arrangement is placed on the conductor AB, so that the knife-edges making contact include between them a length SS' of the rod. TT' is a precisely similar arrangement placed on CD. One terminal of the galvanometer is applied at S, and the resistances r, s adjusted so that a point P on the wire which gives balance is found for the other terminal. The terminal of the galvanometer is shifted to S', and a second point P' found by varying the resistances of the coils from r_1, s_1 to r_1', s_1' in such a manner as to keep the sum $r+s$ constant. Similarly balance is found for TT' with values r_2, s_2, r_2', s_2', for the resistances of the coils, fulfilling the condition that the sum $r+s$ is the same as in the former case. Let a, b, c, d, k denote the resistances between L and S, L and S', L and T, L and T', L and M respectively; α, β, γ, δ the resistance between W and P in the four cases, κ the resistance of the whole wire WW'. We have by (1)

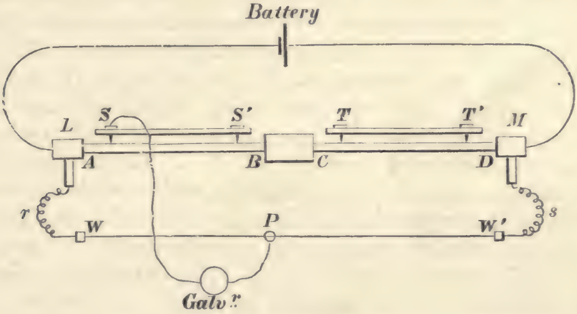

FIG. 98.

$$\frac{a}{k-a}=\frac{r_1+\alpha}{s_1+\kappa-\alpha},$$

and therefore
$$\frac{a}{k}=\frac{r_1+\alpha}{R}, \qquad \text{..............................(28)}$$

where
$$R=r+s+\kappa.$$

Similarly $$\frac{b}{k}=\frac{r_1'+\beta}{R}.$$

Therefore $$\frac{b-a}{k}=\frac{r_1'-r_1+\beta-a}{R}. \quad\text{.......................(29)}$$

In the same way we get

$$\frac{d-c}{k}=\frac{r_2'-r_2+\delta-\gamma}{R} \quad\text{.......................(30)}$$

and combining the last two equations we get for the ratio of the resistances of the conductors between the pairs of knife edges,

$$\frac{b-a}{d-c}=\frac{r_1'-r_1+\beta-a}{r_2'-r_2+\delta-\gamma}. \quad\text{.....................(31)}$$

30. Rayleigh's method of comparing low resistances. This method of Matthiessen and Hockin is not given here as a highly accurate means of comparing low resistances. The arrangement shown in Fig. 98 would be improved by the addition of two finite but not large "ballasting" resistances, one inserted at A, the other at D, to prevent the terminal ratios, *e.g.* $a/(k-a)$, from being exceedingly small. With a sufficiently sensitive reflecting galvanometer of high resistance the differences of potential between S, S', T, T' can be compared with accuracy enough to enable a satisfactory estimate of the comparative conductivities of two pieces of thick copper rod to be obtained.

The following method was given as an alternative by the late Lord Rayleigh [*Collected Papers*, 2, p. 276]. It is founded on the arrangement for obtaining a low resistance for use in the Lorenz method of determining the ohm [see XII. 35, below]. A low resistance p, which is to be measured, is joined in series with a standard coil of resistance q, 1 ohm, or $\frac{1}{10}$ ohm. The coil q is shunted by the coils c and b, of which the ratio of resistances, b/c, is made nearly equal to p/q, while c is fairly large. A high resistance galvanometer is applied to the terminals of p, and the deflection, d, noted. The galvanometer is then applied to the terminals of b, and c is adjusted until the deflection is again d. If the resistance G of the galvanometer be very great we have $p=bq/(b+c+q)$.

The exact equation however is

$$p=\frac{bqG}{(b+c+q)G+bc}.$$

It is essential, if G is not so great as to render the resistance of the wires connecting the galvanometer to the pairs of points in the current circuit to use the same galvanometer leads in taking the two readings. To avoid error from thermoelectric electromotive forces the tests should be repeated with the battery reversed and the mean result taken.

Since p is supposed small it is necessary, in order that the largest available current may not be reduced, that q should not be too large.

It is generally convenient, though some of the sensibility of the arrangement is thereby sacrificed, to take G large. A high resistance reflecting galvanometer is generally sensitive enough to give very considerable accuracy. With $p=\frac{1}{100}$, $q=1$, $b=1$, and G very great c would be 98.

31. Potentiometer method for low resistances. Low resistances may be measured by means of a potentiometer arranged as follows. Two circuits A, B are provided, as shown in Fig. 99. Circuit A contains a resistance p which is to be evaluated, a standard resistance r, an

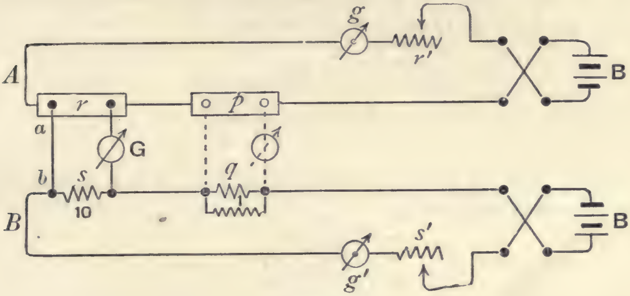

FIG. 99.

additional resistance r', a battery B and a galvanometer g, so that the current γ in the circuit can be observed and controlled. Circuit B is arranged in like manner with resistances q, s, and s', a battery B' and a galvanometer g'.

The terminals of r are connected to those of s, the left-hand terminal directly by means of a massive conductor in mercury cups which form these two terminals, the right-hand terminals through the galvanometer G as shown. The current in the circuit B is now adjusted so that there is no current through G. Currents γ, γ' now flow in A and B.

A similar experiment is made with p and q connected in the same way (but r and s now disconnected from one another), except that q is now shunted with a high resistance to balance the galvanometer. These experiments are repeated until balance for both pairs of resistances are obtained. We have now, if p, q, r, s denote the resultant balancing resistances, $p\gamma=q\gamma'$, $r\gamma=s\gamma'$ so that

$$\frac{p}{r}=\frac{q}{s}.$$

Of course by means of a special rocking arrangement with mercury cups, or a suitably designed key, the two connections are transferred from one pair of coils to the other as quickly as possible, for it is to be remembered that the currents in the circuits are running continuously. The arrangement is described by Mr. A. Campbell [*Phil. Mag.* July 1903] as one in use at the National Physical Laboratory.

The sensitiveness of the arrangement is easily found. Let the connections be as in the diagram. The terminals a, b are at the same potential; if then the total resistances of the circuits A, B are $p+r_1$ and $q+s_1$, the current through G due to a small alteration, dp, of p from balance will be $\gamma\, dp/\{G+pr_1/(p+r_1)+qs_1/(q+s_1)\}$, and the deflection will be proportional to this multiplied by $\sqrt{G}$. It follows that the deflection will be a maximum if the resistance G of the galvanometer have the value $pr_1/(p+r_1)+qs_1/(q+s_1)$.

Putting in this value of G we find that the deflection D is proportional to $\frac{1}{2}\gamma\, dp/\sqrt{G}$, that is, is given by

$$kD=\frac{1}{2}\frac{\gamma\, dp}{\left\{\dfrac{pr_1}{p+r_1}+\dfrac{qs_1}{q+s_1}\right\}^{\frac{1}{2}}}. \qquad \text{......................(32)}$$

Let us take r_1, s_1 very great in comparison with p and q respectively. Then the deflection is given by

$$kD=\frac{\frac{1}{2}\gamma\dfrac{dp}{p}\sqrt{p}}{\sqrt{1+\dfrac{q}{p}}}. \qquad \text{........................(32')}$$

The best resistance of the galvanometer is now $p+q$, or simply p, if q/p is very small.

Of course unless p and r be nearly equal the best galvanometer resistance for the second observation will not be the same as for the first. The reader may compare the sensitiveness in the case of $p=r$ and $q=s$, and verify that, if q/p be small, the sensitiveness is about twice that of the bridge with equal arms, that is with $r_1=r_3=r_2=r_4$. The bridge method has the advantage that in it the keys are only tapped down, while with the potentiometer the currents are kept flowing while the observations are made, so that in the latter case the heating effects are much more serious.

By reference to Fig. 99 it will be seen that if the resistance q be removed and an arrangement be made so that the terminals shown connecting s and r can be swung over so as to connect s and p, r and p can be adjusted to equality. The coil s may have any convenient value, and thus if p be a standard, r may be made a copy of p very readily.

32. Two-step method for low resistances. Mr. Campbell has suggested the following, which he calls a "two-step" bridge method. The arrangement is shown in Fig. 100. p and q are two resistances, r and s two others. A small resistance u is inserted in the position shown, and is shunted so that the galvanometer is in balance in position a. The galvanometer is now placed in position b, and r or s shunted until balance is again obtained. It is not necessary to know u accurately. The connections *c*, *c* are included with *r* and *s*. They may be determined by sending a current round their circuit and comparing the drop of potential for each with that for a known resistance *p* or *q* in the circuit.

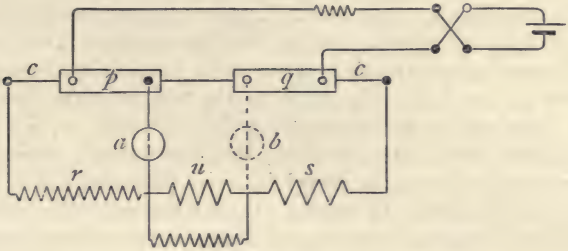

FIG. 100.

We have if u' be the shunted value of u for the position a, and s' be supposed to be the shunted value of s for the position b,

$$\frac{p}{q} = \frac{u' + 2r}{u' + s + s'}. \qquad (33)$$

If for the second position balance is obtained by shunting r down to r', we have

$$\frac{p}{q} = \frac{u' + r + r'}{u' + 2s}. \qquad (34)$$

Since u' is small in comparison with the resistance r and s, and is approximately known, we get p/q.

33. Fall of potential method for low resistances. The following method of comparing resistances is in principle the same as Thomson's bridge with secondary conductors, and Matthiessen and Hockin's method described above, as, like them, it consists in comparing the difference of potential between two cross-sections near the ends of the conductor to be tested with the difference of potential between two cross-sections in a standard conductor, when the same uniform current is flowing in both. It is, however, more readily applicable in practice, and is very useful for a great many practical and commercial purposes, as, for example, in the testing of the armatures or magnet coils of machines, in the estimation of the resistances of contacts, and in the determination of the specific conductivities of thick copper wires or rods. All that is required is a small battery, a suitable galvanometer of sufficient sensibility, and two or three resistance coils of from $\frac{1}{5}$ ohm to 1 ohm. These coils may very conveniently for many purposes be made of galvanized or tinned iron wire of No. 14 or 16 B.W.G., wound round a piece of wood $\frac{1}{2}$ inch thick, from 8 to 10 inches broad, and from 12 to 18 inches long, with notches cut in its sides, at intervals of a quarter of an inch, to keep the wire in position. To avoid any electromagnetic effect which may be produced by the coils if they happen, when carrying currents, to be placed near the galvanometer, the wire should be doubled on itself at its middle point, the bight put round a pin fixed near one end of the board, and the wire then wound double on the board, the two parts being kept far enough apart to insure insulation. Resistance coils made in this way are exceedingly useful for electric lighting experiments, as the thickness of the wire and its exposure everywhere to the air prevent undue heating by strong currents, or, if there is much heating, obviate the risk of damage. For the battery a single cell, as for example a gravity-Daniell, or, if the battery is to be carried from place to place, two hermetically sealed chloride of silver cells, which may be joined in series or in parallel as required, may very conveniently be used. As instrument of comparison a Thomson's centiampere balance used as voltmeter with a resistance in series with its coil, or some sensitive form of voltmeter, is convenient for many practical purposes; but when greater accuracy is aimed at, as when the method is used for the measurement of the (specific) conductivity of short lengths of thick metallic wires by comparison with a standard, a sensitive reflecting galvanometer of resistance great in comparison with that of the conductor between the points at which the terminals are applied should be employed, and the battery should be of as low internal resistance as possible.

The galvanometer is first set up and made of the requisite sensibility by adjusting, as described in 2 above, the intensity of the field in which it is placed.

The conductor whose resistance is to be compared, and one of the coils whose resistance is known, are joined in series with the battery. It is advisable to have the circuit at a distance of a few yards from the galvanometer, so that accidental motions of the wires carrying the current may not have any sensible effect on the needle. One operator then holds the electrodes of the galvanometer so as to include between them, say, first the wire which is being tested, then the known resistance, then once more the wire being tested, in every case taking care not to include any binding screw connection, or other contact of the conductors. The known resistance should, when great accuracy is required, be so chosen that the readings obtained in these two operations are as nearly as may be equal.

Let the mean of the readings for the first and third operations be V scale divisions, for the second V'; let r denote the known resistance, and x the resistance to be found.

Since by Ohm's law the difference of potential between any two points in a homogeneous wire, forming part of a circuit in which a uniform current is flowing, is proportional to the resistance between those two points, we have

$$x=\frac{V}{V'}r. \quad \ldots\ldots\ldots\ldots (35)$$

The resistance of a contact of two wires whether or not of the same metal may be found in the same manner, by placing the galvanometer electrodes so as to include the contact between them, and comparing the difference of potential on its two sides with that between the two ends of a known resistance in the same circuit. Care must however be taken in all experiments made by this method, especially when the galvanometer circuit includes conductors of different metals, to make sure that no error is caused by thermal electromotive forces. To eliminate such errors the observations should be made with the current flowing first in one direction and then in the other in the battery circuit.

The following results of some measurements of the resistance of a Siemens SD_2 dynamo machine, made at Glasgow, may serve to illustrate this method. An iron wire coil, of half an ohm resistance, was joined to one of the terminals of a standard Daniell, and short wires attached to the other terminal of the cell and the free end of the coil were made to complete the circuit through the armature, by being pressed on two diametrically opposite commutator bars, from which the brushes and the magnet connections had been removed. The electrodes of the galvanometer, which was a dead-beat reflecting galvanometer of high resistance, were applied alternately to the same commutator bars, and to the ends of the half ohm, and the readings recorded. The following are the results, extracted from the Laboratory Records, of three consecutive experiments:

EXPERIMENT I.

Operation.	Reading on Scale.	Deflection of Spot of Light.
Galv. zero read	214	
Electrodes on ½ ohm	857	643
,, ,, armature	597	383

EXPERIMENT II.

Operation.	Reading on Scale.	Deflection of Spot of Light.
Galv. zero read	214	
Electrodes on armature	607	393
,, ,, ½ ohm	874	660
,, ,, armature	607	393

EXPERIMENT III.

Operation.	Reading on Scale.	Deflection of Spot of Light.
Galv. zero read	214	
Electrodes on ½ ohm	874	660
,, ,, armature	607	393
,, ,, ½ ohm	872	658

The first experiment gives for x the value, $383 \times \cdot 5/643$, or ·298 ohm The other two experiments, although their numbers are different, give very nearly the same result, which agrees closely with a measurement made about eight months before, by the same method, with another potential galvanometer. The readings show that the galvanometer had ample sensitiveness for the test.

In the ordinary testing of the armatures of machines by this method, the circuit of the battery may be completed through the brushes; but if the machine has been wound on the shunt system, care must be taken previously to disconnect the magnet coils. In every case the galvanometer electrodes must be placed on the commutator bars directly.

34. Differential galvanometer with high resistance coils for low resistance tests. Prof. Tait* used a differential galvanometer† (see 35 below) for this method of determining low resistances. The conductors to be compared were arranged in series, so that the same current flowed through both. The terminals of one coil were then placed at two points on one conductor, the terminals of the other coil at two points on the other, such that the galvanometer deflection was zero. The difference of potential between the points of each pair was therefore the same in the two cases. Hence the lengths of portions of the two conductors of equal resistance were obtained.

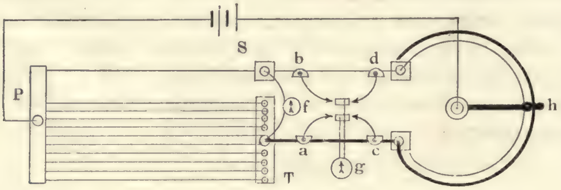

FIG. 101.

The following zero method, due to Prof. T. Gray, is founded on the same principle. The arrangement of apparatus is shown in Fig. 101. One terminal of a battery of one or two low resistance cells is attached to a stud on a thick copper bar *P*, the other terminal to a metallic axis round which the copper bar *h* turns. The bar *h* makes contact at its outer end with a bare wire and a bare rod bent round into concentric circles with centre at the axis of the bar, and having a pair of remote extremities connected with mercury cups or binding terminals, and the other pair of extremities free as shown. To one of these terminals is connected one end of the bar to be tested, to the other one end of the standard bar. The other end of one of these bars, say the standard, is connected to a mercury cup *S*, which is in line with, but is insulated from, a row of mercury cups or a mercury trough cut in a copper bar placed parallel to *P*. Between this bar and the trough are stretched a series of parallel wires all of the same material and length and as nearly as possible of the same resistance ; and a single wire, of the same resistance, material, and length, connects the bar *P* and the cup *S* with which the standard bar is in contact. These wires may be conveniently straight rods of platinoid, an eighth of an inch in diameter, and six feet long, soldered at one end to the bar *P*, and at the other to stout well-amalgamated copper terminals dipping into the mercury cups or trough. The wires may be made of the same resistance by means of a slide-wire bridge, or by the method described below.

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* *Trans. R.S.E.* vol. xxviii. 1877-8.

† For an account of this instrument and its use in the measurement of resistance see Maxwell's *Electricity and Magnetism*, vol. i. Further particulars of the use of differential galvanometers are given in XIV. 55, 56 below.

The cup *S* and the terminals *T* are now brought to one potential by turning the bar *h* round on the circular wire until a sensitive galvanometer, *f*, joining them shows no deflection. This galvanometer is then left connected, and by means of a second sensitive galvanometer, *g*, two pairs of points *a*, *d* and *c*, *d* are found between which in each case no current flows when they are connected by a wire. Each pair of points are therefore at the same potential. Hence if we denote by r_1 the resistance of the standard between *b* and *d*, by r_2 that of the other rod between *a* and *c*, and by *n* the number of wires joining *P* and *T*, we have

$$r_2 = \frac{r_1}{n}. \qquad \text{.................................}(36)$$

A differential galvanometer with two independent pairs of terminals may be employed for this method. One coil may be made to join *a*, *b*, the other *c*, *d*, or one coil may be made to join *b*, *d*, and the other *a*, *c*. In the former case either the effect on each coil must be made zero, or care must be taken to connect the terminals to *a*, *b* and *c*, *d* so that the magnetic effects of the two coils at the needle may be opposed. The resistance of the galvanometer coils, except when the current in each coil is made zero, must be so great as not to cause any sensible alteration of the potentials at the points at which the terminals are applied.

The wires joining *P* to *S* and *T* may be tested for equality as follows. Two nearly equal wires are made to join *P* to *S* and *P* to *T*, and *h* is placed so that the galvanometer *f* shows zero current. The wire joining *P* to *T* is then removed and another put in its place. If the current in *f* still remain zero for the same position of *h* the latter wire and the former are of the same resistance. If not the necessary correction is made and the comparison repeated.

35. Differential galvanometer method for comparison of standards. A comparison of two nearly equal resistances, such, for example, as those of two standard unit coils, or even of two unequal standards, can be made with precision by means of a differential galvanometer. We shall suppose that the two coils of the galvanometer have been adjusted so that the action on the needle is zero when the same current passes through each. This adjustment can be made by putting the coils in series and setting the magnetic action of one against the other.

The current in the circuit flows through both, and either by changing the relative positions of the coils if they are movable, or by adding a turn or turns to the feebler, the adjustment is made. The coils are at the same time made of the same resistance, though this adjustment is not so important as the other. If the coils do not balance for equal currents, a balance obtained by adjustment of resistance will only hold for other cases in which the same currents or currents in the same ratio flow in the coils. For example, if balance is got for $p=1$, $q=10$, balance will also be got for $p=0{\cdot}1$ and $q=1$, if the current γ is made 10 times what it was in the former case. The coils, of which the actual resistances are p, q, are arranged as in Fig. 102 with the two galvanometer coils, which though symmetrically placed with respect to the suspended needle are here shown separate for distinctness.

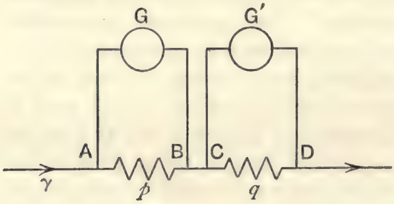

FIG. 102.

We suppose that the resistances of the coils both have the same value G, and that along with these are ballasting coils of known resistances x and y. It is supposed that by means of a mercury-cup commutator, the connections of the derived circuits to the terminals AB and CD may be interchanged. The main current is γ.

The difference of potential on AB is $\gamma p(G+x)/(p+G+x)$, so that the current through the coil of resistance $G+x$ is $\gamma p/(p+G+x)$. Similarly the current through the coil of resistance $G+y$ is $\gamma q(q+G+y)$. The difference of these currents is

$$\gamma\left\{\frac{p}{p+G+x}-\frac{q}{q+G+y}\right\},$$

or

$$\gamma\frac{p(G+y)-q(G+x)}{(p+G+x)(q+G+y)}.$$

If this difference is zero we have

$$\frac{p}{q}=\frac{G+x}{G+y}. \qquad \text{(37)}$$

Let now the coils be interchanged by means of the commutator, and x and y be altered to x' and y' to give balance. We now get

$$\frac{p}{q}=\frac{G+y'}{G+x'}. \qquad \text{(38)}$$

With the previous result this gives

$$\frac{p}{q}=\frac{y'-x}{x'-y}. \qquad \text{(39)}$$

If the same connecting wires are used for the coils employed, the values of $y'-x$ and $x'-y$ are known and the ratio of p to q is determined.

Another mode of proceeding is as follows. It is convenient when p and q are nearly equal. The deflection D_1 of the galvanometer needle is read off. Then the resistance q is altered by a known amount to q' by shunting, and the new deflection D_2 is read off. We get

$$\frac{D_1}{D_2}=\frac{p-q}{p-q'}\,\frac{q'+G+x}{q+G+x}. \quad \text{.........................}(40)$$

Now $G+x$ is usually fairly large in comparison with q, and we suppose that the difference between q' and q is small. Hence we have, very approximately, instead of (40),

$$\frac{D_1}{D_2}=\frac{p-q}{p-q'}. \quad \text{..................................}(41)$$

To obtain an idea of the sensitiveness of this mode of testing, we notice that if p and q be nearly but not quite equal, we have approximately, since $x=y$, for the difference of currents,

$$\gamma(p-q)/(q+G+x)=\gamma\, dp/(q+G+x).$$

The deflection is proportional therefore to this multiplied by $\sqrt{G}$. The best galvanometer resistance is then $q+x$, and the deflection is then proportional to $\frac{1}{2}\gamma\, dp/\sqrt{p+x}$, since p is nearly equal to q.

It appears that at the Physikalische Reichsanstalt in Berlin the arrangement for the testing of mercury standards $G=G'=6$ ohms, $p=q=1$ ohm, $x=10$ ohms.

36. Measurement of specific resistances. In order that the conducting powers of different substances may be compared with one another, it is necessary to determine their *specific resistances*, that is, the resistance in each case of a wire of a certain specified length and cross-sectional area. We shall here define the specific resistance of any substance at any given temperature as the resistance between two opposite faces of a centimetre cube of the material at that temperature.* This resistance has been very carefully determined for several different substances at ordinary temperatures by various experimenters, and a table of results is given below (see Appendix).

To measure the specific resistance of a piece of thin wire, we have simply to determine the resistance of a sufficiently long piece of the wire by the ordinary Wheatstone-bridge method described above, and from the result to calculate the specific resistance. Let the length of the wire be l cm, its cross-section s square cm, and its resistance R ohms. Then the specific resistance of the material would be Rs/l ohms. The length l is to be carefully determined by an accurately graduated measuring-rod; and the area s may be found with sufficient accuracy in most cases by direct measurement, by means of a decimal wire gauge measuring to a hundredth of a millimetre. If, however, the wire be very thin, the cross-section may, if the density is known, be accurately obtained in square cm by finding the weight in grammes of a sufficiently long piece of the wire (from which the insulating covering, if any, has been carefully removed), and dividing the weight by the product of the length and the density. Very thin wires are generally covered with silk or cotton, which may very easily be removed, without injury to the wire, by making the wire into a coil, and gently heating it in a dilute solution of caustic soda or potash. The coating must not in any case be removed by scraping.

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* The reciprocal of this (called below the specific conductivity) may be advantageously called the *electric conductivity* of the substance, if the word conductivity be set free by the general adoption of the word *conductance* for the reciprocal of the resistance of a given conductor.

If the density is not known, it may be found by weighing the wire in air and in water by the methods described in books on hydrostatics. All the weights, from 1 gramme upwards, ordinarily used in weighing are made of brass, and hence when conductors of nearly the same specific gravity as brass are weighed in air, the correction for buoyancy may be neglected. The weighing in water however must be corrected both for expansion of water with rise of temperature and for the weight of air displaced by the weights. For a temperature of 13° C. these corrections are as follows:—for expansion of water an increase of loss of weight in water of 0·059 per cent.; for buoyancy of air a diminution of apparent weight in water of about 0·0143 per cent. Care should be taken in weighing to prevent air bubbles from adhering to the sides of the specimen; and the water used for weighing should first have been carefully boiled to expel the air contained in it. All error of this kind may be avoided by boiling the water with the specimen in it, and then allowing both to cool together.

37. Commercial tests of specific resistances of copper mains. If the wire be thick, and a sufficient length of it to render possible an accurate measurement of its resistance by the ordinary bridge method is not conveniently available, one of the methods of comparing small resistances described above (25...34) is to be used. The most convenient in many practical cases is that described in 33, in which the resistance between two cross-sections of the bar to be tested is compared with that between two cross-sections of a standard rod of pure copper. The cross-sections should, if the distance between them be not thereby made too small, be chosen so as to make the two resistances nearly equal. If we put V for the number of divisions of deflection on the scale of the potential galvanometer, when the electrodes of the galvanometer are applied to the standard rod, at cross-sections l cm apart; V' that when they are applied to the rod being tested, at cross-sections l' cm apart, then we have for the ratio of the resistance of unit length of the wire tested to the resistance of unit length of the standard at the temperature at which the comparison is made, the value $V'l/Vl'$. If s and s' be the respective cross-sectional areas, which in this case are easily determinable by measurement with a screw-gauge, and we assume that the temperature at which the measurements of resistance are made is 0° C., we get for the ratio of the specific resistances at 0° C. the value $V'ls'/Vl's$, and therefore also for the ratio of their specific conductivities $Vl's/V'ls$. This last ratio multiplied by 100 gives the percentage conductivity at 0° C. of the substance as compared with that of pure copper. If, as will generally be the case, the temperature at which the experiments are made be above the freezing-point, the value of $100Vl's/V'ls'$ may be taken as the percentage of the specific conductivity of pure copper at the observed temperature possessed by the substance, and this, if the wire tested is a specimen of nearly pure copper, will be nearly enough the same at all ordinary temperatures.

If in experiments by this method the electrodes are applied by hand to the conductors, the operator should, especially if the electrodes and the conductors tested are of different materials, be careful not to handle the wires, but should hold them by two pieces of wood in strips of paper passed several times round the wires, or by some other substance which conducts heat badly, so that no thermal electromotive force may be introduced into the circuit of the galvanometer (see above, p. 346). He may conveniently make the galvanometer contacts by means of two knife edges fixed in a piece of wood which can be lifted from one conductor to the other without its being necessary to handle the galvanometer wires in any way. This will besides render any measurement of the length of the conductor intercepted between the galvanometer electrodes unnecessary, as l is equal to l'. We have then for the percentage specific conductivity of the substance the value $100Vs/V's'$.

As an example of this method we may take the following results of a measurement (made in the Physical Laboratory of the University of Glasgow) of the specific conductivity of a short piece of thick copper strip. The specimen was joined in series with a piece of copper wire of No. 0 B.W.G. of very high conductivity, in the circuit of a Daniell's cell of low resistance. The electrodes of a high resistance reflecting galvanometer applied at two points 700 cm apart in the copper wire gave a deflection of 153·5 divisions, when applied at two points 500 cm apart in the strip 270 divisions. The weight of the wire per metre was 443 grammes, of the strip per metre 186·3 grammes. Hence the specific conductivity of the copper strip was 96·6 per cent. of that of the wire against which it was tested.

38. Realization of a standard ohm. The accurate realization of a standard ohm, as defined on p. 29 above, involves the determination of the specific resistance of mercury, an operation which requires great care and considerable experimental skill. This determination has been made by several experimenters, among others by Lord Rayleigh and Mrs. Sidgwick and by Messrs. Glazebrook and Fitzpatrick at Cambridge, and by Messrs. Hutchinson and Wilkes at Baltimore. [See Chapter XV. below.]

The value obtained by Lord Rayleigh and Mrs. Sidgwick for the resistance at 0° of a column of mercury 1 metre long and 1 square millimetre in cross-section was ·95412 B.A. unit (XV. 27, below). Messrs. Glazebrook and Fitzpatrick's value for the same resistance is ·95352 B.A. unit, Messrs. Hutchinson and Wilkes found it to be ·95341 B.A. unit. Previous measurements made by Werner Siemens and Matthiessen gave ·9536 B.A. unit and ·9619 B.A. unit respectively for this resistance. It will be noticed that the mean value for this resistance given by the three later measurements quoted lies between these, but much nearer to the former. Messrs. Siemens Brothers for a long time used the resistance of a column of mercury specified as above as the unit of resistance, and standard units were issued by them to experimenters. One of these examined by Lord Rayleigh gave ·95365 B.A. unit for its resistance at the temperature 16·7° at which it was stated to be correct.

39. Copies of the standard ohm. Standard ohms have been made in mercury, by using tubes bent so that the requisite length is obtained in a compact form, but they are not very convenient in use, and are of course liable to breakage. A copy of the standard ohm can however be easily made when the resistance of a column of mercury of definite cross-section and length has been accurately found. Figs. 103 and 103(a) show such copies. Fig. 103 is the usual form of the standard. It is made of platinum-silver wire, wound within the lower cylinder. The space within up to the top of the wider cylinder is filled with paraffin wax. The ends of the coil are attached to two thick electrodes of copper rod, bent as shown and kept in position by a vulcanite clamp. The ends of these when the coil is used are placed in mercury cups in the manner already explained, and should always, before the coil is placed in position, be freshly amalgamated with mercury. The lower cylinder up to the shoulder is placed in water when the coil is in use, and the temperature of the water is ascertained by means of a thermometer in the hollow core of the cylinder. The variation of the resistance of the coil with temperature is known, and hence its resistance at any observed temperature can be obtained. Of course care must be taken not to expose the standard to too strong currents, and to keep the temperature as near as possible to the normal temperature at which the standard is given as correct.

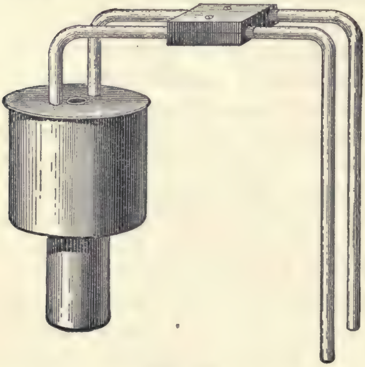
FIG. 103.

Fig. 103 shows a form of the standard constructed by Messrs. Elliott Bros. according to a suggestion made by the late Professor Chrystal. A thermoelectric couple, of which one junction is within and close to the coil, and the other outside the case, is used to determine the temperature of the coil. In the form in which the instrument is now made the external junction is not brought out through the bottom of the case as shown, but the wire is brought out at the top of the case, and then joined to a wire of the other metal which is entirely outside and attached to one of the binding screws. The external junction is of course placed in water the temperature of which is measured, and the thermal current is observed by means of a galvanometer connected to the terminals. This gives the difference of temperatures between the junctions and therefore the temperature of the coil.

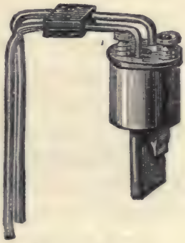

FIG. 103 (*a*).

On account of the uncertainty of the temperature of the coil, and its liability to loss of insulation by deposition of moisture on the upper surface of the cylinder, Prof. J. A. Fleming* has constructed a standard in which the case containing the coil is a hollow circular ring of brass made by screwing together by projecting flanges two square sectioned circular troughs. The electrodes (rods arranged as in Fig. 102) proceed to the ring through two upright brass tubes from 5 to 6 inches in length, from which they are insulated by vulcanite collars at the bottom, and at the top by two vulcanite funnels corrugated on the outside, and projecting above the tubes. Paraffin oil placed in these vulcanite funnels prevents loss of insulation by condensation of moisture on the insulating pieces.

40. Constancy of standards. A careful watch has been kept on the values of the B.A. standards of resistance, for several years at Cambridge by Glazebrook, and more recently at the National Physical Laboratory by various members of the Laboratory staff. The primary standards at the National Physical Laboratory are of mercury; the secondary standards are made of platinum, platinum-iridium alloy, gold-silver, platinum-silver, and manganin. The last mentioned substance is an alloy of 84 p.c. copper, 12 p.c. manganese, and about 4 p.c. nickel. It has a very low temperature coefficient, which however depends on the temperature. From the following table of its values it will be seen that it is positive at ordinary temperatures, vanishes at about 45° C., and is negative at higher temperatures:

Range of Temp.	Mean Temp. Coeff.	Range of Temp.	Mean Temp. Coeff.
10° to 20°	$+25\times10^{-6}$	45° to 50°	-1×10^{-6}
20° ,, 30°	$+14$,,	50° ,, 55°	-2 ,,
30° ,, 35°	$+4$,,	55° ,, 60°	-4 ,,
35° ,, 40°	$+3$,,	60° ,, 65°	-5 ,,
40° ,, 45°	$+1$,,		

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* *Phil. Mag.* Jan. 1889.

If manganin resistances are allowed to approach 100°, their constancy at varying temperatures is injuriously affected.

As regards the constancy of the coils it appears that the mercury standards and the platinum standards have varied little, while the platinum-iridium, the gold-silver, and some of the platinum-silver coils have altered to a relatively large extent. A few of the manganin standards have altered very little, but most of these standards have increased in resistance.

The method of comparison consists in placing the coil to be tested in one arm of a Wheatstone bridge, of which the other arms are manganin resistances. The balance is obtained by shunting, for which the bridge is adapted.

41. Measurement of high resistances. The measurement of a very high resistance such as that of a piece of insulating material cannot be effected by means of Wheatstone's bridge, and recourse must be had in most cases to electrostatic methods, in which the required resistance is deduced from the rate of loss of charge of a condenser, the plates of which are connected by the substance in question. If, however, the resistance of the material be not too great, and a large well-insulated battery of from 100 to 200 cells, and a very sensitive high resistance galvanometer are available, the following method is the most convenient. First join the galvanometer, also well insulated, and the resistance to be measured (prepared as described in 43 below, to prevent leakage) in series with as many cells as gives a readable deflection, which call D. Now join the battery in series with the galvanometer alone, and reduce the sensibility of the instrument to a suitable degree by joining its terminals by a wire of known resistance, and, to keep the total resistance in circuit great in comparison with the resistance of the battery, insert resistance in the circuit. Let E and B denote respectively the electromotive force and resistance of the whole battery, G the resistance of the galvanometer, S the resistance joining its terminals in the second case, R the resistance introduced into the circuit of the galvanometer in that case, and X the resistance to be found; we have for the difference of potential between the terminals of the galvanometer in the first case the value,

$$\frac{EG}{G+B+X}=mD, \qquad (42)$$

where m is the factor by which the indications of the galvanometer must be multiplied to reduce them to volts. In the second case the resistance between the galvanometer terminals is $SG/(S+G)$, and therefore the difference of potential between them is,

$$\frac{E\dfrac{SG}{S+G}}{B+R+\dfrac{SG}{S+G}}=\frac{ESG}{(B+R)(S+G)+SG}=mD_1. \qquad (43)$$

Hence combining equations (42) and (43) so as to eliminate E and m, and solving for X, we get

$$X = \frac{D_1}{D}\left(B + R + G + \frac{(B+R)G}{S}\right) - (B+G). \quad \text{............(44)}$$

If X be great in comparison with the remainder of the resistance in circuit the term $(B+G)$ may be neglected.

This method was used by Mr. T. Gray and the author for the determination of the specific resistances of different kinds of glass. The specimens of glass were in the form of thin, nearly spherical flasks about 7 cm in diameter, with long narrow and thick walled necks. The thin walls of the flask were brought into circuit by filling it up to the neck with mercury, and sinking it to the same level in a bath of mercury, then joining one terminal of the battery to the internal mercury by a wire passed down the long neck, and the other to the mercury in the bath without. This mercury bath was an iron vessel contained in a sand-bath which could be heated to any required temperature. A well-insulated galvanometer (constructed by aid of a grant from the Government Research Fund to a special design*) of high resistance and great sensitiveness was included in the current. A battery of over 100 Daniell's cells was used, and after a reading of the galvanometer in one direction had been taken and recorded, with the corresponding temperature of the glass, the coatings of the flask were connected together until the next reading was about to be taken. For this the current was reversed, and the deflection taken as before, and so on. The "electric absorption" was thus reversed between every pair of readings, and lasted in most cases about three minutes. The resistances were therefore those existing after three minutes' electrification. The result for the glass of highest insulation tested, which was lead glass of density 3·14, was a specific resistance at 100° C. of about 8400×10^{10} ohms. The resistance was halved for each 8·5° or 9° rise of temperature.

A modification of this method, for which a potential galvanometer or voltmeter is very suitable, may be used for the determination of the insulation resistance of the conductors in an electric-light installation.

The conductors are disconnected from the generator and both ends from one another. They are then joined at one end by the potential galvanometer in series with a battery of as many cells as gives a readable deflection. The number of divisions corresponding to this deflection is read off, and the number of divisions which the battery gives when applied to the galvanometer alone is then observed. Call the latter number V and the former V'; and let E divisions be the total electromotive force of the battery. Let the resistance of the battery, which may be determined by the method described below (p. 377), be B ohms, the resistance of the galvanometer G ohms, and the insulation resistance to be found R ohms ; we have plainly,

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* *Proc. R.S.* vol. xxxvi. (1884). See 32 below.

$$V=\frac{EG}{B+G}, \quad V'=\frac{EG}{B+G+R}.$$

Therefore
$$R=(B+G)\left(\frac{V}{V'}-1\right). \quad \ldots\ldots\ldots\ldots\ldots\ldots(45)$$

If B be small in comparison with G we may put

$$R=G\frac{V-V'}{V'}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(46)$$

A shunt-wound generating machine giving sufficient electromotive force may be used instead of the battery, and in this case R is found by equation (46).

The insulation resistance for unit of length is found from this result by *multiplying* by the length of either of the conductors.

This method is applicable to the measurement of the insulation-resistance of cables or telegraph lines, but for details the reader is referred to the manuals of testing in connection with these special subjects.

42. Leakage method for high resistances. In the case of insulating substances the method just described requires the use of so powerful a battery that it is quite inapplicable except when the specimen, the resistance of which is to be measured, can be made to have a large surface perpendicular to the direction of the current through it, and of very small dimensions in that direction. Such a case is that of the insulating covering of a submarine cable in which the current by which the insulation-resistance is measured flows across the covering between the copper conductor and the salt water in which the cable is immersed.

In general, therefore, in the determination of the insulating qualities of substances which are given in comparatively small specimens it is necessary to have recourse to the electrometer method mentioned in 41 above, of which we shall give here a short account.

The most convenient instrument for this purpose is a quadrant electrometer of good sensibility. For a full description of this instrument, and the mode of using it, see the chapter below on *Electrostatic Measurements*. The electrometer, having been carefully set up according to the most sensitive arrangement, and found to be otherwise in good working order, is tested for insulation. One pair of quadrants is connected to the case according to the instructions for the use of the instrument, and a charge producing a difference of potential exceeding the greatest to be used in the experiments is given to the insulated pair by means of a battery, one electrode of which is connected for an instant to the electrometer-case, the other at the same time to the electrode of the insulated quadrants, and the percentage fall of potential produced in thirty minutes or an hour by leakage in the instrument is observed. If this is inappreciable, the instrument is in perfect order. For practical purposes the insulation is sufficiently good when the same battery being applied to charge the electrometer alone as is applied to charge the cable, or condenser formed as described below, there is not a more rapid fall of potential without the cable or specimen than with it; for there can then be no error due to leakage.

43. Details of leakage observations. An air condenser, well insulated by glass stems varnished and kept dry by pumice moistened with strong sulphuric acid, is adjusted to have a considerable capacity, and its insulated plate is connected to the insulated quadrants of the electrometer, and the other to the electrometer-case, to which the other pair of quadrants is also connected. A charge producing as great a difference of potential as before is given to the condenser and electrometer thus arranged, and the fall of potential observed by means of the electrometer. If the loss in a considerable time be also inappreciable, the condenser insulates properly, and its resistance may be taken as infinite.*

The specimen of material to be tested is now placed so as to connect the plates of the condenser. The manner in which this is to be done of course depends on the form of the specimen. If it is a flat sheet, it may be covered on each side, with the exception of a wide margin all round, with tinfoil, and thus made to form itself a small condenser which is to be joined by thin wires in parallel with the large condenser. The edges and margins of the sides of the specimen should be carefully cleaned and dried, and covered with a thin coating of paraffin to prevent conduction along the surface between the two tinfoil coatings, when the condenser is charged. It is advisable, when possible, to coat the whole surface including the tinfoil with paraffin, and to make the contacts with the tinfoil plates by means of thin wires also coated with paraffin for some distance along their length from the tinfoil. The plate condenser thus formed should be supported in a horizontal position on a block at the middle of the lower surface. The upper coating is made the insulated plate.

If the specimen be cup-shaped, as, for example, if it be in the usual form of an insulator for telegraph or other wires, the hollow may be partially filled with mercury, and the cup immersed in an outer vessel containing mercury, so that the mercury stands at nearly the same level outside and inside. The lip of the cup down to the mercury on both sides is to be cleaned and coated with paraffin, as before, to prevent leakage across the surface. A thin wire connected with the insulated plate of the condenser is made to dip into the mercury in the cup, and a similar wire connected with the other plate of the condenser dips into the mercury in the outer vessel. Strong sulphuric acid may, on account of its drying properties, be used with advantage instead of mercury as here described, when the substance is not porous and is not attacked by the acid.

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* A condenser of any other kind, such as those used in cable testing, the insulating material between the plates of which is generally paper soaked in paraffin, may be used instead of an air condenser, but as the resistance of the latter may, if the glass stems be well varnished and kept dry, be taken as infinite, and there is besides no disturbance from the phenomenon called *electric absorption*, it is preferable to use an air condenser if possible.

In every case in which, as in these, the insulating substance and the conductors making contact with it form a condenser of unknown capacity, the condenser used in the experiment must be arranged to have a capacity so great that the unknown capacity thus added to it, together with the capacity of the electrometer, may be neglected in the calculations.

The condenser, if it has been disconnected, is again connected as before to the electrometer. One electrode of a battery of from six to ten small Daniell's cells in good order, is also connected with the electrometer case, and the other electrode is brought for a short time, thirty seconds say, or one minute, into contact with the insulated plate of the condenser at any convenient point, such for example as the electrode of the electrometer connected with the insulated pair of quadrants. The battery electrode is then removed, and the condenser and electrometer left to themselves.

The condenser has thus been charged to the potential of the battery, which will be indicated by the reading on the electrometer scale at the instant when the battery is removed. The deflection of the electrometer needle will now fall, more or less slowly according to the insulation resistance of the condenser with its plates connected by the material being tested. Readings of the position of the spot of light on the electrometer scale are taken at equal intervals of time, and recorded, and this is continued until the condenser has lost a considerable portion, say half, of its potential.

44. Calculation of resistance from leakage. The resistance of the insulating material is easily calculated from the results in the following manner. Let V be the difference of potential between the plates of the condenser at any instant, Q the charge of the condenser at that instant, which may be taken as proportional to the deflection on the electrometer scale, and C its capacity (I. 28). We have $Q=CV$, and therefore $dQ/dt=C\,dV/dt$. But $-dQ/dt$ is the rate of *loss* of charge, that is, the current flowing from one plate to the other, and this is plainly equal by Ohm's law to V/R. Hence $-dQ/dt=V/R$, and therefore

$$C\frac{dV}{dt}+\frac{V}{R}=0.$$

Integrating, we get $$\log V+\frac{t}{CR}=A, \quad \ldots\ldots\ldots\ldots\ldots\ldots(47)$$

where A is a constant. If V be the difference of potential t seconds after it was V_0, we get by putting $t=0$ in (47), $A=\log V$. Hence (47) becomes

$$\frac{t}{CR}=\log\frac{V_0}{V}$$

and

$$R=\frac{t}{C}\frac{1}{\log\frac{V_0}{V}}.^{*} \qquad \text{(48)}$$

If $V=\frac{1}{2}V_0$, we have $R=t/C\log 2$.

If the condenser have a resistance so low as to add materially to the rate of discharge, an additional experiment must be made in the same way to determine the resistance of the condenser alone, with its plates connected only by its own dielectric. Let R_c denote the resistance of the condenser, determined by equation (48) from the results of the latter experiment, and R_i the resistance of the specimen; we have $1/R=1/R_i+1/R_c$, and therefore

$$R_i=\frac{RR_c}{R_c-R_i}. \qquad \text{(49)}$$

If C has been obtained in c.g.s. electrostatic units of capacity, it may be reduced to electromagnetic units by dividing by the square of the number of electrostatic units of capacity equivalent to the electromagnetic unit, that is (see I. 56 and XVI.) by 9×10^{20} nearly.

When an air condenser is used, its capacity can generally be obtained approximately by calculation from the dimensions and area of the plates. For example, if two parallel plates of metal, placed at a distance d apart, very small in comparison with any dimension of either surface, have a difference of potential V, and there be no other conductor or electrified body near, it can easily be shown that the capacity on a portion of area A near the centre of either plate is $A/4\pi d$. Hence, in the example below, we have for the capacity of the disk of area A the value $A/4\pi d$, if we neglect the non-uniformity of the electrical distribution near the edge.

If C has been taken in absolute c.g.s. electromagnetic units of capacity (see I. 33, 41 and Chap. XVII.), we obtain R from (48) in cm per second, which may be reduced to ohms by dividing by 10^9.

When a condenser such as one of those used in submarine telegraph work is used, the capacity C of which is known in microfarads [I. 54], then since a microfarad is $1/10^{15}$ c.g.s. electromagnetic units of capacity, we have for R in ohms the formula

$$R=10^6\frac{t}{C}\frac{1}{\log\frac{V_0}{V}}. \qquad \text{(50)}$$

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[*] It is to be remembered that the logarithms to be used here are Naperian logarithms. The Naperian logarithm of any number is equal to the ordinary or Briggs' logarithm multiplied by 2·302585 ....

The following are results actually obtained in tests of a specimen of insulating material made in the form of an ordinary telegraph insulator. An air condenser consisting of two horizontal brass disks, the distance of which apart could be regulated by means of a micrometer screw, was joined with the insulator made into a small condenser with mercury inside and outside, as described above. The lower disk was of considerably greater diameter than the upper, which had a diameter of 12·54 cm, and the distance between them was adjusted to be 1 cm. The upper disk was connected to the insulated pair of quadrants, and the lower to the electrometer case. Calling A the area of the upper plate, and d the distance between them, we have, neglecting the effect of the edges of the upper disk, for the capacity of this condenser the value $A/4\pi d$ in c.g.s. electrostatic units. Hence in the actual case $C = 9{\cdot}828$. The interior surface of the insulator covered by the mercury was so small, and the thickness of the material so great, that, even allowing the material to have a high specific inductive capacity, the capacity of the condenser which it formed was small in comparison with that of the air condenser. The experiment gave, when the condenser and insulator were joined as described, $V_0 = 251$, $V = 100$, $t = 5640$ seconds. Hence

$$R = \frac{5640}{9{\cdot}828 \times 2{\cdot}303 \times \log_{10} \frac{251}{100}} = 623,$$

in seconds per centimetre (c.g.s. electrostatic units of resistance). As the condenser was not insulating perfectly, a separate test was made for it alone, with the results $V_0 = 239$, $V_1 = 182$, $t = 6120$. Hence

$$R_e = \frac{6120}{9{\cdot}828 \times 2{\cdot}303 \times \log_{10} \frac{239}{182}} = 2286,$$

and therefore by (49) $$R_i = \frac{623 \times 2286}{2286 - 623} = 857,$$

in seconds per centimetre.

Multiplying this result by 9×10^{20} (the approximate value of v^2, see Chap. XVI.), to reduce to electromagnetic units, we get for the resistance of the insulator 7712×10^{20} cm per second, or 771×10^{12} ohms.

45. Measurement of battery resistance. We shall now consider very briefly the measurement of the resistance of a battery. This term is not perfectly definite in meaning, as there is reason to believe that the resistance as well as the electromotive force of a battery depends to some extent on the current flowing through the battery, and further the resistance and the electromotive force, and possibly also the polarization of the battery, are affected by differences of temperature. But the information which in practice we generally require from the test, is really what available difference of potential can be obtained with a certain working resistance in the external circuit. This could be obtained at once by connecting the terminals of the battery by this resistance, and measuring the difference of potential by means of a quadrant electrometer or a potential galvanometer. If we call this difference of potential V, and the electromotive force of the battery when on open circuit E, then putting R for the external resistance we may write

$$\frac{E}{R+r}=\frac{V}{R}=\gamma, \quad \text{.............................}(51)$$

where r is a quantity the definition of which is simply that it satisfies this equation. If the battery had the same electromotive force E, when generating the current γ, as when on open circuit, then r would be the effective resistance of the battery; but, although this is not the case, we may without being led into error still speak of it as the resistance of the battery for the current γ. In fact, the value of r, thus found for a particular value of R, does actually enable us to calculate from the known electromotive force for open circuit, with a moderate degree of approximation in the case of a constant battery, and also, but less surely, in the case of a secondary battery, what available difference of potential will exist between the terminals of the battery when connected by other and somewhat widely differing values of R, and therefore also to find what arrangement of a battery it will be best to adopt in any given circumstances. So far as this practical result is concerned, the numerous methods which have been devised for the determination of the resistance of a battery before any sensible polarization (which requires time to develop) has been set up are, though interesting in themselves, of no practical value, and we shall not here describe any of them.

From equation (51) we have

$$r=\frac{E-V}{V}R. \quad \text{.................................}(52)$$

To determine r therefore we have simply to measure with a potential galvanometer the difference of potential which exists between the terminals of the battery when on open circuit, or connected only by the galvanometer coil, the resistance of which we suppose to be very great in comparison with r, and again to measure in the same way the difference of potential when the terminals are connected by a resistance R, also small in comparison with that of the galvanometer.

If the galvanometer scale be graduated so that readings are proportional to the tangents of the corresponding angles, we have, if D be the deflection in the first case, and D' the deflection in the second case, the equation

$$r=\frac{D-D'}{D'}R. \quad \text{.............................}(53)$$

Instead of a potential galvanometer a quadrant electrometer may be employed if the battery is not too large, and the same formula applies when D and D' are taken proportional to the sines of the angles through which the mirror is turned.

A resistance coil, which may be of german silver wire, constructed as described in 4 above, should be used for the resistance connecting the terminals, and if the current passing through it be considerable its resistance should be determined when the current is flowing. This may be done by including in its circuit a current-galvanometer, and determining the current γ through the wire in amperes, when V is read off in volts on the potential instrument. The resistance of the wire with that of the current-galvanometer is in ohms V/γ, and this is to be used as the value of R in equation (53).

If a galvanometer of high resistance be not available, an approximate test can be made by means of a sensitive galvanometer of low resistance. The battery and galvanometer are joined in series with a resistance R, and again with a resistance R'. Let D and D' be the deflections, which must have a difference comparable with either. Then, supposing E and r to be the same in both cases, and putting G for the resistance of the galvanometer, we have

$$D = m\frac{E}{R+G+r}, \quad D' = m\frac{E}{R'+G+r'},$$

where m is a constant.

Therefore we find $$r = \frac{D'R' - DR}{D - D'} - G. \qquad \text{......................}(54)$$

46. Methods of Mance and Thomson for battery resistance. Mance showed how to determine the resistance of a battery by means of Wheatstone's bridge. The battery is placed in the position BD of Fig. 88 above, and a key is connected between A and B. The resistances r_1, r_2, r_3 are adjusted until the depression of the key produces no alteration in the galvanometer deflection. The galvanometer and the key, with their respective connecting wires, are then conjugate conductors; and it is easy to show that the resistance of the battery is then $r_2 r_3/r_1$. The needle of the galvanometer is kept nearly at zero by means of a small magnet during the adjustment of the resistances, so that it is as sensitive as possible to any alteration of current produced by depressing the key.

This method is so troublesome as to be practically useless, chiefly on account of the variation of the effective electromotive force of the cell produced by alteration of the current through the cell which takes place when the key is depressed. Prof. O. J. Lodge * has discussed the method, and shown how it may be improved by inserting a condenser in series with the galvanometer between C and D. Still it is inconvenient and gives no information which may not be obtained more easily in another way, and we shall therefore not enter into further detail regarding it.

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* *Phil. Mag.* 1877, p. 515.

Lord Kelvin[*] showed how the same mode of operating may be made to give the resistance of a galvanometer when there is no other galvanometer available. The arrangement of Fig. 88 is varied by placing the galvanometer in the position *BD*, and a key in the position there shown as occupied by the galvanometer. The deflection of the galvanometer produced by depressing the battery key is nearly annulled by means of a magnet, and the resistances r_1, r_2, r_3 are adjusted until no alteration of the galvanometer deflection takes place when the key in *CD* is depressed. When this is the case *C* and *D* are at the same potential, since the addition of the conductor *CD* does not disturb the current distribution in the network; and we have for the resistance r_4 of the galvanometer

$$r_4 = \frac{r_2}{r_1} r_3.$$

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[*] *Proc. R.S.* vol. xix. (Jan. 1871).

CHAPTER XII.

GALVANOMETRY AND MEASUREMENT OF CURRENTS.

Section I. Absolute Galvanometry.

1. Standard galvanometers and electrodynamometers. Since currents flowing in a given circuit are taken (V. 3 above) as proportional to the intensities of the magnetic fields they produce, and unit current is defined accordingly, the fundamental determinations of currents in absolute units must be made by some form of standard galvanometer, or standard electrodynamometer, or by the particular form of the latter instrument which is called a current weigher. Various current weighers of very elaborate and accurate construction have been made, and are employed in standardizing laboratories for the direct absolute measurement of currents. A standard galvanometer is an instrument which exerts on a magnetic needle in any given position a couple which can be calculated with sufficient accuracy from the dimensions and arrangement of the coil-system, and the (approximately) known distribution of magnetism in the needle. For absolute measurements of currents by such an instrument it is necessary to know also the intensity, at the needle, of the magnetic field which exists independently of the current in the coil; since that with the field produced by the current gives the resultant-field in which the needle rests in equilibrium if subject only to magnetic action, or the magnetic couple system on the needle if besides magnetic forces, others (such as elastic forces) are effective in producing equilibrium.

A standard electrodynamometer is simply a standard galvanometer with the needle replaced by a movable coil, or coil-system, of such form and arrangement, and so suspended as to enable the system of couples acting upon it to be calculated for any position, or for a certain zero position, to which the movable coil-system is brought back by a proper displacement or distortion of the suspension or otherwise. In this case equilibrium is generally produced by means of a force due to elasticity or to gravity, which can be accurately determined.

The calculation of the magnetic forces has been given in Chapter VII. for the more important arrangements of coils. We have only to consider the general construction and action of such instruments, the modes of suspension adopted for the needle or coil, the calculation or determination of the other than magnetic forces acting on the suspended system, and the practical operations of setting up and using the instruments.

2. Tangent and sine galvanometers. Construction. Dealing first with absolute galvanometers, we notice that according to the mode in which they are used they are classed as *tangent galvanometers* or *sine galvanometers*. In the former the arrangement is such that the current flowing through the coils is (exactly or approximately) proportional to the *tangent* of the deflection of the needle from the undisturbed or initial position, in the latter the current is proportional to the sine of the deflection. We shall consider first the construction of galvanometers.

As stated above, the standard galvanometer should be of such a form that the values of its indications can be easily calculated from the dimensions and number of turns of wire in the coil. Such a galvanometer can be made by any experimenter who can turn, or can get turned, with accuracy a wooden or brass ring with a rectangular groove round its outer edge to receive the wire.

If a wooden ring is made the wood should be hard and perfectly seasoned. It is desirable that the block from which the ring is turned should be built up of pieces of well seasoned wood put together with glue, and under pressure, and arranged so that the grain of the wood offers the maximum resistance to warping (see 5, below).

If a brass ring is made the greatest care should be taken to select brass free from iron, or other magnetizable material. Some account of tests of brass and other materials for freedom from magnetic matter will be given below in connection with the description of various absolute instruments which have been made. In early absolute instruments this precaution was probably insufficiently attended to.

It is to be preferred that the experimenter should at least perform the winding of the coil and the adjustments of the needle, etc., himself, to be sure that errors in counting the number of turns, or in placing the needle at the centre of the coil are not made. If there are to be several layers of wire, the breadth and depth of this groove ought to be small in comparison with its radius, and each should be not greater than $\frac{1}{10}$ of the mean radius of the coil, which should be at least 15 cm.

The gauge of the wire with which the coil is to be wound must depend of course on the purposes to which the instrument is to be applied, but it should be good well-insulated copper wire of high conductivity, and not so thin as to run any risk of being injured by the strongest currents likely to be sent through the instrument. For the exact graduation of current as well as of potential instruments, it is convenient to make it have two coils—one of comparatively high, the other of low resistance. The latter may in some cases in which great accuracy is not required be a simple hoop of say 15 cm radius, made of copper strip 1 cm broad and 1 mm thick. As however the distribution of the current in a massive conductor is uncertain in consequence of want of homogeneity in the material, and it is besides difficult to allow exactly for any irregularity that may exist where the ends are led out, and further, as it is difficult to make such a hoop of perfectly accurate shape, and it is impossible to determine by calculation the exact constant of such a conductor, it is better to use instead several turns of thick wire. Each spire of the coil may then be regarded, as explained above, as a circular conductor coinciding with its circular axis.

To form electrodes to which wires can be attached, the ends of the copper strip or thick wire are brought out side by side in the plane of the ring, with sheet vulcanite or paraffined paper between them. Insulated wires are soldered to the ends of the circle thus arranged, and are twisted together for a sufficient distance to prevent any direct effect on the needle from being produced by a current flowing in them. The end of the wire should be brought from the end of the last winding to the beginning of the first by a step—in an axial plane of the coil for instance—which can have little or no effect on the needle, and then the two wires should be close together for some distance from the coil. If one terminal is a piece of wire well insulated with rubber, and the other is a piece of copper tubing enclosing the wire, the terminals will be non-inductive. But provided the needle is never deflected through a large angle, it will be sufficient to twist the two leading in and leading out wires together, and lay them along a line parallel to the axis of the coil.

In constructing the fine-wire coil the operator should first subject the wire to a moderate stretching force, and then carefully measure its electrical resistance and its length. He should then wind it on a moderately large bobbin and again measure its resistance. If the second measurement differs materially from the first, the wire is faulty and should be carefully examined. If no evident fault can be found, on the removal of which the discrepance disappears, the wire must be laid aside and another substituted. When the two measurements are found to agree the wire may then be wound on the coil. For this purpose the ring may either be turned slowly round in a lathe or on a spindle, so as to draw off the wire from the bobbin also mounted so as to be free to turn round. The wire must be laid on evenly in layers in the groove (which may be done with the utmost uniformity with a self-feeding lathe) and the winding ended with the completion of a layer. Great care must be taken to count accurately the number of turns laid on. Error in counting may be avoided by following the plan used by Maxwell of winding a single layer of thin cord on a long wooden cylinder rigidly attached to the bobbin and therefore turning with it. A pin driven into the cylinder serves to indicate the end of one layer and the beginning of the next. After winding the resistance should be again measured, and if it agrees nearly with the former measurements the coil may be relied on.

The ring carrying the coil thus made should then be fixed to a convenient stand in such a manner that if necessary it can be easily removed. The stand ought to be fitted with levelling screws, so that the plane of the coil may be made accurately vertical. A shallow horizontal box with a glass cover and mirror bottom may be carried by the stand near the level of its centre, and within this the needle and attached mirror or index suspended. Or, what is more convenient in many cases, a platform should be arranged below the level of the centre a sufficient distance to allow a magnetometer (such as one of those described in Chapter II. above) to be placed with the centre of its needle at the level of the centre of the coil.

3. Needle and suspension. Scale and pointer. The needle should be a single small magnet about a centimetre long, hung by a single fibre of unspun washed silk (half a cocoon thread), at least 10 cm long, or, better, by a fine quartz thread from the top of a tube fixed to the cover of the shallow box, or from the suspension head of the magnetometer if that is used, so that the centre of the needle when the coil is vertical is exactly the centre of the coil. To allow of the exact adjustment of the height of the needle, the fibre should be attached to the lower end of a small square screw spindle, raised or lowered, without being turned round, by a nut working round it above the cap of the tube.

If the instrument is to be used with scale and pointer (or, as is desirable in some cases, is to be furnished with scale and pointer as well as mirror), the pointer may be made by drawing out a bit of thin glass tube at the blowpipe into a thread, so thick as to remain nearly straight under its own weight when suspended by its centre. In order that the zero position of the pointer may not be under the coil, the pointer ought to be fixed horizontally with its length at right angles to the needle, so as to project to an equal distance on both sides of it. To test that this adjustment is properly made, draw a couple of lines accurately at right angles to one another on a sheet of paper. Then suspend a long thin straight magnet over the paper, and bring one of the lines into accurate parallelism with it. Remove then the magnet and put in its place the little needle and attached index. If the index is parallel to the other line the adjustment has been correctly made. The needle may then be suspended in position, and the box within which it hangs closed to prevent disturbance from currents of air.

A circular scale graduated to degrees, with its centre just below the centre of the coil, and its plane horizontal, is placed with its zero point on a line drawn on the mirror-bottom of the box at right angles to the plane of the coil, so that when the needle and coil are in the magnetic meridian the index may point to zero. The accuracy of the adjustment of the zero point is to be tested, as explained below, by finding whether the same current reversed produces equal deflections on the two sides of zero.

To test whether the centre of this divided circle is accurately under the centre of the needle, supposed at the centre of the coil, draw from the point immediately under the centre of the needle two radial lines on the mirror-bottom, one on each side of the zero point and 45° from it, thus including between them an angle of 90°, and turn the needle round without giving it any motion of translation. If the index lies along these two radial lines when its point is at the corresponding division on the circle the adjustment is correct. Of course a fairly accurate first adjustment is previously made by placing the circle so that the two points each at distance 45° from the zero lie on these straight lines.

Error from inaccurate centering can be almost completely eliminated by making the pointer extend across the circle and reading both ends of it.

When taking readings the observer places his eye so as to see the index just cover the image in the mirror-bottom of the box, and reads off the number of divisions and fractions of a division, indicated on the scale by the position of the index. Error from parallax is thus avoided.

A mirror rigidly attached to the needle may be used as in the magnetometer, instead of the needle and index, and observed by means of a telescope with attached scale, or, in the manner of an ordinary testing galvanometer, by means of a beam of light thrown by a lamp on the mirror and reflected to a scale. A long fibre magnetometer carried on a platform properly fixed within the bobbin may be used for the needle and attached mirror. A hole, slot, and plane arrangement on the platform for the adjusted position will enable the magnetometer to be taken away and replaced at pleasure. The adjustments of scale, etc., are the same as those described in Chapter II. above.

When a mirror is employed the coil is parallel to the undisturbed position of the needle (the magnetic meridian, when as usual the earth's field only is employed to give the return couple on the needle) when equal deflections on the two sides of zero are produced by reversing any current. The scales used should, if of paper, always be carefully glued to a wooden piece thinly painted over with melted paraffin instead of being, as they frequently are, fixed with drawing-pins, and the scale should then be carefully tested, with a metal or glass scale, for possible stretching in the process of attachment. Preferably however they should be scales ruled on glass by any one of the simple methods now available for copying an accurately engraved standard.

It is to be noticed that a mirror and straight scale placed at right angles to the undeflected position of the ray, and used in the ordinary way, give readings proportional to the tangents of double the angles of deflection.

4. Single-layer tangent galvanometer. The author, about 1884, constructed a standard galvanometer which possessed several advantages over the ordinary form. He had long been of opinion that single-layer coils were much preferable to multiple-layer coils for absolute work, and had advocated their use. This view has been entirely confirmed by the results of the employment of single-layer coils in the various Lorenz apparatus, inductance standards, and current weighers which have been more recently made. The galvanometer referred to consisted of a cylindrical bobbin, about 50 cm in diameter and 25 cm in length, wound with a single layer of fine wire. The needle (1 cm long) was suspended at the centre of the bobbin, and the magnetic field, produced by a current flowing in the wire, was in this arrangement practically invariable over a distance in any direction at the centre considerably exceeding the length of the needle. Very accurate placing of the needle was not necessary, as a displacement of so much as half its length from the central position (an error of adjustment which is practically impossible with the slightest care) produced a quite imperceptible effect on the deflection with any given current.

The distribution of the wire, since there was only one layer, was known with perfect certainty, and hence the constant of the instrument could be calculated with great exactness. At each end of the bobbin was wound one of two equal coils of small transverse dimensions in comparison with their radii. These were of thick copper wire arranged so as to form a Helmholtz double-coil galvanometer of the kind described above (VII. 8), available for strong currents.

When the instrument was being designed it was thought desirable to have the bobbin made of some material which could not possibly contain magnetic substances in sufficient quantity to affect the accuracy of measurements of currents flowing in the wire. The fear then felt by the author that the bobbins of brass ordinarily employed for standard galvanometers might very probably contain iron, in sufficient quantity to cause disturbance through its induced magnetization, was afterwards found by Prof. T. Gray to be entirely justified. The measurements of currents made by a new standard galvanometer were found by him to be so much disturbed by the effect of magnetic substances, contained in the walls of a brass box surrounding the needle, as to be useless.

5. Manner of building up a wooden bobbin. It was resolved therefore to construct a bobbin of wood in such a manner as to avoid risk of serious alteration of figure by warping, or of dimensions through variation in the amount of moisture contained in the wood. A large number of pieces of mahogany were cut from a dry well-seasoned board about ½ inch thick. Each piece was about 4 cm broad, 20 cm in length, and was cut so as to form a segment of a ring the outside diameter of which was about 50 cm and the inner diameter about 8 cm less. Four of these cut so that the grain of the wood ran in different directions in adjoining pieces and placed end to end gave a complete circular ring, or rather cylinder, ½ inch in length. Above that was placed a similar ring with the grain of the wood in the pieces crossing that in the pieces below, and the pieces themselves overlapping the end joints in the preceding ring. Above that was placed another ring, and so on until the whole bobbin, rather more than 25 cm in length, had been built up. The cylinder thus roughly formed was then turned carefully down to cylindrical figure of the size desired, and as nearly truly circular as possible, and the pores all over the surface, inside and outside, filled with spirit varnish to prevent the absorption of moisture.

[A bobbin thus built up of pieces of wood will probably not take or keep so true a figure as one made of metal, but there can be no doubt of its great superiority over the ordinary bobbin of wood, made out of one piece. For all except purposes for which the highest accuracy is required, it may be relied on to give correct results.]

Two edges of wood, projecting slightly beyond the outside cylindrical surface, were fixed at the ends to keep the wire in its place. The coil was then carefully wound, the turns counted, and the wire covered with "American cloth" to preserve it from injury. The two ends of the thin wire coil were brought out together at one end of the coil for connection to two electrodes closely twisted together and several yards in length, by which the instrument could be joined to any circuit in which it might be required. That end of the wire which had to be carried from the further extremity of the coil was (supposing the coil set up in position) brought along horizontally in a vertical plane through the axis of the coil until it met the other extremity at the termination of the last spire of the coil. The current in this part of the wire of course just compensates by its effect on the needle that of the component of current in each element of the spires in the direction of the axis.

6. Tangent galvanometer: sine galvanometer. Principal constant. The couple given in VII. 5 (13) is, if as a first and usually sufficient approximation the first term of the expression only is taken, $2\pi N\gamma \boldsymbol{M} \cos\theta/(a^2+b^2)^{\frac{1}{2}}$, where $\boldsymbol{M}$ is the magnetic moment of the needle, N the total number of turns in the coil, a the radius of the coil, b its half length, and θ the angle which the needle makes with the mean plane of the coil. The return couple given by the permanent magnetic field (horizontal intensity H) is $\boldsymbol{M}H\sin\theta$, if the mean plane of the coil and the axis of the needle are made to coincide when the deflection is zero, by the adjustment explained below. Thus equating these couples we have

$$\gamma = \frac{(a^2+b^2)^{\frac{1}{2}}}{2\pi N} H \tan\theta. \qquad \text{.........................(1)}$$

For the thick wire coils the deflecting couple Θ is given in VII. 5 (13), and for equilibrium we have $\Theta = \boldsymbol{M}H\sin\theta$. If we put $\Theta = \gamma \boldsymbol{M} G\cos\theta$, we get

$$\gamma = \frac{H}{G}\tan\theta, \qquad \text{...............................(2)}$$

where G is the quantity obtained by dividing the multiplier of $\cos\theta$ in the expression for the couple by $\boldsymbol{M}\gamma$. G is sometimes called the galvanometer constant. [The determination of G will be discussed later. It is easy to establish (1) by direct integration.]

7. Sine galvanometer. In a sine galvanometer the coils are made movable round a vertical axis through the centre of the needle, and when the needle is deflected the coils are turned until an equilibrium position is obtained in which the needle and mean plane of the coils are again parallel. Thus $\cos\theta$ in the expression for Θ given in last chapter must be put equal to unity. The deflection θ of the needle is equal to the angle through which the coils have been turned, and is usually measured by observing this angle by means of a finely divided scale provided with verniers and reading microscopes. For such an instrument we have instead of (2)

$$\gamma=\frac{H}{G}\sin\theta. \qquad \text{.................................(3)}$$

In the values of G for the different types of instrument given by the various expressions contained in Chapter VI., the inclination of the needle to the plane of the coil is of course to be put equal to zero.

An instrument capable of being used at pleasure either as a tangent or sine galvanometer was designed by the late Professor G. F. Fitzgerald, and is shown in Fig. 104. Its distinctive peculiarities consist in an arrangement of coils which permits the constant of the instrument to be determined with the coils in position, and a very ingenious arrangement for measuring the deflections of the needle and the coils from the adjusted position for no current. The only drawback is that the suspended system is somewhat heavy, so that a suspension thread the torsional effect of which is considerable must be employed.

The coils are visible through a plate-glass casing and can be measured *in situ.* The deflection of the needle is observed in the following manner on the cylindrical scale shown in the figure. A pair of small totally reflecting prisms, with their reflecting surfaces inclined at 45° to the horizontal, are carried by the magnet, and give images of diametrically opposite parts of this scale, and show on these images of one and the same line or mark. These are seen at the same time in the field of view of a microscope which receives the light from the mirrors. Thus the arrangement is equivalent to, but much more sensitive than, a pointer playing round a graduated circle and read at both ends to eliminate error from inaccuracy of centering.

The coils can be turned round to follow the magnet, and their position observed on the same cylindrical scale; so that a single scale serves for the use of the instrument both as a tangent galvanometer and as a sine galvanometer.

It has been noticed in 3 above that the ordinary method of using the mirror and scale gives with a straight scale properly adjusted the tangent of twice the angle of deflection. In Professor Fitzgerald's instrument, besides the arrangement just described for reading the deflection, a mirror is provided attached at 45° to the axis of suspension. A vertical ray of light falling upon this mirror is sent out horizontally through one of the plate-glass sides of the case to a horizontal scale. As the mirror turns round the plane of reflection turns with it, and

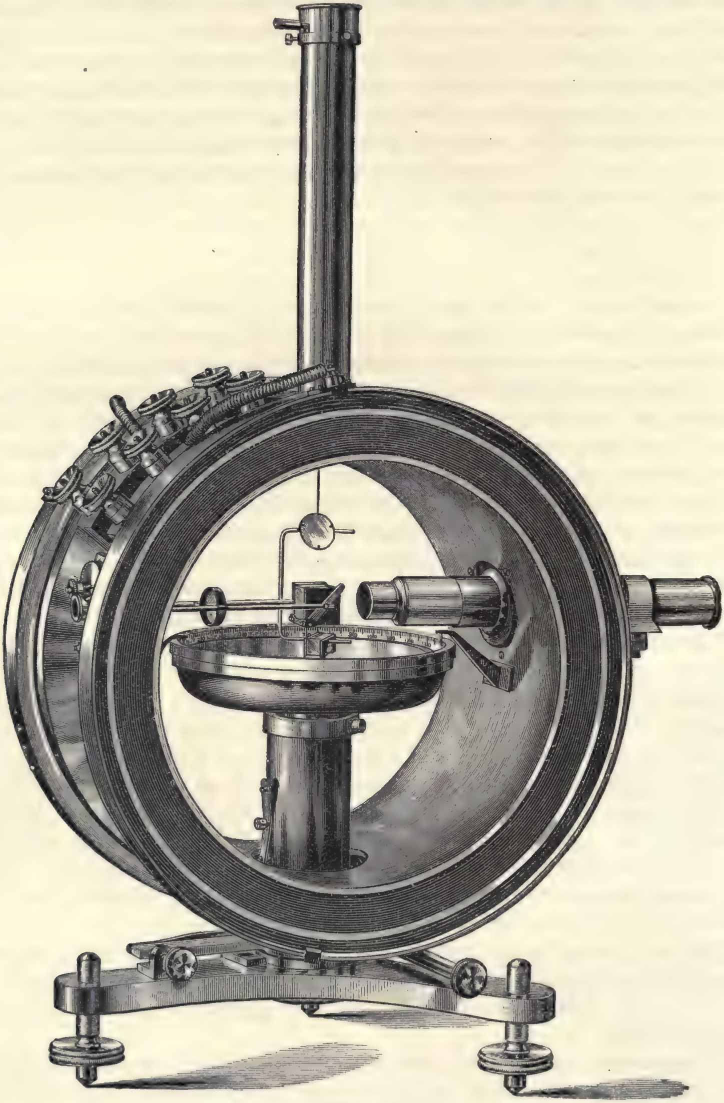

FIG. 104.

through the same angle, so that with a straight scale placed at right angles to the undisturbed position of the ray, the readings on the scale are proportional to the tangents of the actual deflections.

8. T. Gray's sine galvanometer. Fig. 105 shows a sine galvanometer designed by the late Prof. T. Gray. A single layer of wire is wound on a tube of 10 cm (or preferably greater) diameter, and at least ten diameters in length. If the coil be uniformly wound with n turns per unit of length, and l be its half-length and a its radius, the force f per unit of current at the centre is (see VII. 14 above) $4\pi nl/(a^2+l^2)^{\frac{1}{2}}$. This becomes $4\pi n$ if l be great in comparison with a; for example if l is ten times a, the value of f is only $\frac{1}{2}$ per cent. less than $4\pi n$, as is shown by the equation

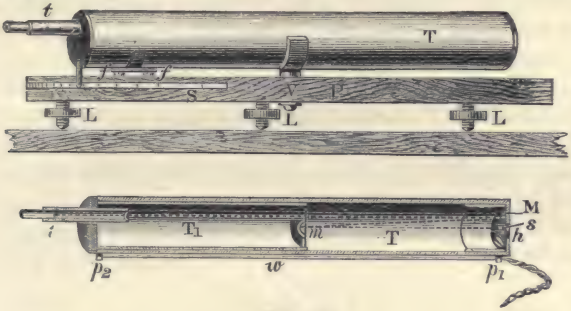

FIG. 105.

$$f=4\pi n\left(1-\frac{1}{2}\frac{a^2}{l^2}+\frac{3}{8}\frac{a^4}{l^4}-\dots\right)$$

$$=4\pi n\left(1-\frac{1}{200}+\frac{3}{80000}-\dots\right) \quad (l=10a).$$

Thus the very exact determination of the radius is not a matter of very great importance, and if the coil be very uniformly wound over the middle part, and very fairly regularly elsewhere, the value of f will be given with great accuracy by the first two terms of the series. The uniformity of the winding can be made almost quite perfect by laying on the wire under a moderate tension by means of a self-feeding lathe.

The coil is wound on the tube T (Fig. 105). The ends of the wire are attached to pins p_1, p_2, and a wire w running parallel to the axis of the coil connects p_1 to a third pin p_3 close to p_1. A pair of flexible electrodes well twisted together connects p_1p_3 to a pair of terminals on the platform P. The tube is mounted, as shown, on the circular platform P, which is furnished with levelling screws L, L, L, and can be turned round the vertical axis V, the supports f, f sliding on the platform and maintaining the tube in a horizontal position. The scale S on the edge of the platform enables the angle through which the coil is turned to be measured.

The needle is suspended at the centre of the tube, and may be either a light polished disk, or a plane or concave mirror with attached steel magnets. The arrangement preferred is as follows :—At one end of the tube is a short scale *s* facing towards the mirror (which is plane) and illuminated by light entering a small hole at that end of the tube, and thrown on the scale by a reflecting prism or inclined mirror. At the same end of the tube is a fixed mirror *M*, also turned towards the suspended mirror *m*. By means of the telescope *t* at the other end of the tube, fixed above the centre with its vertical cross-wire as nearly as may be in the medial vertical plane of the coil, the scale *s* is seen by light which has suffered two reflections, one at *m*, the other at *M*, and thus the angle through which the needle has been turned can be obtained.

For the scale *s* may be substituted a narrow slit, or, preferably a wide slit, or hole, crossed by a wire, in front of which within the tube is fixed a lens, and for the telescope a sheet of obscure glass. An image of the slit or wire is focused by the lens on the obscure glass, and the position of this can be read from without on a scale fixed to or engraved on the glass.

Or, the plane mirror *m* may be replaced by a concave spherical mirror as in an ordinary Thomson's galvanometer, and the obscure glass carried by a sliding tube which can be pushed out or in to give a sharp image of the slit or wire.

The method of using the instrument is as follows : It is placed in a well-lighted room, and the platform *P* is levelled by means of the screws *L*. The coil is then turned until the central division of the scale *s* coincides with the cross-wire of the telescope (or the zero of the scale on the obscured glass), and the reading on the scale *S* is taken. Then a steady current is passed through the coil, and the angle noted through which the tube has to be turned to bring the central division of *s* again to the cross-wire of the telescope. The current is then reversed, and the scale *s* moved if necessary until the angles on the two sides of zero are equal. If θ is this deflection on the scale *S* the current is given by the equation

$$\gamma = \frac{H \sin \theta}{4\pi n \left(1 - \frac{1}{2}\frac{a^2}{l^2}\right)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The angle θ can evidently be attained with great accuracy by very accurate division of the scale *S*, and reading it with a vernier and microscope.

9. Theory of a tangent galvanometer. We now discuss shortly some general propositions regarding the action of galvanometers, their adjustment and sensibility.

We shall suppose to begin with that the forces acting are wholly magnetic, and that the suspension is such as to prevent other than horizontal forces from affecting the needle. When no current is flowing the needle rests horizontal with its axis parallel to the permanent magnetic field, or to its horizontal component. The needle will take up a new position making an angle θ with the plane of the coil. The angle which the needle now makes with its initial position is $\theta - \alpha$, say. The couple, Θ, acting upon the needle is given by the equations set forth in VII. 5. If M be the magnetic moment of the needle, and H the horizontal component force of the permanent field, we have for the return couple $MH \sin(\theta - \alpha)$. Hence

$$\Theta = MH \sin(\theta - \alpha).$$

But we may write $\Theta = \gamma MG \cos\theta$, and therefore

$$\gamma = \frac{H}{G} \frac{\sin(\theta - \alpha)}{\cos\theta}. \quad \text{.............................}(5)$$

G, as shown by (13), VII. above, in general depends on θ. If the needle however be sufficiently short the terms depending on θ disappear. G is the *galvanometer constant* referred to in 6.

If α is zero (5) becomes

$$\gamma = \frac{H}{G} \tan\theta, \quad \text{..............................}(5')$$

and if G is independent of θ the current is proportional to the tangent of the deflection. Hence the name of the instrument.

It is to be observed that the magnetic moment of the needle is in general affected by the earth's magnetic field, and also by the current in the coil. When however there is equilibrium between the deflecting and the restoring couple the magnetic moment of the needle enters as a factor in both couples, and the condition of equilibrium is independent of that magnetic moment. It is quite otherwise however in the "ballistic" use of a galvanometer, and errors from this fact may arise. [See 40 below.]

10. Adjustment of instrument. The instrument is generally set up so that α is zero or very nearly so. This adjustment may be made as follows. Supposing the stand of the coils fitted with a level by means of which the coils can be placed in a vertical position, the instrument is thus levelled and placed by guess with the mean plane of the coils as nearly as may be parallel to the needle. The coil is then joined up with a voltaic cell and reversing key so that a current can be sent in either direction through it. A current is sent through the coils, and the deflection θ of the needle is observed by means of the mirror or pointer attached to the needle. The current is then reversed and the opposite deflection observed. If this is the same as before the coil is properly placed. If not let the numerical value of the first deflection without regard to sign be θ, and of the second θ', and let α be the (unknown) angle which the mean plane of the coils makes with the needle. Supposing G the same in both cases, which it will approximately be if θ is nearly the same as θ', we have, by (5),

$$\frac{\sin(\theta - \alpha)}{\cos\theta} = \frac{\sin(\theta' + \alpha)}{\cos\theta'}.$$

This gives
$$\tan \alpha = \frac{\sin(\theta - \theta')}{2\cos\theta\cos\theta'},$$
which shows that if $\theta > \theta'$ the coil is turned through an angle α, in the direction of the first deflection; if $\theta < \theta'$ the coil deviates from the position of the needle by an angle α in the direction of the second deflection.

The actual value of α can thus be calculated, and if the coils can be turned through any required angle the correction of position can at once be made. If, however, there is no provision for turning the coils through a definite angle, the correction must be made by guess from the direction of the greater deflection, then the new position tested, and if necessary corrected, and so on.

11. Coil at 45° to meridian. The galvanometer is sometimes set so that $\alpha = 45°$, and the current then made to flow so that the deflection is towards the coil. Then by (5) (changing the sign of the right-hand side to keep γ positive)

$$\gamma = \frac{H}{G}\frac{\sin\left(\frac{\pi}{4} - \theta\right)}{\cos\theta} = \frac{\sqrt{2}}{2}\frac{H}{G}(1 - \tan\theta). \qquad (6)$$

It is to be noticed that here θ is to be taken positive when it is on the same side of the coil as the initial position of the needle, and negative when it is on the opposite side. The deflection of the needle may thus be as great as 90° from the initial position. For this value of the deflection the current is $\sqrt{2}H/G$.

The adjustment to this position may be made by first placing the galvanometer as described above so that its mean plane is parallel to the undisturbed position of the needle, and then turning the instrument round through exactly 45°. This mode of using the instrument, though it gives a wider range, is attended with the inconvenience that the deflection if considerable can only be taken in one direction.

12. Sensibility of galvanometer. The sensibility of a galvanometer may be defined as the reciprocal of the current required to produce a definite small angular deflection of the needle, or, which comes to the same thing, it may be taken as measured by the angular deflection produced by a specified current, for example, a micro-ampere (one millionth of an ampere). Frequently if the galvanometer be a reflecting one it is regarded as inversely proportional to the current required to produce a deflection of one division of the scale, but this of course is a function of the arrangement of mirror and scale, and not merely of the coil.

The sensibility can be determined by sending through the coil, arranged as will generally be necessary with some considerable resistance in circuit, and shunted, if need be, by a resistance the ratio of which to the resistance of the coil is known, a current from a cell of known electromotive force, calculating the current, and observing the deflection.

The actual merit of the instrument cannot however be completely determined by such a process, as that depends on length of period of the needle, steadiness of zero, etc., which are not here taken account of.

13. Sensibility for different positions of needle. The sensibility of a galvanometer,* for different positions of the needle, is the ratio of the increase of deflection to the increase of the current, or $\delta\theta/\delta\gamma$. This is a maximum in the case of a tangent galvanometer for zero deflection.

When however the deflection is 45° a given percentage of increase or diminution of the current produces a maximum increase or diminution of deflection, that is to say $\delta\theta/(\delta\gamma/\gamma)$ is then a maximum; and hence the instrument is sometimes (erroneously) stated to be "most sensitive" when the deflection is 45°. The only importance in making the deflection 45° lies in the fact that with this deflection a given small error in reading the angle will have a minimum effect on the estimation of the current.

To prove these propositions we observe first that by (2)

$$\frac{d\theta}{d\gamma}=\frac{G}{H}\,\frac{1}{1+\tan^2\theta},$$

and this is obviously a maximum when $\theta=0$.

Again let the reading be in error $\delta\theta$ when the deflection is really θ. Then the current is estimated by (2), and if γ is the true current the estimated current is $\gamma\pm\delta\gamma$, or $\gamma\pm d\gamma/d\theta\,.\,\delta\theta$. The error in estimation of the current is $\delta\gamma/\gamma$ or $d\gamma/d\theta\,.\,\delta\theta/\gamma$. But

$$\frac{1}{\gamma}\,\frac{d\gamma}{d\theta}\,\delta\theta=\frac{1+\tan^2\theta}{\tan\theta}\,\delta\theta.$$

This is a minimum when $(1+\tan^2\theta)/\tan\theta$ is a minimum, that is when $\tan\theta=1$, or $\theta=45°$.

14. Torsion of suspension fibre. In every properly constructed absolute galvanometer the torsion of the suspension ought to be negligible, and if a quartz thread, or a sufficient length of properly prepared silk fibre be used, it will be negligible. The amount of torsion may however be estimated as follows. Let the needle supposed initially in the magnetic meridian be turned once or more times completely round, and let its deflection from the magnetic meridian in its new position of equilibrium be noted by means of index and divided scale, or mirror and scale or telescope provided for the purpose. If α be the angular deflection of the magnet from the magnetic meridian produced by turning the magnet once round, the angle through which the thread has been turned is $2\pi-\alpha$. The couple produced by this torsion has for moment $MH\sin\alpha$. Hence by Coulomb's law of the proportionality of the couple due to torsion to the twist given, the couple corresponding to deflection θ is $MH \sin \alpha . \theta/(2\pi - \alpha)$. Thus if a current γ produces the deflection θ the equation of equilibrium is

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* An elaborate comparison of sensibilities of galvanometers is given in a paper by Messrs. Ayrton, Mather, and Sumpner, *Phil. Mag.* July 1890.

$$\gamma G \cos \theta = H\left(\sin \theta + \frac{\theta}{2\pi - \alpha} \sin \alpha\right),$$

and therefore

$$\gamma = \left(1 + \frac{\theta}{2\pi - \alpha} \frac{\sin \alpha}{\sin \theta}\right) \frac{H}{G} \tan \theta \,. \quad \text{......................(7)}$$

15. Bilateral and Unilateral deflection of a galvanometer needle by alternating current. Before leaving for the present the subject of galvanometers we give here a short discussion of the action of an alternating current in the coil of such an instrument on the magnetic needle. As suggested a long time ago by Lord Rayleigh, the magnetic moment of the needle must be altered more or less by the current in the coil, to an extent depending on the deflection. As stated above, this effect in no way influences the results of galvanometer measurements in an important class of cases. There are other cases however in which it produces striking and more or less puzzling phenomena. The first case we take is that of an alternating current of fairly high frequency flowing in the coil, which would at first sight naturally be expected to produce no effect.

Let θ be the angle which the needle makes with the mean plane of the coil at time t, θ_0 the initial value of θ. If γ be the current in the coil, there will be a component of magnetic force in the direction of the needle which is proportional to $\gamma \cos \theta$. The change in the magnetic moment we suppose to be also proportional to $\gamma \cos \theta$, so that the couple on the needle due to this is $C\gamma^2 \sin \theta \cos \theta$, where C is a constant. The whole couple on the needle will therefore at time t be

$$MG\gamma \cos \theta + C\gamma^2 \sin \theta \cos \theta,$$

where M is the magnetic moment of the needle and G the galvanometer constant. Hence if mk^2 be the moment of inertia of the suspended system about the suspension thread, κ the friction coefficient, and H the horizontal intensity of the field in which the needle hangs, supposed to be parallel to the initial position of the needle, the equation of motion of the needle is

$$mk^2\ddot{\theta} + 2\kappa\dot{\theta} + MH \sin(\theta - \theta_0) = MG\gamma \cos \theta + C\gamma^2 \sin \theta \cos \theta \,. \quad \text{......(8)}$$

If the current in the coil be represented by $\gamma = A \sin(nt - \epsilon)$, the mean value of γ^2 is $\frac{1}{2}A^2$. Thus we may take $A/\sqrt{2}$ as the effective current, γ_m say. Hence the mean couple due to the periodic variation of the magnetic moment of the needle is $C\gamma_m^2 \sin \theta \cos \theta$. Let it be supposed that the value of θ_0 is zero or very small. Let the value of γ_m be gradually increased, by diminishing the resistance in the alternating circuit. At first there is little if any deviation of the needle from its zero position, but its free period is increased. Thus at first, since the frequency is high, there are only small values of θ, and $\cos\theta$ is nearly unity. The equation of the vibrational motion is

$$mk^2\ddot{\theta}+2\kappa\dot{\theta}+(MH-C\gamma_m^2)\theta=0,$$

approximately. Thus the period, $2\pi\{mk^2/(MH-C\gamma_m^2)\}^{\frac{1}{2}}$, increases as γ_m is increased. Clearly, however, as $C\gamma_m^2$ approaches more and more nearly to the value MH, the equilibrium tends to become unstable, and the spot of light finally moves off to one side or the other. Thus there is bilateral deflection.

On the other hand, if θ_0 be of sensible amount $\theta-\theta_0$ must have the same sign as θ, since we suppose that $\cos\theta_0$ is positive. For while γ_m is below the critical value, the spot of light will oscillate about the position determined by the two couples, which do not alternate in direction, that is the position given by the equation

$$\frac{\sin(\theta-\theta_0)}{\sin\theta}=C\gamma_m^2\cos\theta,$$

or

$$\frac{\theta-\theta_0}{\theta}=C\gamma_m^2, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

approximately. Hence, since $\theta-\theta_0$ and θ have the same sign, and θ_0 is always in the direction to make $C\gamma_m^2\sin\theta_0$ have the same sign as MH, the final deflection is in that direction. Thus we have unilateral deflection, in the direction in which the needle has been turned initially from zero.

The effect of the change produced by the current in the coil, in the ballistic use of a galvanometer, we must leave until later in this chapter we deal with that subject. Bilateral and unilateral deflection was discussed by the late Professor Chrystal in 1876.* The reader should consult this paper for further experimental particulars.

16. Electrodynamometers. We now consider absolute electrodynamometers. The first instrument of this kind seems to have been invented by W. Weber, and used by him in his researches on the mutual action of currents. Electrodynamometers have advantages over galvanometers (1) in having no magnet, and therefore avoiding altogether uncertainty as to distribution of magnetism; (2) in not involving for the reduction of their indications any knowledge of the intensity of the earth's field; but they are inferior in point of sensibility, and as the return couple is generally given by a bifilar or torsion suspension the accurate estimation of its value may be a matter of some difficulty.

The galvanometer designed by Professor Fitzgerald and described above could, as he has pointed out, easily be adapted for use as an electrodynamometer. All that is required is the substitution of a proper suspended coil, and a bifilar suspension for the needle. The same arrangement of mirrors and cylindrical scale would be available to give the deflections.

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* *Phil. Mag.* 2 (1876). See also a paper by Alexander Russell, *Phil. Mag.* 12 1906).

Other electrodynamometers have since been made, and the conditions for their accurate use are now better understood and realized. Current weighers have also come into use as standard instruments for accurate work.

17. B.A. Committee's electrodynamometer. We shall describe the general arrangement and mode of using an electrodynamometer first with reference to the instrument made by Mr. Latimer Clark for the British Association Committee on Electrical Standards, and illustrated in Figs. 106, 107. The design of this instrument was excellent in several respects.

The first of these figures shows the general arrangement of the instrument, the second the details of the suspension.

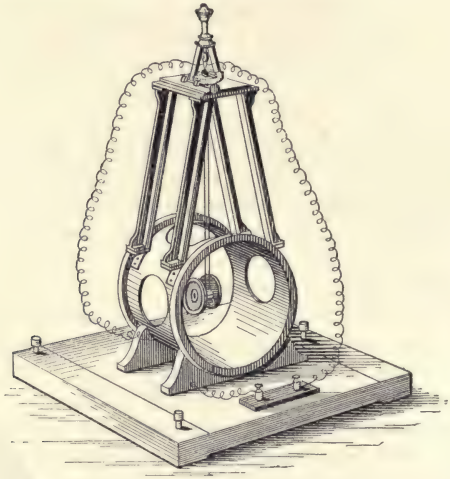

FIG. 106.

The bifilar consists of two wires the tension of which is maintained the same by their being attached to a piece of silk thread which passes over a pulley, as shown in Fig. 107. The distance between the threads is adjusted by two guide pulleys which can be set at any required distance apart. The current is led into the suspended coil by means of the suspension wires. Arrangements are also made whereby the current can be sent in either direction through each coil.

The instrument has both its stationary and movable coil systems constructed on Helmholtz's plan of two equal parallel coils at a distance apart equal to their radii. The suspended coil system is hung so that it is concentric with the fixed coils, and when there is zero deflection their planes are at right angles to one another.

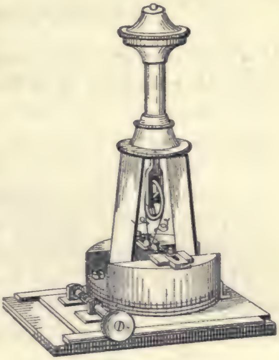

FIG. 107.

When the axis of the suspended coil makes an angle $\pi/2-\phi$ with the plane of the fixed coil, the couple Θ due to the currents and *tending to increase* the deflection, θ, has the expression given in (40) or (45), VII. 22, 23, with sign changed. Again the suspended coil is acted on by a couple due to the earth's magnetic force H, and tending to diminish $\pi/2-\phi$. Thus the equation (45) just referred to gives for the former couple $4Nn\gamma\gamma' G_1 g_1 \cos(\pi/2-\phi)$, since ${}_\phi Z'_1=1$; and for the other couple $2n\gamma' g_1 H \sin\theta'$, where N, n, γ, γ', are the numbers of turns and the currents in the fixed and movable coils respectively, and θ' is the angle which the axis of the movable coil makes with the magnetic meridian. Thus if L be the return couple due to the suspension, and the plane of the fixed coil make an angle a with the magnetic meridian, and an angle β with the axis of the movable coil in the undisturbed position, we have for equilibrium $\theta'=\theta+\beta+a$, and

$$4Nn\gamma\gamma' G_1 g_1 \cos(\theta+\beta) - 2n\gamma' g_1 H \sin(\theta+\beta+a) - L = 0.$$

The value of L, if θ be small, is proportional to $\sin\theta$, so that $L=F\sin\theta$.

$$F\tan\theta = 4Nn\gamma\gamma' G_1 g_1(\cos\beta - \tan\theta\sin\beta)$$
$$-2n\gamma' g_1 H\{\tan\theta\cos(a+\beta)+\sin(a+\beta)\},$$

and if a and β be both small and $2n\gamma' H$ be small compared with F,

$$\tan\theta = \frac{1}{F}\{4Nn\gamma\gamma' G_1 g_1 \cos\beta - 2n\gamma' g_1 H\sin(a+\beta)$$
$$-\frac{1}{F}(16N^2n^2\gamma^2\gamma'^2G_1^2g_1^2\sin\beta + 8NnH\gamma\gamma'^2G_1g_1^2)\}. \;\ldots\ldots \; (10)$$

18. Methods of using the instrument. Now a direction of the current in the coils being assumed as positive, the currents are sent through the two coils according to the adjoining scheme and produce the corresponding deflections θ_1, θ_2, θ_3, θ_4.

	γ	γ'
θ_1	+	+
θ_2	−	−
θ_3	+	−
θ_4	−	+

Thus we get by substitution in (10) and reduction

$$\gamma\gamma' = \frac{1}{4}\frac{F}{4NnG_1g_1\cos\beta}(\tan\theta_1 + \tan\theta_2 - \tan\theta_3 - \tan\theta_4)\,. \;\ldots\ldots(11)$$

If $\gamma = \gamma'$ this gives the value of γ^2.

By this method H is eliminated, and it is the best method to adopt when readings have to be obtained quickly, as when the current is varying. If however the current is constant enough the head of the bifilar suspension may be turned round until the suspended coil is brought back to its original position after deflection. When this is the case the angle θ through which the coil is deflected from its equilibrium position is clearly equal and opposite to the angle χ, through which the head of the bifilar has been turned round from the position of parallelism with the plane of the coil. We have thus $\theta = -\chi$. For equilibrium we have the equation

$$F\sin\chi = -4Nn\gamma\gamma' G_1g_1 + 2n\gamma' g_1 H\sin\alpha.$$

Taking four deflections according to the above scheme, we get four readings of the head of the bifilar $\beta_1, \beta_2, \beta_3, \beta_4 = -\theta_1, -\theta_2, -\theta_3, -\theta_4$, and so

$$F\sin\chi_1 = -F\sin\chi_3 = -4Nn\gamma\gamma' G_1g_1 + 2n\gamma' g_1 H\sin\alpha,$$

$$F\sin\chi_2 = -F\sin\chi_4 = -4Nn\gamma\gamma' G_1g_1 - 2n\gamma' g_1 H\sin\alpha.$$

Hence $$\gamma\gamma' = -\frac{F}{4NnG_1g_1}(\sin\chi_1 + \sin\chi_2 - \sin\chi_3 - \sin\chi_4), \;\ldots\ldots\ldots(12)$$

in which again H does not appear.

19. The Gray absolute electrodynamometer. An absolute electrodynamometer may be constructed, as described above (VI. 25), of two single-layer coils placed with their centres in coincidence, as shown in Fig. 108. If the ratio of length to radius be as proposed above in each case $\sqrt{3}/1$, the value of the couple due to the action of the currents will be as given in VI. 25 (76′),

$$\{8\pi^2 nn'\gamma\gamma' a^2 x\xi/\sqrt{a^2+x^2}\}\cos(\pi/2-\phi),$$

where n, n' are the numbers of turns per unit length in the two coils, x, ξ, a, α, their respective half-lengths and radii, γ, γ_1' the currents in them, and $\pi/2-\phi$ the angle which the axis of the movable coil makes with the mean plane of the fixed coils. This with $\pi/2-\phi$ replaced by $\theta+\beta$ is to

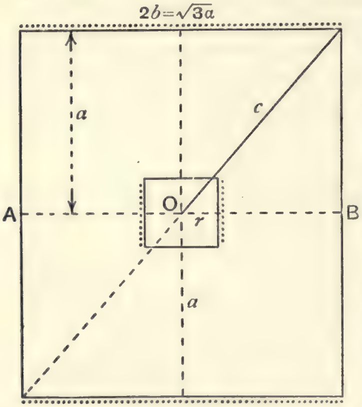

FIG. 108.

be used in the formulae given above, instead of

$$4Nn\gamma\gamma' G_1 g_1 \cos(\theta+\beta).$$

Thus the equations replacing (11), (12) for this case are

$$\gamma\gamma' = \frac{1}{4}\frac{F\sqrt{a^2+x^2}}{8\pi^2 nn' a^2 x\xi}(\tan\theta_1 + \tan\theta_2 - \tan\theta_3 \tan\theta_4), \ldots\ldots(13)$$

$$\gamma\gamma' = -\frac{1}{4}\frac{F\sqrt{a^2+x^2}}{8\pi^2 nn' a^2 x\xi}(\sin\beta_1 + \sin\beta_2 - \sin\beta_3 - \sin\beta_4). \quad \ldots(14)$$

An instrument fulfilling the conditions set forth in VII. 25 has been constructed with great care and skill at the Bureau of Standards, at Washington, and used by Messrs. Patterson and Guthe, and Carhart and Guthe, in determinations there made of the absolute electromotive forces of standard cells. The instrument is shown in Fig. 109.

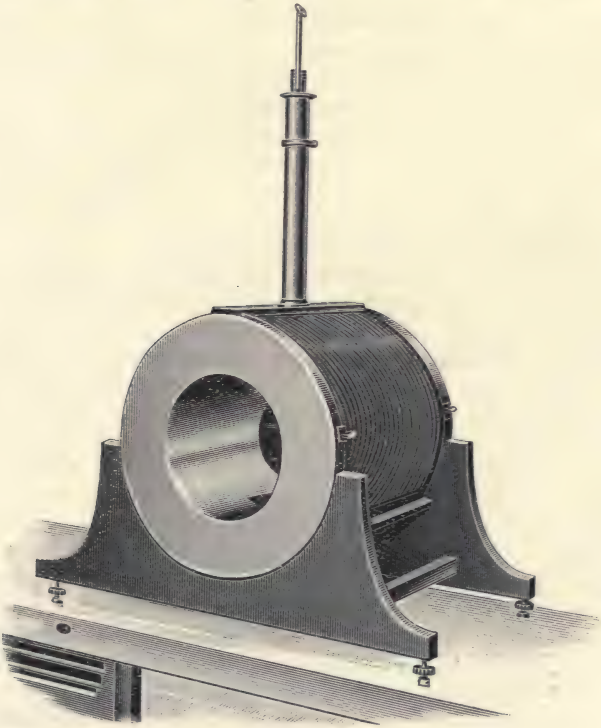

FIG. 109.

It consists of a cylindrical coil of thin wire wound in a single layer, on a cylinder of carefully selected plaster of Paris, cast and ground accurately to shape, with a smaller coil hung in its interior. Two smaller coils of different sizes were constructed, and were wound on porcelain cylinders made at the Royal Porcelain Factory at Berlin, and ground to exact size and shape at the Bureau of Standards. The effective diameter of the stationary coil was 49·9624 cm at 25° C. The two smaller coils made had at that temperature average diameters 9·93333 cm and 7·52157 cm.

The ratio of the radius of each coil to its length was made, for the reason stated above, $1/\sqrt{3}$. With this ratio of radius to length, and coincidence of centres, the couple Θ exerted on the movable coil by the outer one, when their axes are at right angles, takes the form

$$\Theta = 2\pi^2 \frac{r^2}{c} N_1 N_2 \gamma_1 \gamma_2, \quad \text{.........................(15)}$$

where r is the radius of the suspended coil, c the half diagonal of the fixed coil, or $(a^2+b^2)^{\frac{1}{2}}$ if a be the radius and $2b$ the length of the coil (that is $c=\sqrt{7}a/2$), N_1, N_2 are the numbers of turns in the fixed and suspended coils, respectively, and γ_1, γ_2 the currents in these coils. Of course these may or may not be the same current.

With this arrangement all the terms in the series of products $K_1k_1+K_2k_2+\ldots$, between the first and the seventh, disappear, and the seventh and succeeding terms are only small correction terms, which are not appreciable unless the suspended coil is made large, and can be easily and quickly calculated in any actual case.

The following detailed account of the instrument follows the description and discussion given by Mr. E. B. Rosa, in the *B.B.S.W.* 2, p. 71.

20. Value of the couple in the Gray dynamometer. The expression for the couple Θ may be conveniently considered as the product of two factors $H\gamma_1$ and $A\gamma_2$, where $H=2\pi N_1/c$, $A=\pi r^2 N_2$. The first $H\gamma_1$ is the magnetic force at the centre of the fixed coil due to the current γ_1 flowing in its windings, and A is the sum of the areas of the different turns of wire in the suspended coil. Hence the couple is the same as if the latter coil were hung in a perfectly uniform magnetic field of intensity $H\gamma_1$. As Mr. Rosa states, the field of the large coil is not uniform, as the centre is a point of maximum intensity on the axis and a point of minimum intensity for a line along the axis of the other coil. The couple is, however, for a small coil half the radius of the other, the same to 1 part in 27,000 as it would be if the field were perfectly uniform and of intensity equal to that at the centre.

The instrument was found extremely accurate in precision work. The quantity which had to be exactly measured, and which was therefore found most difficult of exact determination, was the couple; and we shall give here the results of Mr. Rosa's determination of the different sources of error from the point of view of theory.

21. Corrections. Calibration of windings, etc. In the first place we have to inquire, to what degree of accuracy the field H at the centre of the coil due to a current sheet of n turns per cm is equivalent to that produced at the same point by the single layer of windings of wire of N_1 $(=2bn)$ turns of wire carrying the same current. We have

$$H=2\pi\frac{N}{c}.$$

Suppose the coil of wire replaced by a single layer of flat thin strip of breadth $2b/N$ laid round edge to edge, without actually touching, so that there is the same current per cm of length at every part of the cylinder. As used by Guthe the instrument had about 20 turns per cm, so that the covered wire had a diameter of 0·05 cm. Thus for a single turn at the centre the axial magnetic force is

$$H_1=2\pi\times 20\left[\frac{x}{(a^2+x^2)^{\frac{1}{2}}}\right]_{-\cdot025}^{+\cdot025}=\frac{4\pi}{a}\,\frac{0\cdot5}{(1+0\cdot001^2)^{\frac{1}{2}}}=\frac{2\pi}{a}\times\cdot9999995.$$

A single turn of infinitely thin wire, at the centre of the coil, would give a magnetic force $2\pi/a$. A single turn of the strip would give a force less than this by one part in 2,000,000. If the strip were 1 cm wide the difference would be 400 times as much, but still only 1 part in 5000.

On the other hand, for the strip ·05 cm wide, wound *on edge*, the magnetic force per unit current is, if 2α be the breadth of the strip,

$$H_1=2\pi\int_{-\alpha}^{+\alpha}\frac{dy}{2\alpha}\,\frac{1}{a+y}=\frac{2\pi}{a}\left(1+\frac{1}{3}\,\frac{\alpha^2}{a^2}\right)=\frac{2\pi}{a}\times 1\cdot0000003.$$

Thus the force in this case is greater than $2\pi/a$ by 1 part in 3,000,000.

Thus the effects of thickness of the wire in giving breadth to the equivalent strip, and in giving increase of diameter, are opposite, and together make the field of one turn differ from that of a single turn of infinitesimal thickness, and radius a, to less than 1 part in 1,000,000.

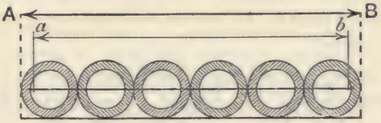

FIG. 110.

Similarly the difference between the magnetic field at the centre due to a turn of wire round one end of the coil and the field due to the corresponding part of the current sheet is inappreciable (about 1 part in 5,000,000), and moreover, if necessary, all these differences could be allowed for. It is to be observed that the current-sheet length of such a coil is the overall length of the winding, including the insulating covering [see Fig. 110]. This amounted to about 1 part in 5000 in excess of the length ab.

The coil was calibrated for the possible irregularities of winding, by measurements of the breadths on the cylinder covered by each 50 turns; and was divided into corresponding sections of which the magnetic fields were computed by the formula (derived from Fig. 111)

$$H = \frac{2\pi n}{x_1 - x_2}\left\{\frac{x_1}{(a^2 + x_1{}^2)^{\frac{1}{2}}} - \frac{x_2}{(a^2 + x_2{}^2)^{\frac{1}{2}}}\right\},$$

where $n = 50$, and (for the end section) $x_1 = 21{\cdot}6382$, $x_2 = 19{\cdot}1437$, $a = 24{\cdot}9812$, all in centimetres. The other sections were dealt with in a similar way. There were 17 sections of 50 turns each and one of 22 turns, 872 turns in all.

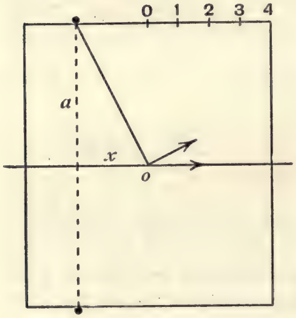

Fig. 111.

The magnetic force at the centre of the coil was 165·992 c.g.s., while the assumption of uniform winding would have given 165·778 c.g.s. The difference was about 1 part in 800. This of course is an error to which all coils are subject.

The effect of the spirality of the winding was also computed, for it will be seen that one half of a given winding is on the whole nearer, and the other half farther, from the centre than the mean distance. The effect was found not to be more than 1 part in 2,000,000.

It is easy to show that the error dH/H in the magnetic field is $-\frac{4}{7}da/a$, where da/a is the error in measurement of the radius, and the corresponding error from inaccuracy in measuring the length of the cylinder is $-\frac{3}{7}db/b$. The error in field due to 0·1 millimetre error in a was 1 part in 8,750, for the same error in b, the error in field was 1 part in 10,100. For the plaster of Paris cylinder on which the coil was wound the effect of 1° C. change of temperature was 1 in 40,000; for marble it was found to be 1 in 100,000. The error in field due to change of temperature is numerically the same as the error in linear dimensions caused by the change.

An outside value of the effect of displacing the wires to provide an opening for the suspension was found by considering the displacement of four wires through a distance of 2 millimetres on each side of the central plane and half a millimetre radially in order that they might be above the adjacent wires. The effect was 1 part in 150,000,000 of the whole force for the lateral displacement, and 1 part in 2,000,000 for the radial displacement.

With regard to the couple Θ, the error due to inaccuracy in the measurement of r, the radius of the smaller coil, was $2dr/r$, since the couple is proportional to the square of r. It was therefore important to measure r with great accuracy, and to allow in every way for wire thickness, irregularities of winding, etc., as in the case of the fixed coil. Two smaller coils of different dimensions were made as stated above, to give a check on the measurements.

Further particulars, as to the mode of using the instrument, are to be found in the *Physical Review* in the account there given of the work of Guthe, Patterson, and Carhart on the electromotive forces of cells.

22. Non-absolute galvanometers and dynamometers. Choice of gauge of wire. Galvanometers and electrodynamometers which by themselves are not capable of giving measurements of currents in absolute units are very frequently used. Such instruments are "calibrated" by some reliable method, so that the absolute values of the currents corresponding to any given deflections are known. In general they differ very much from the so-called absolute instruments in the arrangement of their coils, etc., which in non-absolute instruments has had chiefly in view the attainment of the greatest possible sensibility.

We shall distinguish between instruments which have in their coils a great many turns of fine wire, so that the resistance of the coil system amounts to at least several hundred ohms, and those instruments the resistance of which is comparatively low. The former are frequently called "potential" instruments or voltmeters from their use in determining the difference of potential between two points in a circuit at which the terminals are applied; the latter are called low resistance or "short coil" instruments, and sometimes (when their resistances are so low that one of them can be placed in series with the working circuit without materially increasing its resistance) "current meters" or amperemeters.

First taking galvanometers, we shall establish some general theorems regarding the arrangement of their coils, then very shortly discuss their graduation for absolute measurements, and finally deal with graduated electrodynamometers.

In the first place, let the galvanometer have a certain cylindric channel which is to be filled with wire, and let it be required to find the gauge of wire with which it ought to be wound if it is to be used in circuit with an electrical generator of given electromotive force and resistance. Let a be the radius of cross-section of the wire employed, c the thickness of the covering, and S the cross-section of the channel made by a plane through the axis. The portion of the cross-section occupied by each turn will be $(2a+2c)^2$ if the turns are arranged in square order in the cross-section, and $(2a+2c)^2\sqrt{3}/4$ if they are arranged in triangular order. This includes the space occupied by the covering and the vacant spaces between the spires.

Considering at present the first case only, we see that the number of turns is $S/(2a+2c)^2$, if any inaccuracy introduced by its being impossible to fit an exact number of turns into a complete layer is neglected. If r be the mean radius of the cross-section of the channel, the whole length of wire is approximately $2\pi rS/(2a+2c)^2$. But ρ denoting the specific resistance of the wire, the resistance per unit length is $\rho/\pi a^2$, and the whole resistance R of the coil is $\frac{1}{2}\rho r S/(a+c)^2 a^2$. For a given current the magnetic force at the needle is proportional to the number of turns, and the magnetic force parallel to the axis may therefore be written $AS\gamma/(a+c)^2$, where A is a constant. If E be the electromotive force of the generator, and R' the resistance of the generator and wires connecting it to the galvanometer bobbin, we have

$$\gamma = \frac{E}{\dfrac{\rho r S}{2a^2(a+c)^2} + R'},$$

and for the axial component of magnetic force

$$F = \frac{ASE}{\dfrac{\rho r S}{2a^2} + R'(a+c)^2}. \quad \text{.........................(16)}$$

Since the numerator is constant, this has its maximum value when the denominator is a minimum. Calculating in the usual manner the necessary condition, we find the equation

$$a^4 + ca^3 = \frac{\rho r S}{2R'}, \quad \text{.............................(17)}$$

a biquadratic for the determination of the corresponding value of a. But for the reciprocal $1/R$ of the resistance of the bobbin we have the value $2(a+c)^2 a^2/\rho r S$, and this used with the last equation gives

$$\frac{R}{R'} = \frac{a}{a+c}, \quad \text{.............................(18)}$$

or the resistance of the bobbin should have to the resistance of the generator and connecting wires the ratio of the radius of the wire when bare to its radius when covered.

If the spires are arranged in triangular order, the equation of condition corresponding to (17) is

$$a^4 + ca^3 = \frac{2\rho r S}{\sqrt{3}R'}, \quad \text{.............................(19)}$$

and since, in this case, $1/R = \sqrt{3}a^2(a+c)^2/2\rho r S$, we have the same result as before.

It may be remarked here that the magnetic effects of a given bobbin wound with wire of different gauges, the thickness of coating in which bears a constant ratio to the diameter of the wire, and traversed in each case by the same current, are proportional to the square root of the resistance of the coil. For we have then $(a+c)/a = k$, or $a+c = ka$. Thus, by what has been set forth above, the magnetic effect is proportional to $1/a^2$, and the resistance to $1/k^2 a^4$; hence the magnetic action varies as $\sqrt{R}$.

It is very carefully to be observed that this comparison of magnetic effects holds for a given current in the coil. The matter may be looked at also as follows. For a given ratio of diameter of wire to thickness of insulating coating, the number of turns on the coil is directly proportional to the length of wire in the coil, which is inversely proportional to the cross-section of the wire. But the resistance of the coil is directly proportional to the length of the wire and inversely proportional to the cross-section, that is the resistance is proportional to the square of the length of the wire. The length and therefore the number of turns of wire are thus proportional to the square root of the resistance.

It is obvious that this is also true when the thickness of the covering is so small as to be negligible.

23. Best shape of section of bobbin. The best shape of cross-section for the bobbin of an ordinary galvanometer is shown in Fig. 112. The curve forming the external boundary of the cross-section is given by the equation,

$$r^2 = p^2 \sin\theta, \quad \ldots\ldots\ldots\ldots(20)$$

where r is the distance of any point P of the surface from O the centre of the coil, θ the angle POM which OP makes with the axis OM, and p a constant.

To prove this, note that the axial magnetic force due to a single turn of wire of radius a, is proportional to a/r^3, that is to $\sin\theta/r^2$. Let now this turn be transferred to any point outside the surface, fulfilling equation (20), on which it lies. Then whatever the radius of the circle into which it is now bent, the length of arc which it furnishes is the same as before, and so the axial magnetic force is proportional to the new value of $\sin\theta/r^2$. But for every point of the cross-section outside the boundary fulfilling (20) the value of $\sin\theta/r^2$ is smaller, and for every point within the boundary is greater, than for a point of the surface. Thus a given length of wire produces a greater or less axial magnetic force according as it is wound within or without this surface. If then a coil be wound of any shape of cross-section the external boundary of which does not fulfil (20), by removing the wire from one part of the coil to another, the cross-section may be brought to this shape, and the axial magnetic force increased.

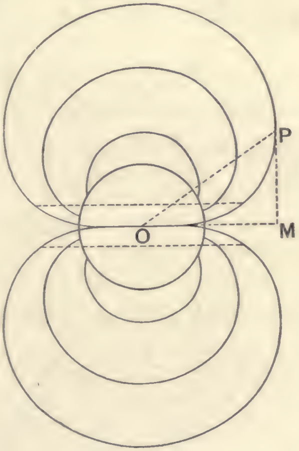

FIG. 112.

Fig. 112 shows curves for different values of p, and the two parallel dotted lines indicate a cylindrical chamber left for the needle.

24. Effect of grading the gauge of wire in bobbin. In the investigation given in 22 above of the best gauge of wire with which to fill a given channel, when the bobbin is to be used with a generator of known electromotive force, it has been assumed that the wire must be of uniform thickness; and we have just seen what is the best form of cross-section to give a coil which is to contain a given volume of wire. When a coil is wound, however, each additional turn of wire, though it increases the axial magnetic force for a given current, also increases the resistance in circuit, and thereby diminishes the current produced by a given electromotive force. We shall now inquire whether by winding the outer layers of thicker wire the effect of increased resistance can be reduced to a minimum.

The volume of the coil supposed without chamber for the needle is

$$2\pi y \times \tfrac{1}{2}\int_0^\pi r^2\,d\theta,$$

where y is the distance of the mean point of the cross-section from the axis. Now

$$y = \frac{\iint r^2 \sin\theta\,dr\,d\theta}{\tfrac{1}{2}\int r^2\,d\theta},$$

the limits of integration being 0 and $p(\sin\theta)^{\frac{1}{2}}$ for r, and 0 and π for θ. Hence, on the supposition already made,

$$\text{volume of coil} = \tfrac{2}{3}\pi p^3 \int \sin^{\frac{5}{2}}\theta\,d\theta$$

$$= \tfrac{1}{3}Np^3, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

if $N = 2\pi\int_0^\pi \sin^{\frac{5}{2}}\theta\,d\theta$, which does not depend on the dimensions or shape of the coil. The chamber containing the needle should be made as small as possible,* as the part of the coil immediately surrounding the magnet is the most valuable; but it will always cut away a part of the coil depending on p, which may be denoted by $f(p)$. The actual volume of the coil is thus $\tfrac{1}{3}Np^3 - f(p)$.

25. Theory of a graded coil. If now dl be an element of length of the wire composing the coil, and p the parameter of the generating curve of the surface on which it lies, then since $1/p^2 = \sin\theta/r^2$, the axial magnetic force at the centre is, by the law of magnetic force due to elements of the circuit, $\gamma\int dl/p^2$ ($=\gamma G$, say), where p is a function of the whole length, l, of wire in the coil from some chosen point, say the inner end, to dl. We shall suppose the wire to be of a different gauge at different places in the coil. If its radius at dl be a, the thickness of the covering there c, and the winding be in square order, the volume occupied by dl is $dl(2a+2c)^2$, so that the whole volume is $4\int dl(a+c)^2$, where a

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* For the manner of winding the space close to the magnet see 26 below.

(and c if not constant) is a function of l, and the integral is taken throughout the whole length of wire in the coil.

Let the coil be considered as made up of layers each fulfilling the equation $r^2 = p^2 \sin\theta$, but each for its own value of p, so that a is a function of p. We have thus for the volume of the space between the layers corresponding to p and $p + dp$ the expression

$$Np^2 dp - f'(p)\,dp = (2a + 2c)^2 dl,$$

if dl be now put for the length of wire in this space. Thus

$$dl = \{Np^2 dp - f'(p)\,dp\}/(2a + 2c)^2,$$

and we get (since we have put $G = \int dl/p^2$)

$$dG = \frac{Np^2 - f'(p)}{p^2 4(a + c)^2} dp, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

$$dR = \frac{\rho}{\pi a^2} \frac{Np^2 - f'(p)}{4(a + c)^2} dp. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

If the generator have as before an electromotive force E, and R' denote as before the resistance of the generator and connecting wires, we have $\gamma = E/(R + R')$ and $\gamma G = EG/(R + R')$. To make γG or $G/(R + R')$ a maximum by properly grading the wire, we have to choose the diameter for each layer so that the contribution of the layer to $G/(R + R')$ shall be as great as possible. Now imagine any layer to be taken away from the coil, everything else remaining the same. G becomes $G - dG$, and R, $R - dR$. Thus $G/(R + R')$ changes by

$$\{(R + R')\,dG - G\,dR\}/(R + R')^2.$$

If we make the thickness of the layer very small, $G/(R + R')$ will be the same whatever layer is removed, and may in that case be regarded as a constant for all parts of the coil, and as we are considering only the effect of a particular layer we consider $R + R'$ as a constant. We have, then, to find the value of $a + c$ for which the effect

$$dG - G\,dR/(R + R')$$

is a maximum. If $a + c$ be denoted by u the necessary condition is

$$\frac{d}{du} dG - \frac{G}{R + R'} \frac{d}{du} dR = 0$$

or

$$\frac{\dfrac{d}{du} dR}{\dfrac{d}{du} dG} = \frac{R + R'}{G}.$$

Performing the differentiations on the values of dG and dR given in (22) and (23) above, we find

$$\frac{\rho p^2}{\pi a^2}\left(1 + \frac{u}{a}\frac{da}{du}\right) = \frac{R + R'}{G} = \text{constant}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(24)$$

If the radius of the wire and the thickness of its covering have always the same ratio, that is if u/a is constant, we have

$$a/u = da/du \quad \text{or} \quad u/a \,.\, da/du = 1.$$

Hence in this case a is in simple proportion to p.

On the other hand, if the thickness of the covering is always the same, $da/du = 1$, and we have $p^2(2a+c)/a^3 = \text{constant}$.

On the first supposition, denoting a by αp and $a+c$ by βa, where α and β are constants, and putting $-N/q$ for the integral of the term depending on the chamber in which the mirror hangs, we find from (22)

$$G = \frac{N}{4\alpha^2\beta^2}\left(\frac{1}{q} - \frac{1}{p}\right), \qquad \text{........................(25)}$$

where p is the greatest parameter used for the coil. In general q depends also on this value of p, but, as will be seen from Fig. 112, is nearly constant if the chamber is not large. It is a quantity of the order of magnitude of the internal dimensions of the chamber, and may be regarded as the parameter of the curve which would generate by revolution round the axis a volume equal to that of the needle chamber. The resistance has the value $G\rho/\pi\alpha^2$.

We see from (25) that very little is gained, when this mode of winding with graded wire is adopted, by making p large in comparison with q.

26. The needle and needle chamber. If the chamber in which the needle hangs is cylindrical and runs right through the coil, the needle is shorter than the diameter of the smallest spires, and every spire in the coil produces an effect in the same direction on the needle. If

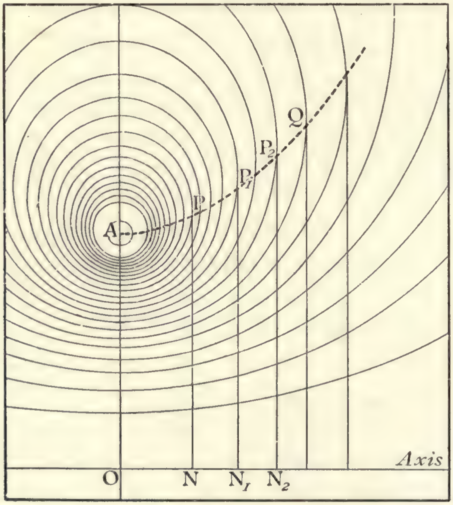

FIG. 113.

however the space in which the needle hangs is not made cylindrical, the shape of it is of some importance, as it is possible to place spires in positions in which they produce a magnetic effect opposed to that of the coil generally.* To see this it is only necessary to consider the diagram of lines of force (Fig. 113) due to a single turn of wire of radius *OA*. Take any line of force and draw a tangent, *PN*, to it at right angles to the axis. Then it is clear that a uniformly magnetized needle at right angles to the axis, half of which is represented in position and length by *PN*, will not be acted on by any couple, since the force on each pole is in the direction of the length of the magnet. If however the magnet be at a greater axial distance, the force upon it is in the same direction as it would be if the needle were very short. Thus on a needle of the length and in the position here specified two turns, one smaller, the other larger in radius than the turn shown in the diagram, and in the same plane with the latter, would, if traversed by currents in the same direction, produce opposite couples. The smaller turn

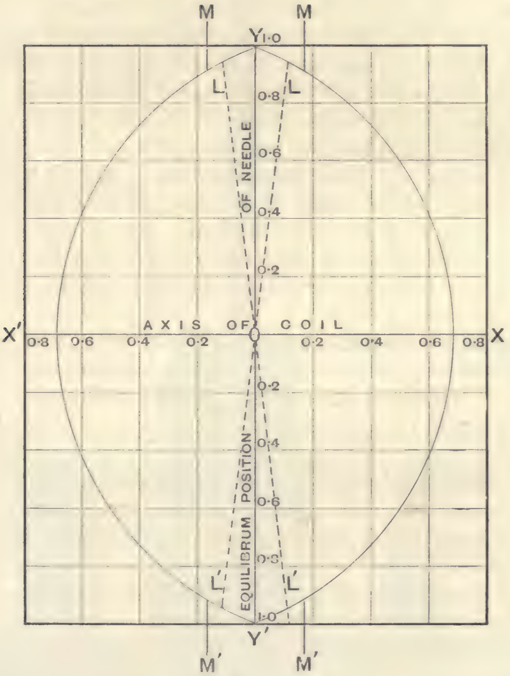

FIG. 114.

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* This is pointed out in Messrs. Ayrton, Mather, and Sumpner's paper, *Phil. Mag.* July 1890.

would however produce a couple in the same direction as the larger, if carried off to a sufficient axial distance from the needle.

For a needle of given length it is easy to draw a curve of limiting positions for the spires. For draw the line APQ through the points of contact of tangents perpendicular to the axis, then the axial distances ON_1, ON_2 of these tangents from the plane of the spire are the limiting distances of the spire from magnets of the half length N_1P_1, N_2P_2, etc. Then by supposing the scale of the diagram reduced in the ratio of N_1P_1 to N_2P_2 we shall have a spire of radius $OA \times N_1P_1/N_2P_2$ in the position to exert zero couple on a needle of half length N_1P_1 when at an axial distance $ON_2 \times N_1P_1/N_2P_2$, and so for other points.

It is therefore clearly undesirable to fill with spires wound in the same direction as the rest of the coil the space near the needles, beyond the limits indicated by these considerations. Figure 114[*] shows the form of the cavity which ought to be left. If it is possible to fill any of this space with wire, it should be done, but the spires made to run in the opposite direction, so that the couples due to their magnetic action may be in the same direction as that due to the rest of the coil.

27. Wiedemann's aperiodic galvanometer. A form of galvanometer very convenient in many respects is that invented by Wiedemann[†] (Fig. 115). A circular disk, or ring, of steel about 2 cm in diameter, magnetized parallel to a diameter, is suspended with its magnetic axis horizontal and forms the needle of the instrument. This needle is attached to the lower end of a bar of aluminium, which also carries the mirror (made of thin glass); and is hung within a damping chamber of copper, by a cocoon fibre, from a torsion head above, by means of which the effect of the torsion of the fibre can be estimated. The mirror is fixed so far above the needle that it is clear of the coils, and is viewed through a telescope in the ordinary manner. The suspension fibre, aluminium bar, and attached mirror are protected by means of a glass tube and case fixed above the damping chamber.

A pair of coils is arranged, one on each side of the damping chamber, with their axes in line through the centre of the needle; and are attached to sliding pieces so that their distances from the needle can be increased or diminished and the sensibility altered accordingly. The openings in the coils are large enough to allow the bobbins to slide over the damping box close up to the needle, leaving, in the closest position, between them only the narrow space necessary for the tube down which passes the fibre.

Two or three sets of pairs of coils suitable for different purposes are provided with the instrument. When the needle moves in the damping box of copper its motion is resisted by the action of the induced currents produced, so much so that it hardly oscillates about a new position of equilibrium.

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[*] From Messrs. Ayrton, Mather, and Sumpner's paper, *Phil. Mag.* July 1890.

[†] *Die Lehre v. d. Elektricität,* vol. iii. p. 289.

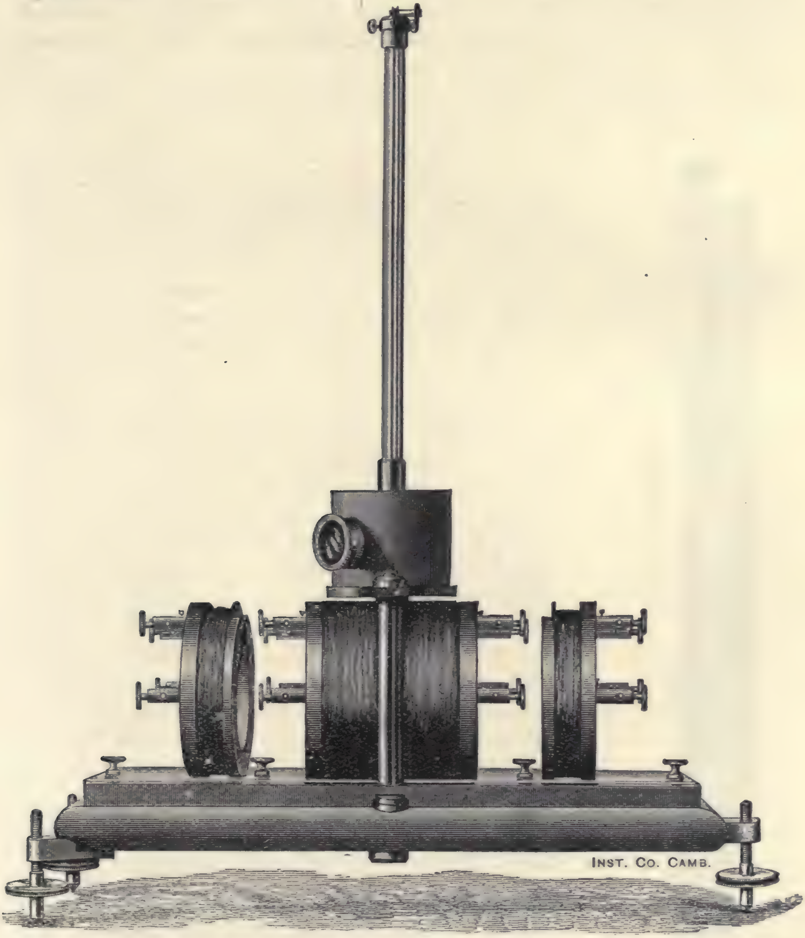
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FIG. 115.

28. Siphon-recorder arrangement used for galvanometer. In the late Lord Kelvin's siphon-recorder for registering signals sent through a submarine cable, a coil of wire is suspended between the poles of a magnet so as to be free to turn round a vertical axis passing through its centre [Fig. 116]. Within the coil is fixed an iron core which serves to concentrate the field of the coil. When the coil is in the undeflected position the planes of its spires are parallel to the direction of the magnetic field, but when a current is sent through the coil it turns, in a direction depending on that of the current, so as to increase the magnetic induction through its circuits. A return couple is provided for the recorder by means of a bifilar suspension. The magnet is either a permanent horse-shoe magnet, or an electromagnet excited by a local current. The current from the sending station passes round the coil, which, turning in one direction or the other according as a "dot" or "dash" is being indicated, actuates the writing siphon.

The ordinary dead-beat reflecting galvanometer invented by Lord Kelvin for cable signalling and ordinary testing is described in the chapter above on the comparison of resistances.

The application of the siphon-recorder arrangement as a galvanometer was referred to in the original patent of the instrument and was pointed out in the first edition of Maxwell's *Electricity and Magnetism*, and has occurred to and been used by several experimenters. MM. d'Arsonval and Deprez have however brought such instruments into general use for several purposes connected with practical electric work. The coil is hung by or rather strung on a stretched metallic wire, by which the current enters and leaves, and the torsion of this wire gives the required return couple. A core of iron is sometimes used within the coil as in the siphon-recorder. This, if used at all, should be quite independent of the coil, so that the coil may be adjusted relatively to the core and pole-faces of the magnet. A mirror attached to the coil enables the deflections to be measured in the ordinary way.

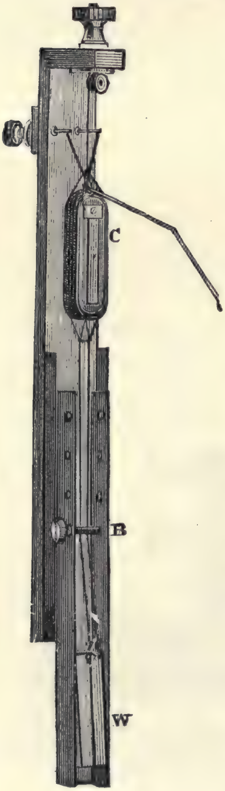
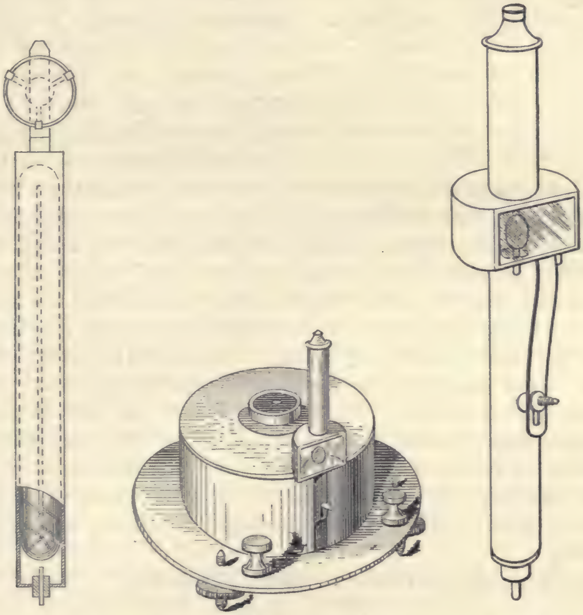


FIG. 116.

This form of galvanometer possesses some advantages. It can be made very sensitive by increasing the intensity of the field, and the coil possesses dead-beat quality in a high degree in consequence of the damping action of the induced currents produced in it when it is moving in the field. (See Chap. XV. 14.) It is moreover only to a slight extent directly affected by external magnetic bodies, since these, unless very highly magnetized, can only slightly affect the field in which the coil is placed. Its action in different cases however requires very careful consideration. Some of these cases will be examined below.

An improved form due to Messrs. Ayrton and Mather is shown in Fig. 117. The coil is enclosed in a silver tube hung by a flattened wire of phosphor-bronze, with spiral of phosphor-bronze for lower connection.

It is desirable that the magnetic field of such a galvanometer should be as little disturbed as possible, in a manner at least which cannot be completely taken account of, and hence the use of iron cores in the suspended coils is inadvisable. Messrs. Ayrton, Mather, and Sumpner* found it possible to make such a galvanometer give deflections proportional to deflections by dispensing with the iron core, and fitting iron pole-pieces to the stationary magnets, so shaped that the moving coil cut lines of force always at the same rate as the deflection varied.



a. Coil Tube and Mirror : full size.

General view of instrument, showing circular permanent magnet with gap between poles for coil.

b. Outer brass tube : full size.

FIG. 117.

29. Best shape of coils in moving coil galvanometers. It has been pointed out by Mr. T. Mather† that in instruments such as this in which suspended coils are used in magnetic fields, these coils should be long and narrow, and that the cross-section at right angles to the axis should be two equal circles touching on the axis. To prove this, it is to be observed first, that if the magnetic moment contributed by any portion of the wire be made greater by increasing the breadth of the spire in which it is placed, the moment of inertia of that part is increased in a greater ratio, and thus the period of free vibration of the coil is increased. The period of the coil is generally limited by practical requirements, and we have therefore to consider what the form of the coil should be, so that for a given moment of inertia there may be a maximum magnetic moment, or for a given magnetic moment a minimum moment of inertia. The solution is the same for both these cases. Consider (Fig. 118) an element E, of area dS, of a cross-section in a plane at right angles to the axis A, and let n be the number of turns per unit of area. If γ be the current in each the current crossing dS is $\gamma n\,dS$. The couple round the axis exerted on unit of length of this part of the coil parallel to the axis is $\gamma n dS\,.\,Hr\sin\theta$, where H is the intensity of the magnetic field, r the distance of the element from the axis, and θ the angle between AE and H. If ρ be the average density of the coil, the moment of inertia of unit length parallel to the axis, and having the section dS, is $\rho r^2 dS$. The ratio of couple to moment of inertia for this part is thus $\gamma nH\sin\theta/\rho r$, and this is to be made a maximum for every element of the coil. Thus $\sin\theta/r$ is to be made a maximum, since the other quantities are constant. The ends of the coil are ineffective as regards magnetic action, and hence so far as they are concerned it is desirable to make the distance of each element from the axis as small as possible. It is also desirable that the poles should be close in order to ensure with ordinary magnets as intense a magnetic field as possible.

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* *Phil. Mag.* July 1890.

† *Phil. Mag.* May 1890.

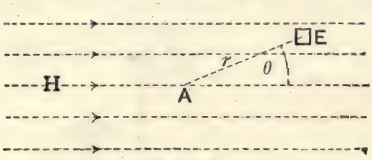

Fig. 118.

Consider now the curve the equation of which is

$$r = c\sin\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

where c is a constant. A family of such curves can be drawn for different values of c, and they are all circles touching in the point A. Now let an element of wire be carried from the surface fulfilling this equation to a point lying outside. For such a point $\sin\theta/r$ has a smaller value. For a point lying inside $\sin\theta/r$ is greater. Thus, if the cross-section of the coil be filled up within any circle $r = c\sin\theta$, a diminution of the value of $\sin\theta/r$ would be produced by transferring any portion of the wire to any other unoccupied position.

The coil should therefore be made long in the direction of the axis, and have the form of cross-section shown in Figure 119, namely, two circles touching on the axis at the point A. The pole-faces should also be correspondingly long, and be broad enough to give a nearly uniform field at the coil, if they are not shaped so as to accomplish the object stated above.

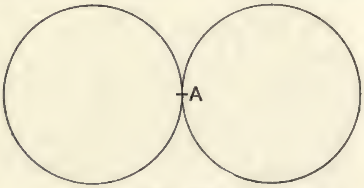

Fig. 119.

30. Suspension of coils. The passage of the current along the suspension wire is apt to affect seriously the constant of the instrument, by altering its torsional rigidity. Suspensions made of twisted strips of thin phosphor-bronze were used by Professors Ayrton and Perry in several of their instruments. These have small torsional rigidity and great radiating surface, and are therefore peculiarly well adapted for use as torsion suspensions which at the same time act as conductors.

It was pointed out in this connection by Messrs. Ayrton, Mather, and Sumpner that by making both coil and suspension of platinum-silver compensating effects as regards changes of torsional rigidity are produced. If the rise of temperature were the same both in the coil and the suspension there would be exact compensation, since the percentage increase of resistance of platinum-silver is nearly equal to its percentage diminution of torsional rigidity.

Moving coil galvanometers should have their constants redetermined at fairly short intervals, for the magnetization of the field magnets and also the elastic constant of the suspension are subject to change. They should also be calibrated for a range of currents, to take account of any change of magnetic field that may result from the action of currents in the coil on the field magnets.

The temperature variation of resistance is very slight in the case of the alloy called platinoid, now much in use for galvanometer and other coils, and on this account Mr. Mather* strongly recommended its use for the suspended coils of D'Arsonval voltmeters, and of rheostats for use with such coils.

The "ballistic" use of moving-coil galvanometers will be considered later in the present chapter.

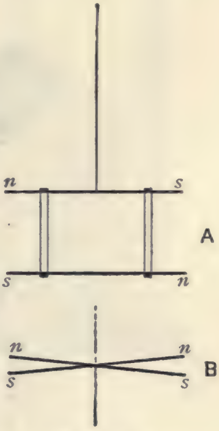

FIG. 120.

31. Astatic galvanometers. In order to obtain sensibility, galvanometers are frequently made with astatic needles, that is suspended needle-systems which, in a uniform field, are either in equilibrium in any position or experience only a comparatively slight directive action. An astatic system generally consists of two similar horizontal needles of equal magnetic moment arranged parallel to one another with their poles turned in opposite directions, as at *A*, Fig. 120, so that the resultant couple on the system is zero or very nearly so. Most commonly the needles are placed horizontally, as nearly as possible in the same vertical plane, with their centres in the same vertical line. In general however the needles are not quite parallel, and the system behaves like a needle of very small magnetic moment with its axis parallel to the line bisecting the obtuse angles between the projections of the needles on a horizontal plane as shown at *B* in Fig. 120. It has therefore been supposed that this is the manner in which an astatic system properly acts, but this is absurd, for if it were so the sensibility of the arrangement would be entirely a matter of accident. Moreover when the system is so used it is affected by the slightest external magnetic influence, and is a source of great trouble through the difficulty of maintaining a definite zero position.

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* *Electrician*, Jan. 8, 1892.

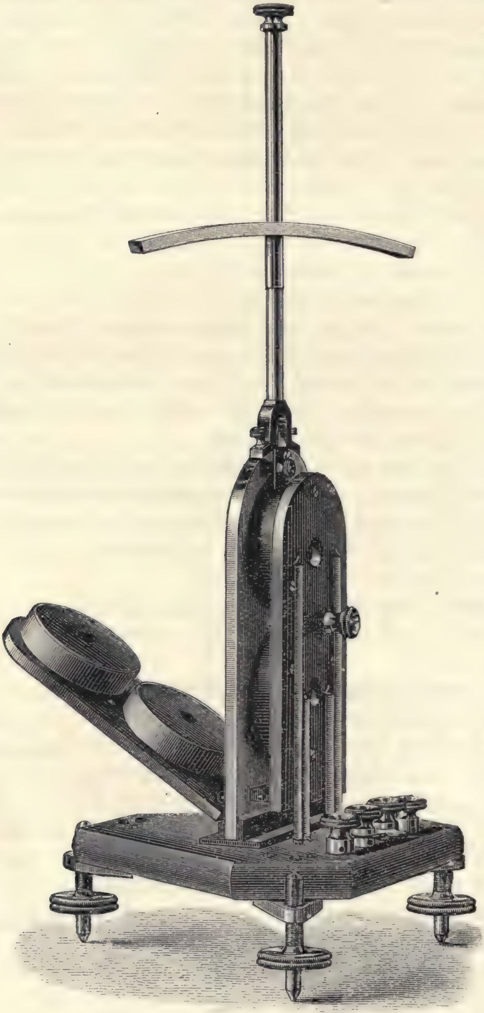

FIG. 121.

An astatic system when quite accurately made has the needles exactly in one plane, and has almost perfect astaticism in a uniform field, and the sensibility is obtained by producing, by means of a magnet placed at some distance, a resultant magnetic field which is not uniform over the needle-system, and therefore gives a differential action which furnishes the necessary directive force on the needles, and which can be made of any desired amount. An astatic galvanometer with directing magnet is shown in Fig. 121.* The instrument illustrated is a form of astatic reflecting galvanometer usually attributed to the late Lord Kelvin. The details of the supports of the coils, needles, etc., will be clear from the figure: the coils, as will be seen, are hinged so as to turn back to allow the suspended system to be easily got at. Each needle-system is a group of short needles, and there are two sets of coils, one containing each group of needles, and joined in such a way that the actions on the needles conspire. Sometimes a single coil only is used enclosing one of an astatic pair of needles. In this case, although the coil exerts couples in the same direction on both needles, the principal turning action is exerted on that which is inside the coil.

32. Gray's astatic galvanometer. Another arrangement of astatic galvanometer is shown in Fig. 122. It is a slight modification of one adopted by Prof. T. Gray and the author for a very sensitive galvanometer constructed for the determination of the specific resistance of glass.† The needles are a pair of horseshoes of hard steel as shown in Fig. 123, and are arranged in two parallel vertical planes so that the poles of one enter the cores of one pair of the four coils C, C, the poles of the other the cores of the other pair of coils. The four coils are fixed in a plate with their axes parallel, and their faces in one plane; and the horseshoes are connected by a curved bar of aluminium so that one enters from one side of the coil system, the other from the other side as

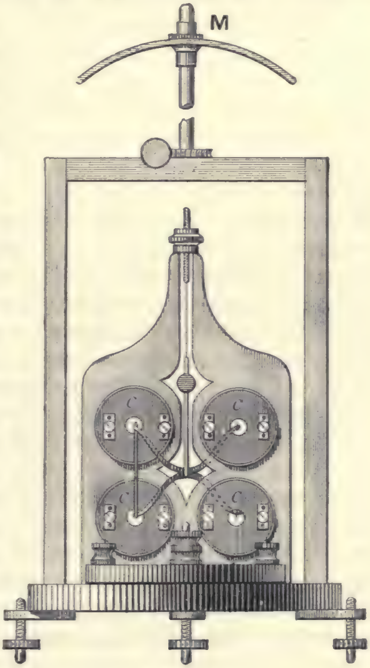

FIG. 122.

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* This cut has been kindly supplied by the Cambridge Instrument Making Co.

† *Proc. R.S.* No. 230, 1884. A similar arrangement of needles has, it appears, been used also by Herr Rosenthal and by Lord Rayleigh. See Ayrton, Mather, and Sumpner's paper, *loc. cit. supra.*

shown by the horizontal section in Fig. 123. The instrument is supported on a plate of vulcanite standing on vulcanite feet to give insulation, and the coils were wound on vulcanite bobbins. The coils are joined so that when a current passes both horseshoes are dragged further into their coils, or both pushed out at the same time. The needle-system is thus turned, and the deflection is measured by means of a mirror and scale in the usual manner. The total resistance of the four coils was approximately 30,000 ohms; and the highest sensibility obtained when the instrument was set up was such that a current $1/10^{11}$ ampere produced a deflection of 1 division on a scale at about a metre distance. The period of the coil was however for many purposes inconveniently long.

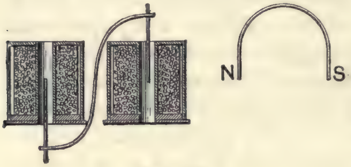

FIG. 123.

A very elaborate instrument on this principle was made for the Central Institution, London, from drawings made by Prof. Ayrton in consultation with Prof. T. Gray.* A full description will be found in the paper of Messrs. Ayrton, Mather, and Sumpner above referred to.

The chief advantage of the arrangement of coils and needles described above is that a great portion of the wire of the coils is placed very near to the poles of the needles, and in a very favourable position for exerting the electromagnetic action required. The instrument, particularly the form shown in Fig. 122, is very easily made, and it does not cost more than an instrument of the ordinary kind. Of course a single horseshoe, or *S* or *Z* shaped bar, might be placed horizontally, and acted on by a pair of coils, and the principle thus applied to a single-needle non-astatic instrument. In astatic instruments, however, of this form it is decidedly preferable, as shown below, to use vertical needles.

33. Vertical astatic needles. It seems to have been pointed out first by T. and A. Gray (*Proc. R.S.* 36, 1883-84, p. 287) that if the line joining the poles or centres of gravity of magnetic polarity in each horseshoe be vertical, the system is always very nearly perfectly astatic for a uniform field, for each vertical horseshoe is itself perfectly astatic. If the needles are equal straight bars placed vertical with a rigid connection they are perfectly astatic, as each needle is perfectly astatic. The pair of horseshoe needles can thus be adjusted to have as nearly as may be perfect astaticism in a uniform field, and thus made to preserve a nearly constant zero when under directive force, a result which it is exceedingly difficult to obtain in the ordinary arrangement of horizontal needles, and which certainly rarely exists when a horizontal magnet or magnets placed above or in an unsymmetrical position relatively to horizontal needles is employed to regulate the sensibility, as then one of the needles must be magnetized and the other demagnetized to a greater or less extent, depending on the position of the magnet. According to this latter arrangement, if we suppose the needles to be parallel or nearly so, and $\mathbf{H}$ to be the magnetic field intensity at the upper needle, $\mathbf{H}'$ that at the lower needle in the same direction, m the magnetic moment of the upper needle, m' that of the lower needle, γ the current flowing, θ the deflection produced, and K a constant, we have

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* See *Phil. Mag.* July 1890.

$$\gamma = K\frac{\mathbf{H}m - \mathbf{H}'m'}{m + m'}\tan\theta. \quad\text{.........................(27)}$$

The sensibility of an astatic instrument with horizontal needles as measured by the tangent of the deflection-angle for a given current is thus very great, as $\mathbf{H}m - \mathbf{H}'m'$ can be made, and is generally, very small. According to the values of m, m', $\mathbf{H}$, $\mathbf{H}'$, the instrument may or may not be seriously affected by external magnets, accidentally displaced in the neighbourhood of the instrument, or by slight changes otherwise caused in the magnetic field. It has been argued that since $\mathbf{H}$, $\mathbf{H}'$ (which are nearly equal) have each a considerable value, any slight magnetic disturbance producing only a very small percentage of change in each of these quantities cannot sensibly affect the value of the sensibility.

This however is a fallacy, as when the instrument is very sensitive, and $\mathbf{H}m - \mathbf{H}'m'$ is therefore very nearly zero, an exceedingly feeble magnetic disturbance changing $\mathbf{H}$ and $\mathbf{H}'$, as it will generally do, by nearly the same absolute amount, and hence in very slightly different proportions, may suffice to alter $\mathbf{H}m - \mathbf{H}'m'$ by an amount comparable with its former value. The equilibrium position of the needles, for zero or any given current, will thus be subject to variation.

Slight changes in all or any of the quantities m, m', $\mathbf{H}$, $\mathbf{H}'$ may, therefore, affect the constant of the ordinary imperfectly astatic instrument very seriously, and as a matter of fact its constant has to be continually redetermined, for it is very sensitive to magnetic disturbances in the neighbourhood.

34. Advantages of vertical needles. In the case, however, of horseshoe needles adjusted to be accurately vertical these disadvantages do not exist. The needles retain their astaticism for uniform field and cannot be affected in the same way by directing magnets. Then $\mathbf{H}$, $\mathbf{H}'$ being the horizontal field intensities at the upper and lower extremities of the needles, γ the current strength, θ the deflection of the needles, and K a constant depending on the coils, we have approximately

$$\gamma = K(\mathbf{H} - \mathbf{H}')\sin\theta. \quad\text{.........................(28)}$$

The sensibility of the instrument can, therefore, be increased to any desired extent by placing the magnet M (Fig. 122) at a greater distance from the needles (or by counteracting its action by a smaller magnet placed nearer to the needles) so as to make $\mathbf{H} - \mathbf{H}'$ sufficiently small. Further, variations of the strength of the horseshoe needles produce no effect unless they consist of changes of magnetic distribution, which may produce a deviation from perfect astaticism. When the instrument is properly adjusted and the needles are as nearly as possible uniformly magnetized, but little disturbance of this kind can be produced by the magnetizing action of the coils, since both poles of each have their magnetism augmented or diminished at the same time in the arrangement of Fig. 122, or both poles of one are magnetized more intensely in some degree, and both poles of the other weakened if both needles enter the coils from the same side.

Another possible arrangement of this system of needles is with like poles above and below. The system will still be perfectly astatic if properly adjusted; and to give a return couple towards a zero position a magnet may be used, placed, for example, horizontally in the vertical plane at right angles to the front of the instrument, in a line passing through the suspension thread. If this magnet be placed nearer to say the lower ends than the upper ends of the needles, and the polarity of the end turned towards the needles be of the *same* name as that of the nearer ends of the needles, they will have a position of stable equilibrium when no current is flowing, with a horizontal line joining a pole of each needle at right angles to the direction of the magnet. The accurate law of variation of deflection with current is, however, in this case more complicated, and the instrument in some cases might have to be graduated by experiments with known currents of different amounts. Any change also of the magnetic distribution of the controlling magnet would affect the indications of the instrument.

It is to be observed that, in consequence of the horseshoe needles being placed in these instruments at a considerable distance from the axis of suspension, a very small value of $\mathbf{H} - \mathbf{H}'$ is sufficient to give the needle system such a directing force as to prevent any great error due to the rigidity or the viscosity of the suspending fibre.

The needle system may be hung in a uniform field and a small needle rigidly connected with it, but placed so as not to be perceptibly affected by the coils, used to give directive force to the magnetic system. This small needle may be hung in such a way that it can be turned round a horizontal axis at right angles to its length, and also round a vertical axis, so as to enable both the sensibility and the zero of the instrument to be adjusted. When the galvanometer is not intended for ballistic experiments, the frame on which the small needle is mounted may conveniently be immersed in a liquid and made to act as a vane for bringing the needle system quickly to rest. This arrangement, of course, would not be astatic, but would give great sensibility on account of the leverage of the horseshoe needles as arranged.

Thus if m denote the magnetic moment of the small needle, H the horizontal component of the earth's magnetic force, k a constant depending on the coils, p the strength of pole of each of the horseshoes (supposed of equal strength), and d the distance of these poles from the suspension thread, we have, since the deflection is small, for the turning couple exerted by the coils $4Ck\phi d$, and for the return couple $mH\theta$, and therefore

$$C=\frac{m\theta H}{4k\phi d}. \quad \ldots\ldots\ldots\ldots(29)$$

Of course this arrangement is applicable whether like or unlike poles are turned in similar directions. It has the disadvantage that any change of m or ϕ or of both would affect the constant of the instrument.

The sensibility of any of these arrangements might also be increased by bringing out a very light arm, say from the middle of the cross-bar connecting the horseshoes, or from any other convenient point, and hanging the mirror by means of a bifilar, one thread of which is attached to the outer extremity of this arm, and the other to a near fixed point. The distance between the fibres being small in comparison with the length of the arm, small deflections would be greatly multiplied. This device would, no doubt, render a greater degree of skill and delicacy of manipulation necessary in the operator or experimenter, but it or some similar plan might in some cases be adopted, and the construction of these instruments renders its application to them very easy.

35. Astatic system with straight vertical needles. The astatic galvanometer described above may be modified as follows. Instead of a set of four coils with hollow cores and horseshoe needles as described, eight coils are used—one set of four arranged in rectangular order in a vertical plane facing a second set of four similar coils in a parallel plane at a small distance from the first. Two *straight* needles of thin steel wire connected together as rigidly as possible by very light bars of aluminium are so chosen as to length and so arranged that they hang from a single silk fibre with their lengths vertical and a magnetic pole as nearly as may be in the line joining the centres of each mutually opposite pair of coils. A magnet giving a differential field at the needles, if their like poles are turned in dissimilar directions, or any other arrangement may be used to give directive force, and a current sent through the coils in any desired way by means of a distributing plate or otherwise.

Astatic galvanometers of the usual pattern are generally made with two coils, one above the other, split into four by a narrow vertical space in which the needle system is suspended, and which admits of the ready removal of the needles for adjustment. In this space may be hung, in a plane nearly parallel (when no current is flowing) to the two coils, two thin magnetic needles of steel wire side by side, kept with their lengths accurately vertical, and at a short distance apart (say $\frac{1}{4}$ or $\frac{3}{8}$ of an inch) by light aluminium, or other non-magnetic bars. Such a system of needles with unlike poles turned in similar directions would plainly experience a similar magnetic action to that exerted by the coils on the needles in the ordinary so-called astatic combination. But two straight vertical needles would plainly be perfectly astatic in a uniform magnetic field, and this astaticism for uniform field would not be liable to disturbance from any arrangement of magnets applied to give directive force to the system, as, for example, one or more magnets directing the system by means of a more powerful action at one end of the needle system than at the other, as shown in Figs. 121 and 122, or magnets arranged symmetrically with respect to both ends of the needles. An instrument with such a system of needles ought therefore to be subject to but slight, if any, disturbance in ordinary circumstances of sensibility when masses of steel or iron are being moved about at some little distance, and would we think be found useful in such cases, as for example in cable testing rooms.

36. Ballistic galvanometers. A ballistic galvanometer is an instrument designed for the purpose of measuring the whole quantity of electricity which passes in a current of short duration. It is so called because the moment of inertia of the needle-system is made so great, and consequently the free period of vibration so long, that the current has begun and ended before the needle has sensibly moved from its initial position ; just as in a ballistic pendulum the change of momentum of an impinging bullet has entirely taken place before the massive bob (though set into motion) had time to be deflected from the position of stable equilibrium which it has under the action of gravity.

The arrangement of needles takes many different forms. For example Professors Ayrton and Perry constructed a ballistic galvanometer in which the needles were each a built-up sphere of small magnets, and therefore had a considerable moment of inertia ; the form of galvanometer referred to in 32 above was constructed for ballistic use, and several others on the same principle have been made for the same purpose ; in other cases the needle is a disk of steel carefully polished to serve as mirror, and magnetized parallel to a diameter which is made horizontal when the needle is suspended.

The coil should always be set up so that the needles rest exactly at right angles to its axis. This enables the needle if the deflection is kept small to be only slightly affected by the magnetizing action of the current in the coil. This adjustment is extremely important for ballistic use of the instrument, as will be seen from the investigation given in 40 below.

The arrangements of coils is the same as in galvanometers for steady currents, except that on account of the influence of induced currents produced by the moving magnets the coils should be made with non-metallic cores or tubes ; or if metallic tubes are used they should be slit longitudinally from end to end.

The siphon-recorder (or d'Arsonval Deprez) arrangement may also be used for ballistic purposes.

37. Approximate theory of the ballistic galvanometer. Let a be the initial angle which the needle makes with the plane of the coil, and θ_1

the angle which the needle would make with its initial position at the extremity of its deflection if there were no damping action. If μ be the magnetic moment of the needle, supposed short, and $G\gamma$ the magnetic force at the needle produced by a current γ in the coil, the turning couple on the needle is $\mu G\gamma \cos a$. Hence if mk^2 be the moment of inertia of the needle, we have, when the current is γ and the deflection from zero is a,

$$\frac{d^2\theta}{d^2t} = \frac{\mu G\gamma}{mk^2} \cos a, \quad \text{......................} (30)$$

if we neglect for the present the action of the current in the coil in changing the magnetic moment of the needle.

If the whole current passes before there is any sensible deflection, we have, integrating over the whole time during which the current lasts,

$$\frac{d\theta}{dt}_{\theta=0} = \frac{\mu G \cos a}{mk^2} \int \gamma dt = \frac{\mu G \cos a}{mk^2} Q, \quad \text{...............} (31)$$

if Q be the whole quantity of electricity which flows in the transient current.

Hence the kinetic energy given to the magnet is [μ = magnetic moment]

$$\tfrac{1}{2} mk^2 \left(\frac{d\theta}{dt}\right)^2_{\theta=0} = \tfrac{1}{2} \frac{\mu^2 G^2 \cos^2 a}{mk^2} Q^2. \quad \text{................} (32)$$

This kinetic energy, as the magnet swings round and comes to rest in the magnetic field of horizontal intensity H, not necessarily that of the earth, is changed into magnetic energy of amount (see p. 54 above) $\mu H(1 - \cos \theta_1)$. Equating this to the value of the kinetic energy just found, we get

$$Q^2 = \frac{2mk^2 H(1 - \cos \theta_1)}{\mu G^2 \cos^2 a}.$$

If T be the complete period of free vibration of the needle, we have $T = 2\pi\sqrt{mk^2/\mu H}$, or $mk^2/\mu = HT^2/4\pi^2$. Thus the last equation becomes

$$Q = \frac{HT}{\pi G} \frac{\sin \frac{1}{2}\theta_1}{\cos a}. \quad \text{......................} (33)$$

38. Damping of oscillations by air-friction. To take into account the damping action exerted on the needle by the air, etc., and by the induced currents produced in the coil by the motion of the needles, we may proceed by the following direct method of observation which has suggested itself to almost all experimenters with galvanometers. Let successive swings of the needle towards the two sides of zero be θ_1, θ_2, θ_3, ... be observed. Then it will no doubt be found that, approximately, $\theta_1/\theta_2 = \theta_2/\theta_3 = \ldots = r$. We have then $r\theta_2 = \theta_1$, $r\theta_3 = \theta_2$, or $r^2\theta_3 = \theta_1$, and so on. Thus $r = (\theta_1/\theta_3)^{\frac{1}{2}} = (\theta_1/\theta_5)^{\frac{1}{4}} \ldots$, and the undamped deflection is, nearly, $\theta_1(\theta_1/\theta_5)^{\frac{1}{4}}$. The usual theory of damped small oscillations is, however, as follows. We suppose the deflection to be small enough to allow the sine of the deflection to be taken as equal to the angle, and take the retarding couple as proportional to the angular speed, as it will be if the velocity is not too great. This theory will be sufficient, as the angular deflection can always be kept small, and nevertheless be read with accuracy; its smallness moreover prevents the angular velocity from becoming too great.

Let then the magnet make a small oscillation in the field of intensity H, and under the influence of the damping couple $\kappa\, d\theta/dt$. The equation of motion is

$$\frac{d^2\theta}{dt^2}+\frac{\kappa}{mk^2}\frac{d\theta}{dt}+\frac{\mu H}{mk^2}\theta=0, \qquad (34)$$

or if we write k for $\kappa/2mk^2$ and n^2 for $\mu H/mk^2$,

$$\frac{d^2\theta}{dt^2}+2k\frac{d\theta}{dt}+n^2\theta=0, \qquad (34')$$

of which the solution, if T_1 be the observed period under the influence of the damping, is

$$\theta=Ae^{-kt}\sin\frac{2\pi}{T_1}t, \qquad (35)$$

if t be reckoned from an instant when $\theta=0$, and the vibrator is passing through the undisturbed position in the positive direction. The period T_1 is given by

$$T_1=\frac{2\pi}{(n^2-k^2)^{\frac{1}{2}}}. \qquad (36)$$

39. Logarithmic decrement of ballistic deflection. Equation (35) indicates simple harmonic motion of range diminishing in geometric progression as t increases by successive intervals each equal to $\frac{1}{2}T_1$. The Naperian logarithm of the ratio of any one amplitude to that which succeeds after an interval $\frac{1}{2}T_1$ is $\frac{1}{2}kT_1$. This is called the logarithmic decrement of the motion, and is generally denoted by λ.

From (35) we obtain

$$\frac{d\theta}{dt}=Ae^{-kt}\left(\frac{2\pi}{T_1}\cos\frac{2\pi}{T_1}t-k\sin\frac{2\pi}{T_1}t\right)$$

or

$$\frac{d\theta}{dt}=\frac{2\pi}{T_1}\sec\epsilon\,.\,e^{-kt}A\cos\left(\frac{2\pi}{T_1}t+\epsilon\right), \qquad (37)$$

where $\tan\epsilon=kT_1/2\pi$.

Now if there were no damping the period would be $T=2\pi/n$. Hence

$$T_1=T\frac{n}{(n^2-k^2)^{\frac{1}{2}}}=T\sec\epsilon. \qquad (38)$$

But, by (31), when $t=0$, $d\theta/dt=MGQ/mk^2$, for $\alpha=0$, so that the last equation gives

$$A=\frac{MGQ}{mk^2}\frac{T_1}{2\pi}.$$

Thus

$$\frac{d\theta}{dt}=\frac{MGQ}{mk^2}\sec\epsilon\,.\,e^{-kt}\cos\left(\frac{2\pi}{T_1}t+\epsilon\right). \qquad (39)$$

Putting in this $d\theta/dt=0$, we get the value of t when the first deflection (or "throw") θ_1' has just been completed. Thus $t=T_1(\pi/2-\epsilon)/2\pi$. Hence (35) becomes for this value of t

$$\theta_1' = \frac{T_1}{2\pi}\,\frac{MGQ}{mk^2}\exp\left\{\left(-\frac{\pi}{2}+\epsilon\right)\tan\epsilon\right\}\cos\epsilon$$

$$= \frac{T_1}{2\pi}\,\frac{MGQ}{mk^2}\exp\left(-\frac{\lambda}{\pi}\tan^{-1}\frac{\pi}{\lambda}\right)\cos\epsilon. \quad\text{..................(40)}$$

But if the oscillation were unretarded, and T the free period, we should have

$$\frac{MH}{mk^2}=\frac{4\pi^2}{T^2}=\frac{4\pi^2}{T_1^{\,2}}\sec^2 e=\frac{4}{T_1^{\,2}}(\pi^2+\lambda^2)$$

or

$$mk^2=\frac{MHT_1^{\,2}}{4(\pi^2+\lambda^2)}.$$

Substituting this value of mk^2 in (40), and solving for Q, we get finally

$$Q=\frac{HT_1}{2G}\,\frac{\theta_1'}{\sqrt{\pi^2+\lambda^2}}\exp\left(\frac{\lambda}{\pi}\tan^{-1}\frac{\pi}{\lambda}\right). \quad\text{................(41)}$$

This gives the first actual elongation θ_1'. If the damping be very slight so that λ is very small, we get approximately from (41) or directly from first principles, the equation

$$Q=\frac{HT_1}{2\pi G}(1+\tfrac{1}{2}\lambda)\theta_1'. \quad\text{......................(41')}$$

We shall have in chapters which follow numerous examples of correction of observations of the effects of damping.

40. Uncertainty of ballistic action. Theory of its cause. It is to be noticed that there is some uncertainty as to what the action of the air actually is when the needle of the ballistic galvanometer is suddenly set into motion. Also any magnetizing or damagnetizing action on the needles must be as far as possible guarded against in the arrangement and use of the instrument. The deflection, on this account, ought to be always kept as small as possible, so that on the one hand the needle may never deviate far from the direction of the permanent field in which it is placed, and may on the other be always nearly at right angles to the axis of the coil, and thus only slightly exposed to magnetizing action in the direction of its length.

The following brief discussion illustrates the great importance of having the needle at zero when exactly in the mean plane of the coil. Let us suppose that a condenser of capacity K is discharged through the galvanometer. Though theoretically the current falls off exponentially, so that the time of discharge is infinite, the whole charge to within a very small fraction has, in all ordinary cases, passed through the coil in a small fraction of a second, before the needle has undergone any appreciable displacement. Now in the discharge the energy of the charged condenser is transformed into heat in the coil of the galvanometer, with the possible exception of a very small portion which may be spent in the needle, in eddy currents or otherwise. We shall assume this latter part to be zero. Thus if the whole charge of the condenser be Q, we have the equations

$$R\int_0^\infty \gamma^2 dt = \frac{Q^2}{2K},\quad Q = \int_0^\infty \gamma\, dt, \quad \text{......................(42)}$$

where R is the effective resistance of the discharging circuit.

Now going back to (31) above, multiplying by dt and integrating over the time of discharge, which we suppose so short that we may put $\int \dot{\theta} dt = 0$, we get for the angular speed ω with which the needle is started,

$$mk^2\omega = MGQ\cos\theta_0 + C\frac{Q^2}{2KR}\sin\theta_0\cos\theta_0. \quad \text{...........(43)}$$

The second term on the right arises from magnetization effect of the current on the needle, as already explained in 15 above. It is even possible that the presence of this term may render ω zero, so that the needle does not move. This will occur if

$$\sin\theta_0 = -\frac{2MG}{C}\frac{R}{V}, \quad \text{.........................(44)}$$

where $V = Q/K$, the difference of potential to which the condenser was charged. The sign of V may be positive or negative; hence it is imperative, if no such effect as that here discussed is to occur, that θ_0 should be zero.

Dr. Alexander Russell has determined for various galvanometers values of θ_0, for which with chosen values of R the "throw" is zero. These he calls the "dead points" of the instrument. [See his paper, *Phil. Mag.* 12 (1906).] As he suggests, the positions of the dead points give a means of determining the internal resistance of a condenser. If R_1 be this resistance, and Γ be that of the galvanometer and leads, we have $(R_1 + \Gamma)\int_0^\infty \gamma^2 dt = Q^2/2K$. Let the dead point, when only the resistance Γ is in circuit, be at a distance D_1 from the symmetrical point, and at a distance D_2 when a resistance R is in series with Γ. Then $R_1 = RD_1/(D_2 - D_1) - \Gamma$. The reader should consult Dr. Russell's paper for further particulars and results.

41. Elimination of constant, etc., for ballistic galvanometer. The value of the ratio HG may be found by sending a steady current of known amount γ (determined by electrolysis as explained in p. 464 below, or by a standard galvanometer, or current balance) through the instrument and observing the deflection of the needle. If the indications follow the tangent law, and θ be the deflection, then $H/G = \gamma/\tan\theta$.

If the indications do not follow the tangent law the instrument can be calibrated by sending steady currents of different values through the coil, observing the deflections and interpolating for other currents by means of a curve plotted from the observations, or otherwise.

A condenser of known capacity C charged to a difference of potential V measured by some proper arrangement, may be discharged through the galvanometer and the deflection observed. This gives a known value of Q, and the value of HT_1/G can therefore be obtained by (41) or (41′).

These methods and others will be exemplified below, especially in Chapters XVI. and XVII.

42. Observations by the method of recoil. In cases in which the transient current can be repeated when desired, successive observations may be made without waiting for the needle to come to rest, by using the method of recoil proposed by Weber. The current is first sent in the positive direction round the coil, and the needle thereby caused to swing to its maximum deflection in the positive direction, then through zero to the negative side and back again to zero. At the instant when the needle arrives at zero the second time, the transient current is repeated but in the negative direction, thus reversing the motion of the needle, which swings to a maximum deflection on the negative side, then back again through zero to the positive side. When the needle returns to zero from the positive side, the transient current is repeated, but in the positive direction and so on, a fresh impulse being given in the opposite direction to motion every time the needle arrives at the zero position after a complete free swing from side to side. The angular deflections are shown in Fig. 124.

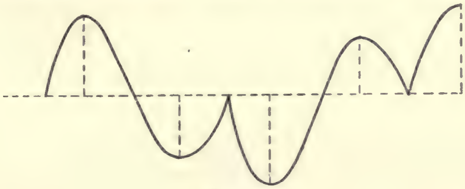

FIG. 124.

By equation (40) the first deflection θ_1 is given by the equation

$$\theta_1 = \frac{2GQ}{HT_1}\sqrt{\pi^2+\lambda^2}\exp\left(-\frac{\lambda}{\pi}\tan^{-1}\frac{\pi}{\lambda}\right) = KQ. \quad \text{..........} \quad (45)$$

When the magnet swings over to the other side, the numerical value of the deflection θ_2 will be given by

$$\theta_2 = KQe^{-\lambda}.$$

By (39) the angular speed with which the needle starts is MGQ/mk^2, and that with which it returns to zero is $MGQe^{-\lambda}/mk^2$. Hence its (positive) angular speed, when it returns to zero the second time, is

$MGQe^{-2\lambda}/mk^2$. The negative angular speed given then is MGQ/mk^2, so that the speed is now numerically $MGQ(1-e^{-2\lambda})/mk^2$ in the negative direction. This will give a deflection in the negative direction of amount θ_3, where
$$\theta_3 = KQ(1-e^{-2\lambda}).$$

The next following amplitude will be positive, and will have the value
$$\theta_4 = KQ(1-e^{-2\lambda})e^{-\lambda}.$$

Lastly, the velocity with which the needle returns to zero from the positive side is $MGQ(1-e^{-2\lambda})e^{-2\lambda}/mk^2$, and the positive velocity then imparted being MGQ/mk^2, the velocity towards the positive side is $MGQ\{1-(1-e^{-2\lambda})e^{-2\lambda}\}/mk^2$, and the deflection θ_5 is given by
$$\theta_5 = KQ(1-e^{-2\lambda}+e^{-4\lambda})e^{-\lambda},$$
and so on.

43. Combination of results of method of recoil. We have for the first group of four deflections
$$\frac{\theta_4-\theta_2}{\theta_3-\theta_1} = e^{-\lambda}, \quad \text{.............................}(46)$$
and the same thing will be given by every succeeding group of four deflections. Hence, taking all such groups into account, we find
$$\frac{\Sigma\theta_4-\Sigma\theta_2}{\Sigma\theta_3-\Sigma\theta_1} = e^{-\lambda} \quad \text{...........................}(47)$$
which gives the logarithmic decrement.

Again, from the values of the deflection found above, we have
$$KQ(1+e^{-\lambda})(1-e^{-2\lambda}) = \theta_3+\theta_4,$$
$$KQ(1+e^{-\lambda}) = \theta_1+\theta_2.$$
Hence
$$KQ(1+e^{-\lambda}) = (\theta_1+\theta_2)e^{-2\lambda}+\theta_3+\theta_4,$$
$$KQ(1+e^{-\lambda}) = (\theta_3+\theta_4)e^{-2\lambda}+\theta_5+\theta_6,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$KQ(1+e^{-\lambda}) = (\theta_{4n-3}+\theta_{4n-2})e^{-2\lambda}+\theta_{4n-1}+\theta_{4n},$$
supposing $4n$ deflections to be observed. Adding the last set of equations, we obtain
$$4nKQ(1+e^{-\lambda}) = \sum_{j=1}^{j=4n}\{(\theta_{j-3}+\theta_{j-2}+\theta_{j-1}+\theta_j)(1+e^{-2\lambda})-\theta_1-\theta_2 - (\theta_{4n-1}+\theta_{4n})e^{-2\lambda}\}, \quad \text{..........}(48)$$
which enables Q to be found from a combination of all the observations made.

It is to be observed that this method cannot be conveniently used if the damping of the needle is very small, as then a regular repetition of successive sets of nearly the same amplitudes would be difficult to obtain. By observing the successive pairs of free elongations any change of zero which takes place during the experiments can be followed.

Formulae are easily obtained for taking into account the interval occupied in the passage of the current, if that is in the least comparable with the free period of the needle; but, as these are rarely necessary, we shall only give them if the need arises in connection with any electrical measurement described below.

We only note further here that when a galvanometer is used for the measurement of a steady current, it may sometimes be desirable, in order to eliminate any variation of zero due to variation in the direction of the earth's force, to read the galvanometer as follows. The current sent round the coil of the galvanometer in the positive direction deflects the needle, which swings about the new position of equilibrium. The first, second, and third elongations are observed; then contact is broken for about half a whole period, so as to let the needle swing beyond zero, next the current is sent in the opposite direction to that in which it was sent at first, and the three first elongations on the other side observed; then the contact is broken, the current reversed, and so on as before.

If the numerical values of the first six deflections are $\theta_1, \theta_2, \ldots, \theta_6$ we have for the deflection due to the steady current

$$\theta = \frac{\theta_1 + 2\theta_2 + \theta_3}{4} = \frac{\theta_4 + 2\theta_5 + \theta_6}{4}$$

or

$$8\theta = \theta_1 + 2\theta_2 + \theta_3 + \theta_4 + 2\theta_5 + \theta_6, \quad \ldots\ldots\ldots\ldots\ldots\ldots(49)$$

and so for any such series of six deflections.

Some account of methods of measuring currents, differences of potential, etc., in alternating circuits will be given in a later chapter. Many particular devices and arrangements which might have legitimately found a place in this chapter will be much more conveniently described in connection with the experiments in which they were originally used.

44. Moving coil galvanometers with iron cores in the coils. The reader is reminded that in the above discussion of the ballistic action of galvanometers, the moving coil instruments are supposed to have no iron cores in their coils. If they have iron cores the relation of current to deflection may be such as to modify the formulae given. If the rectangular coil have a cylindrical iron core and symmetrical cylindrical pole pieces on the magnet the field will be nearly radial, so that for a steady current on the coil the couple will be independent of the deflection. The return couple is due to torsion of the suspension. In a short time dt a quantity dQ of electricity passes through the coil. This is of course $\gamma\, dt$, and if $C\gamma$ be the couple on the coil the angular momentum produced in dt is $C\, dQ$. Hence $C\, dQ = mk^2 d\omega$ if mk^2 be the moment of inertia and $d\omega$ the increment of angular speed.

For a deflection θ the return couple is $\tau\theta$, where τ is the torsion constant. Hence the period of oscillation of the coil without current in it is $2\pi\sqrt{mk^2/\tau}$. Moreover when equilibrium has been reached at deflection θ the work done on the coil is all represented by $\frac{1}{2}\tau\theta^2$, and so $\frac{1}{2}\tau\theta^2 = \frac{1}{2}mk^2\omega^2$. If ϕ be the steady deflection due to a constant current γ in the coil $C\gamma = \tau\phi$, and so

$$Q = \frac{mk^2}{C}\omega = \frac{T\tau\theta}{2\pi C} = \frac{T\theta}{2\pi\phi}\gamma. \quad \text{......................(50)}$$

The other form of moving coil galvanometer, in which the coil has no iron core, has been discussed above. If there is an iron core the instrument requires more exact calibration for use with currents of considerable strength than seems to be contemplated in the above discussion, which is that given in a paper by Prof. H. A. Wilson, *Phil. Mag.* 12 (1906).

45. Current weighers for the absolute measurement of currents. We shall now give an account of the absolute measurement of currents by the form of electrodynamometer in which the force between two coils, a fixed coil and a movable one, is determined from the weight required to equilibrate the latter in a certain zero position, and the dimensions and windings of the coil. This method seems to have been first used by Joule. Lord Rayleigh and Mrs. Sidgwick have used in their researches on the electrochemical equivalent of silver a form of electrodynamometer balance, or current-weigher, in which the fixed and movable coils were placed with their axes coincident, and in such relative positions that the pull along the axis exerted by one coil-system on the other was a maximum. The fixed coils were the large coils of the British Association electrodynamometer described above, and between these was placed a coil of silk-covered wire wound on a ring of ebonite. The arrangement is shown in Fig. 125, which explains itself. We shall show that this coil placed midway between the two fixed coils was in the position to have maximum force exerted upon it by each of the latter coils.

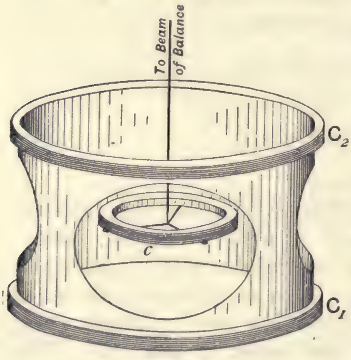

FIG. 125.

The use of a current-weigher such as this has some advantages over either the galvanometer or ordinary electrodynamometer. As here arranged, the accuracy of the constant depended, in the main, only on the determination of the ratio of the radii of the coils; the necessity for finding H and taking account of its variations is avoided; and no difficulty as to the elastic or bifilar constant of suspensions exists. The actual observation of the indications is, however, a somewhat more elaborate process than in these other instruments, involving as it does an exact weighing. It can, however, be carried out with great accuracy by a skilled experimenter.

46. Attraction between two parallel coils. The mutual electrokinetic energy T_m of a system of two coils carrying a current γ is given by the equation

$$T_m = nn'\gamma^2 M, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(51)$$

where n, n' denote the numbers of turns in the two coils, and M denotes for the present the mean mutual inductance of a pair of turns one in each coil. Thus if x is the distance between the coils, the force F exerted by one on the other is given by

$$F = nn'\gamma^2 \frac{\partial M}{\partial x}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(52)$$

It is well to notice here that $\partial M/\partial x$ is a mere number, and depends therefore only on the ratios α/a, x/α, (or x/a), of the radii of the coils, and of the radius of either to their distance apart. Thus if we write

$$\frac{\partial M}{\partial x} = f(a, \alpha, x), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(53)$$

f is a homogeneous function of zero dimensions in a, α, x. Thus we have

$$df = \frac{\partial f}{\partial a}da + \frac{\partial f}{\partial \alpha}d\alpha + \frac{\partial f}{\partial x}dx, \quad \ldots\ldots\ldots\ldots\ldots\ldots(54)$$

with, by Euler's theorem of homogeneous functions, the condition

$$a\frac{\partial f}{\partial a} + \alpha\frac{\partial f}{\partial \alpha} + x\frac{\partial f}{\partial x} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(55)$$

If the coils are so placed that the action between them is a maximum $\partial f/\partial x = 0$, and (55) gives

$$a\frac{\partial f}{\partial a} + \alpha\frac{\partial f}{\partial \alpha} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(56)$$

Thus by (54) equal (proportional) errors in the estimation of a and α produce no effect on the value of f provided the coils are in this position. Hence $\partial f/\partial x$ being zero there is (to quantities of the second order) no effect produced by errors in the estimation of x, and therefore the action between the coils depends only on the ratio α/a. This ratio as will be explained below, can be determined electrically, without direct measurement of either α or a.

The value of M for different arrangements of coils is given in Chaps. VI. and VII. above. We shall use at present the expression given in VI. 21 (63) for the mutual induction of two coaxial circles of radii a, α, and distances x, ξ, from a fixed point on the axis. We have thus

$$\frac{\partial M}{\partial \xi} = \pi^2\frac{a^2\alpha^2}{r^4}\left\{1\,.\,2\,.\,3\frac{x}{r} + 2\,.\,3\,.\,4\frac{x^2-\frac{1}{4}a^2}{r^3}\xi\right.$$
$$\left. + 3\,.\,4\,.\,5\frac{x(x^2-\frac{3}{4}a^2)}{r^5}(\xi^2-\tfrac{1}{4}\alpha^2) + \ldots\right\}, \quad \ldots\ldots(57)$$

where $r^2 = a^2 + x^2$. Here α, ξ are supposed to belong to the small coil, and to be considerably less than a, x, respectively. Thus if α/a is not large, the value of $\partial M/\partial\xi$ will be given to a high degree of approximation by the first term alone of this series. Thus writing f' for $\partial M/\partial\xi$ we have, taking the first term only,

$$f' = 1\,.\,2\,.\,3\pi^2 a^2 \alpha^2 \frac{x}{r^5},$$

and therefore

$$\frac{\partial f'}{\partial x} = 1\,.\,2\,.\,3\pi^2 a^2 \alpha^2 \frac{a^2 - 4x^2}{r^7}.$$

Thus $\partial f'/\partial x$ vanishes and the force is a maximum if $a^2 = 4x^2$ or $2x = a$, that is when the distance between the circles is half the radius of the larger.

Neglecting the second and third terms in (57) which involve ξ, and taking into account the part of the third term which involves α^2, differentiating and putting $x^2 = \frac{1}{4}a^2$ in all factors multiplying α^2, we get as a second approximation to the value of x for a maximum,

$$x = \tfrac{1}{2}a\left(1 - \frac{9}{10}\frac{\alpha^2}{a^2}\right). \qquad \text{(58)}$$

47. Force on movable coil between two fixed coils. For two fixed coils at equal distances on opposite sides of the suspended coil the odd terms vanish, and we have (still supposing that the coils can be regarded as circles) for the action between *one* of the fixed coils and the movable one,

$$\frac{\partial M}{\partial \xi} = \pi^2 \frac{a^2\alpha^2}{r^4}\left\{1\,.\,2\,.\,3\frac{x}{r} + 3\,.\,4\,.\,5\,\frac{x(x^2 - \frac{3}{4}a^2)}{r^5}(\xi^2 - \tfrac{1}{4}\alpha^2) + \ldots\right\}. \qquad \text{(59)}$$

The coils might then be arranged so that $x^2 = \frac{3}{4}a^2$, and thus to terms of the fourth order in α, ξ, the value of $\partial M/\partial\xi$ would be given by

$$\frac{\partial M}{\partial \xi} = 1\,.\,2\,.\,3\,\frac{2^4\sqrt{3}}{7^{\frac{5}{2}}}\pi^2\frac{\alpha^2}{a^2} = \cdot 21375 \times 6\pi^2\frac{\alpha^2}{a^2}. \qquad \text{(60)}$$

On the other hand, if, as was actually the case, $x = \frac{1}{2}a$,

$$\frac{\partial M}{\partial \xi} = 1\,.\,2\,.\,3\,\frac{2^4}{5^{\frac{5}{2}}}\pi^2\frac{\alpha^2}{a^2} = \cdot 2862 \times 6\pi^2\frac{\alpha^2}{a^2}, \qquad \text{(61)}$$

a considerably larger value. This equation multiplied by γ^2 gives a rough estimate of the force which would be produced by a given current with two single turns, and therefore of the force to be expected between one of the fixed coils and the movable coil.

By equation (57), when $a = 2x$, we have, including two terms so as to find the effect of ξ^2 when this is not zero,

$$\frac{\partial M}{\partial \xi} = \cdot 2862 \times 6\pi^2\frac{\alpha^2}{a^2}\left(1 - 3\cdot 2\frac{\xi^2}{a^2}\right). \qquad \text{(62)}$$

In the current-weigher used a was 25 cm, so that $\xi = 1$ mm, and neglected could only give rise to an error of about 1/20,000. Thus the instrument with ordinary care as to adjustment could be regarded as quite free from error due to inaccurate placing of the suspended coil.

48. Force on movable coil in terms of ratio of coil-constants. As the ratio of the galvanometer constants was determined experimentally, and therefore was used in the calculations, we write down here the approximate expression for the force between one fixed coil and the suspended coil in terms of this ratio and the numbers of turns. Putting β for the value of the ratio we may write approximately

$$\beta = \frac{\frac{n}{a}}{\frac{n'}{\alpha}} = \frac{n}{n'} \cdot \frac{\alpha}{a}$$

or

$$\beta^2 \frac{n'^2}{n^2} = \frac{\alpha^2}{a^2}.$$

Thus approximately, by (62) and (52),

$$F = {\cdot}2862 \times 6\pi^2 \frac{n'^3}{n} \beta^2 \gamma^2.$$

An error in the estimation of n', the number of turns in the suspended coil, or, what is the same, any defect in the insulation of that coil, is thus of greater importance than a similar inaccuracy in the estimation of n.

The ratio β enabled the mean radius of the suspended coil to be calculated. The attraction between the coils was then found by an expression easily obtainable by differentiation from the value of M given in elliptic integrals in VI. 10 above, for two coaxial circular conductors. Thus we have

$$\frac{\partial M}{\partial x} = \frac{\pi x \sin \eta}{\sqrt{a\alpha}} \{2G - (1 + \sec^2 \eta) H\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (63)$$

where $\sin \eta = k$. G and H have been calculated by Legendre, and were used by Lord Rayleigh in the formation of a table of values of

$$\pi x \sin \eta \, \{2G - (1 + \sec^2 \eta) H\}$$

for values of η proceeding by intervals of 6′ from 55° to 70°.

The value of $\partial M/\partial x$ was then found for the actual coils of axial breadths $2b, 2\beta, 2d, 2\delta$ by employing the following formula of quadrature,* and multiplying by nn' the product of the numbers of turns.

---

* This formula, it may be here remarked, is applicable not only to M/x, but to any function of a, α, the mean radii. Thus it is used in XIII. 31, to give M for two coils for which $f(a, \alpha, x)$ denotes its value for the mean radii.

Thus $f(a, a, x)$ being the value of $\partial M/\partial x$ for a pair of mean turns, we have for the whole coils,

$$\frac{\partial M}{\partial x} = \tfrac{1}{6}nn' \left\{ \begin{array}{c} f(a+d, a, x) + f(a-d, a, x) \\ + f(a, a+\delta, x) + f(a, a-\delta, x) \\ + f(a, a, x+b) + f(a, a, x-b) \\ + f(a, a, x+\beta) + f(a, a, x-\beta) \\ -2f(a, a, x) \end{array} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (64)$$

[Maxwell, *El. and Mag.*, § 706, App. 2; see also XIII. 31 below].

49. Tests of insulation and particulars of coils. We can now proceed to give an abstract of the experimental processes and results.

The suspended coil, C, of the current-weigher, which had been carefully wound with silk-covered wire on a ring of ebonite, was tested for insulation. The method adopted first was to make as nearly as possible an exact copy of the coil, then to place the coil and its copy side by side with their axes in coincidence, and join them in series so that a current could flow through them in opposite directions. A galvanometer with a needle of long period of free vibration was included in their circuit One pole of a very long steel magnet was then thrust suddenly through the opening of the coils, and produced in them opposite induced currents, which, if the insulation had been perfect in both coils, ought to have together produced no effect on the needle of the galvanometer.

It was found however that the copy decidedly preponderated in magnetic effect; a result which pointed to faulty insulation in the ebonite coil. A comparison of the ratios of the self-inductions of the separate coils to the mutual induction of the pair in a fixed position confirmed this conclusion, and the coil was thereupon rewound.

After rewinding it was tested for insulation by a Hughes' induction balance. This consisted of two pairs of coils, one pair at some distance apart in one horizontal plane being joined up with a source of variable current in a primary circuit, the other pair in positions opposite the primary coils, and at distances finely adjustable by means of screws, being joined up with a telephone as a secondary circuit. When the coils had been adjusted to exact balance the introduction of a small circlet of copper ·004 inch in diameter between a primary and a secondary coil gave a very distinct sound.

The ebonite coil placed between one of the primary coils and its opposite secondary gave an audible sound, but much less than that occasioned by the copper circlet. When the ends were joined by a megohm of resistance the increase of sound was quite distinct; which showed that the insulation-resistance was decidedly greater than a megohm, and therefore amply sufficient.

The particulars of the suspended coil were as follows :

Number of turns	242.
Radial depth 2δ	·9690 cm.
Axial breadth 2β	1·3843 cm.
Mean radius, found electrically as described below	10·2473 cm.

The coil was made of copper wire insulated with silk saturated by paraffin wax. Its resistance was about 10½ ohms.

The particulars of the fixed coils, C_1, C_2, as derived mainly from a record in Clerk Maxwell's handwriting in the Cavendish Laboratory note-book, were as follows :

Number of turns in each	225.
Mean radius, a	24·81016 cm.
Distance of mean planes, $2x$	25·000 cm.
Radial depth, $2d$	1·29 cm.
Axial breadth, $2b$	1·50 cm.
Resistance of each coil (about)	14½ B.A. units.

By measuring the distances from outside to outside, and from inside to inside, of the grooves filled with wire, the distances of the mean planes was found to be 25 cm exactly. The half-difference between these distances gave $2b = 1{\cdot}5024$ cm. The mean radius and number of turns could not be verified, but the recorded value of the former agreed with the outside circumference, and the check on the counting of the number of turns given by the device adopted when the coil was being wound, of at the same time winding string on a drum turning with the coil, almost absolutely ensures the accuracy of the number given.

50. Experimental determination of ratio of coil-constants. The ratio of the radii was found as follows. One of the dynamometer coils, and the suspended coil, were made concentric and coaxial with their planes vertical in the magnetic meridian, and a small needle was hung at the common centre. A diagrammatic sketch of the arrangements is shown in Fig. 126. *D* is the dynamometer coil, *E* the ebonite coil, *N* a resistance box. When the thick copper piece *P* was made to join the mercury cups *F*, *H*, the current from a cell *A* was divided between the two coils, which were joined so that the current flowed round them in opposite directions. The reversing key *B* enabled the current to be sent first in one direction then in the other through the double arc.

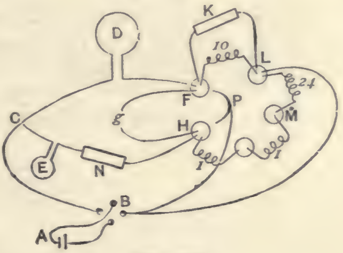

FIG. 126.

By means of *N* the resistances of the arcs *CDP*, *CEP*, joining *C* and *P*, were adjusted so that no deflection of the needle took place. It was found that the resistance taken from *N* which gave balance could not be exactly determined, owing to inductive effects produced by the reversal of the current. Readings of the deflections of the needle were therefore taken for imperfect adjustments, with values of the resistances on opposite sides of the required value, and the value for balance was obtained from these by interpolation.

The ratio of the resistances of the double arc was then obtained by making the two arcs adjoining branches of a Wheatstone bridge. This was done by withdrawing the copper piece P, which had the effect of converting the arrangement into a Wheatstone bridge of which one pair of adjoining branches were D and E, N, connected at C, the other pair a series of three resistance coils (composed of two single units and a 24-unit coil) and a coil of 10 units with its terminals connected by a high-resistance coil K. These branches were connected with one another at L, and with the other pair at the cups F, H. The battery terminals were attached at C, L, and those of a sensitive testing galvanometer, g, at F, H. Thus the ratio of the resistances was determined, and for one dynamometer coil was found to be on three different occasions 2·60087, 2·60098, 2·60113, or a mean of 2·60099. The same coil tested with another set of resistances gave on two occasions in like manner 2·60046, 2·60026, or a mean of 2·60036. The mean was thus 2·60067. For the other coil 2·60072 was found.

If G_1, G_1' be the galvanometer constants of the two coils, γ, γ', the currents flowing in them when their conjoint magnetic effect at the centre was zero, we have $nG_1\gamma$ and $n'G_1'\gamma'$ for the magnetic effects due to the coils, and $nG_1/n'G_1' = \gamma'/\gamma$. But if R, R', be the resistances of the branches, $\gamma/\gamma' = R'/R$, and therefore

$$\frac{nG_1}{n'G_1'} = \frac{R}{R'}. \qquad \text{.............................. (65)}$$

But using for each coil the value of G_1 given at p. 386 above, putting $x=0$, $\xi=0$, since it is the magnetic forces at the common centre that are in question, we find

$$G_1 = \frac{2\pi}{a}\left(1 + \frac{1}{3}\frac{d^2}{a^2} - \frac{1}{2}\frac{b^2}{a^2}\right),$$

$$G_1' = \frac{2\pi}{\alpha}\left(1 + \frac{1}{3}\frac{\delta^2}{\alpha^2} - \frac{1}{2}\frac{\beta^2}{\alpha^2}\right)$$

or

$$\frac{a}{\alpha} = \frac{nR'}{n'R}\,\frac{1 + \frac{1}{3}\frac{d^2}{a^2} - \frac{1}{2}\frac{b^2}{a^2}}{1 + \frac{1}{3}\frac{\delta^2}{\alpha^2} - \frac{1}{2}\frac{\beta^2}{\alpha^2}}. \qquad \text{......................(66)}$$

Now the known values of a, b, d, β, δ, and the approximately known value of α gave at once the value 1·001296 for the second fraction on the right of the last equation. Hence

$$a = \frac{225}{242} \times 2{\cdot}60070 \times 1{\cdot}001296 \times \alpha = 2{\cdot}42114\alpha \qquad \text{...........(67)}$$

51. Adjustment of suspended coil. Final result. The suspended coil was adjusted in position in the current-weigher by first suspending it in a horizontal position, and then levelling and otherwise adjusting the positions of the dynamometer coils. A movable piece stood on three feet on the top of the upper dynamometer ring, and in every position touched its inner cylindrical face in other two points. This piece was moved round the coil, and carried with it a pointer which thus described a circle coaxial with the fixed coils. When the latter were properly placed the pointer just played exactly round the outer surface of the suspended coil.

The level of the suspended coil was adjusted by carrying along the upper face of the upper dynamometer ring a straight rule provided with a pointer which just reached down and touched the upper surface of the suspended bobbin when that was in the proper position. The level of the dynamometer coils was changed until this point when moved about just scraped over the upper surface of the suspended coil.

The value of $f(a, a, x)$ was $\pi \times 1{\cdot}044576$. From this, by the table of values of the elliptic integral expression referred to above, the terms of the expression on the right of (64) were calculated and gave

$$\frac{\partial M}{\partial x} = \pi n n' \times 1{\cdot}044627, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(68)$$

where n, n' are the *total numbers* of turns in the two coils.

If in any experiment the current was γ, the attraction or repulsion between each fixed coil and the suspended coil was $n n' \gamma^2 f$. If m denote the observed difference of the weights applied before and after the reversal of the current,

$$4 n n' \gamma^2 f = mg \times {\cdot}99986,$$

where ·99986 is the correcting factor for the air displaced by the weights m, and g is the acceleration produced by gravity at the place of experiment. This was taken as 981·2822 in centimetre-second units. Hence, m being taken in grammes,

$$\gamma^2 = \frac{981{\cdot}2282 \times {\cdot}99986}{4 \times 225 \times 242 \times 1{\cdot}044627} \frac{m}{\pi}$$

or

$$\gamma = {\cdot}037048\sqrt{m}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(69)$$

52. Current balance of the Bureau of Standards. A very exact Rayleigh current balance has been made at the Bureau of Standards at Washington, and is described in the *Bulletin* of the Bureau, 8 (1912), in a paper by Messrs. Rosa, Dorsey, and Miller. The chief parts of the balance and its arrangement are shown in Figs. 127, 128, 129. The balance was a 2 kg precision balance by Rueprecht with a 30 cm beam. Certain magnetic portions of the balance were removed and replaced by brass or phosphor bronze. All parts of it were tested and found to be non-magnetic. The weight of the moving coil and suspension system was about 1 kg, and with this load the time of a single swing was 15 seconds. It will be seen that only one suspended coil placed in the position for maximum force between two fixed coils was used. It occupied one side of the balance, the scale pan for the weights occupied the other side.

The coils were all wound on brass bobbins with enamel insulated wire. The section of the fixed coils is shown in Fig. 127. The winding was bifilar (or double) to enable the insulation to be tested by a determination between the two windings, and to allow the coils to be joined in series or in parallel, so that different currents might be used with the same heating effect, and, finally, to allow the coils to be set against each other, so that the full heating effect could be produced without any magnetic action.

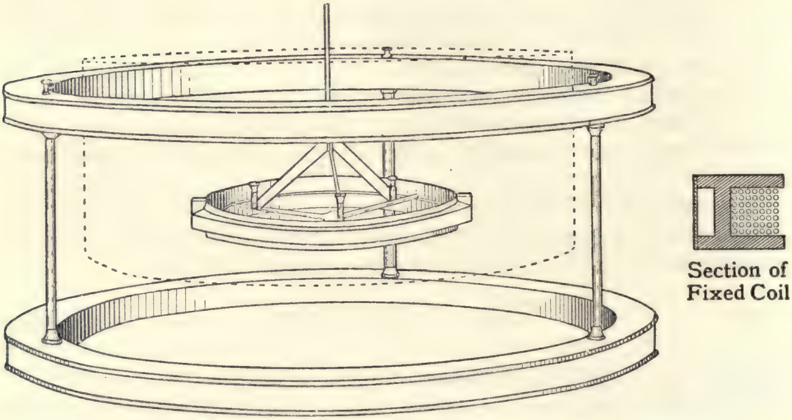

FIG. 127.

Three pairs of fixed coils were made. The details of one section are shown in Figs. 129, 130, 131. In each of the larger coils were 36 layers of 18 double turns in each layer, and in each of the smaller coils 28 layers of 14 double turns. The enamel insulated wire was of uniform thickness and could be wound as regularly as bare wire. When the coil was dry the insulation was very good. At first the coils were not sealed airtight; the channels for the wire were lined with paper attached with thin shellac, and each coil was covered with a strip of glazed paper ·05 mm thick. Finally, the outer paper covering was saturated with melted paraffin, and wrapped round first with muslin then with binders' cloth, all saturated with melted paraffin, so that the coils were effectually sealed against the absorption of moisture from the air.

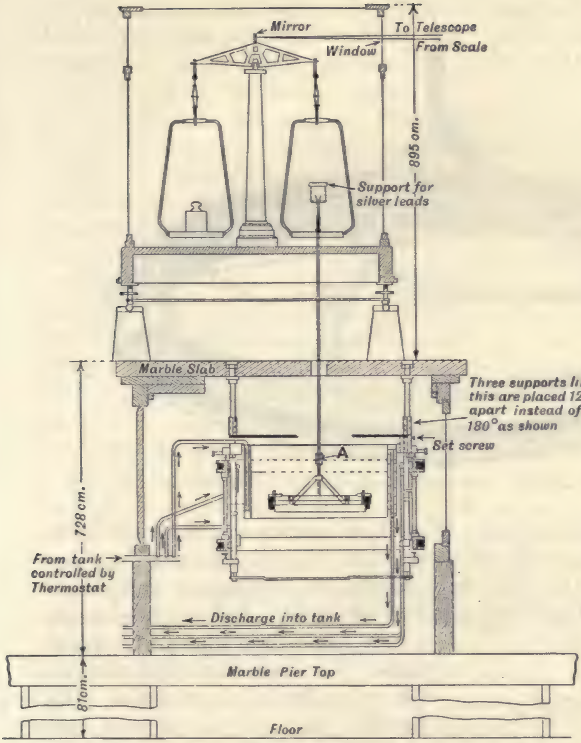

FIG. 128.

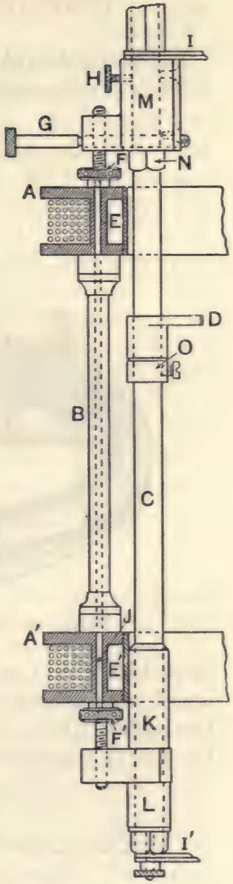

FIG. 129.

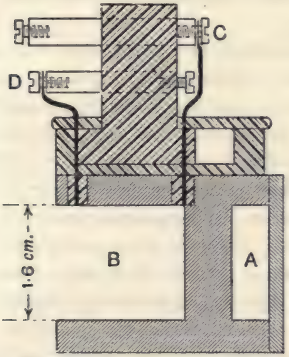

FIG. 130.—Section of small fixed coil showing first form of terminal block. *A* is water channel, *B* the channel for the wire.

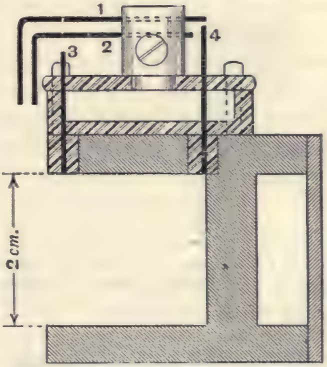

FIG. 131.—Section of large fixed coil showing second form of terminal block. Connections are made by drops of solder.

53. Particulars of coils. Four moving coils were built at various times. Like the fixed coils they were all wound double, on brass bobbins finished dead black, and had the form of section shown in Figs. 130, 131, which also shows the mode of connection between the windings and the leads. Three of the coils were wound on bobbins of cast brass, the fourth was wound on a bobbin of rolled brass. The windings in the latter were not uniformly distributed, but were crowded towards the two sides of the channel. Hot paraffin was painted on and into each layer. The construction of the moving coil ring is shown in Fig. 132.

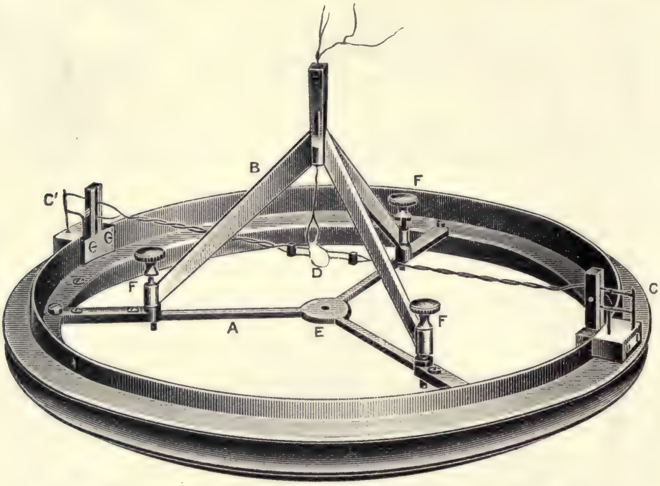

FIG. 132.—Moving coil, showing the leads, the star, and the tripod.

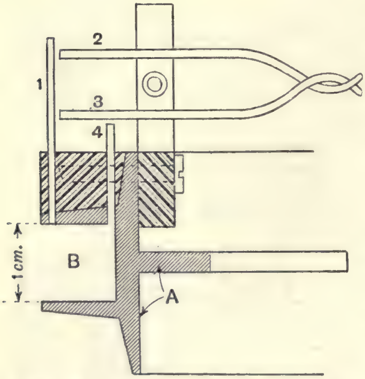

FIG. 133.—Form of moving coil showing third form of terminal block. Connections are made by drops of solder. *A* are stiffening flanges, *B* is channel for wire.

The Rayleigh form of balance has the great advantage that it is possible to provide both fixed and movable coils with channels, behind the slots filled with wire, through which water can be made to circulate. Water cooling was provided in this way for all the coils, and water was forced through the channels by an electrically driven turbine pump, with three pipes, supplying the coils in parallel. The temperature of the water was thermostatically controlled, and it was found that the temperatures of all three coils could be kept practically the same, and constant. During the weighings the moving coil was surrounded with a cylindrical copper jacket double walled at the sides, closed at the bottom and covered with a lid with a hole about 1 cm in diameter at the centre through which passed the tube from which the moving coil was suspended. The space between the walls of this enclosing cylinder was filled with circulating water.

The coils were enclosed in a case resting on a marble slab, 152 by 76 by 7·5 cm, supported by heavy oak piers standing on the floor. White enamelled brick, instead of oak, was tried at first, but was found, as also the sand in the cement, to be slightly magnetic. The iron in the floor construction was found to produce no effect on the measurements of currents.

The two windings of each fixed coil were connected in parallel by a pair of enamel insulated wires, closely twisted, which passed halfway round the outside of the coil: a similar pair of leads ran from the nearer terminals to binding posts set in the wall of the coil case, and connected with a commutator on the outside, by which the coils could be joined up in any desired way.

54. Theory of the balance. Comparison of coils. The theory of such a balance as that described is exactly the theory given above for Lord Rayleigh's instrument. The ratio of the galvanometer constants G_1, G_2 of the two coils was determined by the method described in 50 as used by Lord Rayleigh, with one modification. The link used by Rayleigh to convert the resistances of which the ratio is desired into two adjacent arms of a Wheatstone bridge was omitted, so that a simultaneous balance of both the bridge and the magnetometer (the small needle hung at the centre as described above, *loc. cit.*) was obtained. Thus at the instant of balance the ratio of the resistances in the coil arms was exactly that of the other two arms of the bridge. These had to be of low resistance, as they carried the full currents in the coils, have a low temperature coefficient, and be capable of fine adjustment. To meet the difficulty of fulfilling these conditions, an arrangement was made for quickly transferring the coil arms, to a second bridge in which they could be measured against precision resistances. This process however was found slow, and trouble was given by heating in the coils, and the current being alternately on and off, never allowed the coils to attain a stationary temperature. Consequently the ratio of the two currents was measured finally by the potentiometer method. This will be found described in the chapter on the Comparison of Resistances, where the method used for this balance will be given for illustration, and all the needful adjustments will be described in detail.

55. Calculation of forces. The calculation of the force for the coils, supposed arranged at the distance for maximum force according to 46 above, was carried out as follows. Two assumptions were made in the computation, (1) that the coils were equivalent in their action to coils of square cross-section and of the same mean radii and sectional area as the actual coils, (2) that any such coil produced the same effect as the "equivalent circular current" as defined by Lyle (see XIII. 32 below). That is to say, each coil-current was regarded as a circular current in the mean plane of the coil, having a radius A_e given by

$$A_e = A + \frac{bd}{6A}, \qquad (70)$$

where A is the mean radius and $2b$, $2d$ are the dimensions of cross-section.

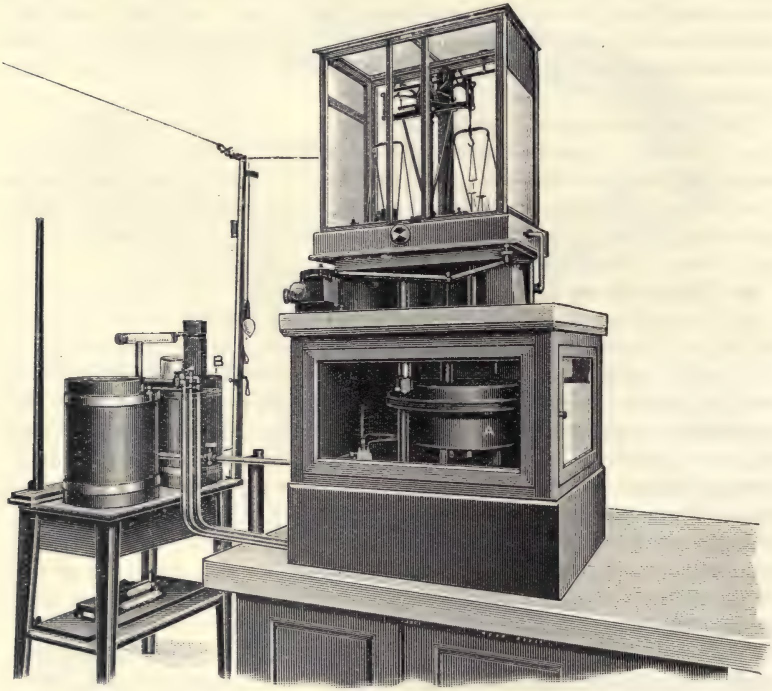

FIG. 134.—General view of current balance as used in the final measurements. The moving coil is suspended from the pan of the balance.

A is the water jacket surrounding the moving coil.

B is the tank containing the thermostat which controls the temperature of the water.

The forces were computed for all the combinations of coils used, and are given in the paper, and variation coefficients were calculated for slight deviations from the maximum force positions, and for errors in sectional dimensions, etc. For the results reference must be made to the paper.

56. Manipulation and weighings. In the manipulation of the balance the weight was changed, and the current reversed, without arresting the balance, to avoid slight changes of zero. The weight was lifted and placed in position without opening the balance case, the current reversed by a circular reversing switch with which the balance was provided, and the beam acted on by an air-jet produced by squeezing a rubber ball, all in such a way that the balance received no jar in the process. Any vibration of the scale pan and coil was stopped by touching with a camel's hair brush the tube by which the coil was suspended from the pan.

Weighings were made with the weight alternately on and off, for the current in the fixed coils alternately in one direction and in the other. This eliminated error from drifting of zero due to temperature changes, and the effect of the earth's force on the moving coil. The effect of the earth's field was thus reduced to a slight permanent displacement of the zero of the balance. The drift was eliminated by noting the successive rest-points, which generally lay on two parallel straight lines slightly inclined to the time axis.

Weighings were made for at least three positions of the movable coil, of which two were nearly equidistant from and on opposite sides of the position corresponding to the maximum force. Electrostatic force due to the differences of potentials on the coils was avoided by connection of the windings to the water jacket, by a wire from the commutator, so that the jacket and the metal framework were all kept at one potential.

57. Value of g. Accuracy of current measurement. The value of g was known from its value at the gravity pier of the United States Coast and Geodetic Survey, which had been carefully compared by means of pendulum observations with the value of g at Potsdam, so that the value of g at the Bureau was referred to Potsdam. It was estimated that at the balance in the Bureau Laboratory the value of g was 981·091 cm/sec^2.

The double force required for reversal of the current was, with one set of coils, $mg = 6000$ c.g.s. for ·84 ampere; with another set of coils, $mg = 6000$ c.g.s. for ·7759 ampere.

To obtain an idea of the accuracy of the work it may be stated that the electromotive force of the mean Weston cadmium cell as constructed at the Bureau, was found to be 1·01822 semi-absolute volts, that is 1·01822 × the difference of potential between the terminals of an international ohm when that carries a current of one absolute ampere. The probable error was estimated as about 3 parts in 1,000,000. It is believed by the authors that a cautious estimate of the uncertainty might be 2 parts in 100,000. See *Standard Cells* in the Appendices.

The ratio of the radii of the coils in the current balance, or rather of their galvanometer constants, was determined by a potentiometer method, which we shall here sketch as an example of potentiometer working. In the diagram the connections of the coils for the measurement are shown. The two coils—moving and fixed—are denoted by M and F. They are of course really coaxial and concentric, being arranged as if they were galvanometer coils, with a needle hung at their common centre. R_1, R_2 are two standard resistances from which leads run to the potentiometer by which the currents are compared.

[The elementary theory of a potentiometer is supposed to be understood. A current from a service battery is sent through a series of known resistances arranged to be varied by switches working round dials. Consequently there is a fall of potential along these resistances.

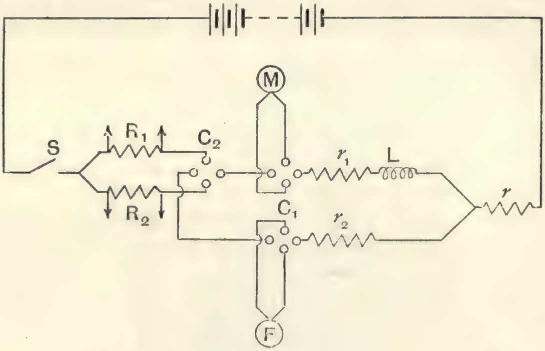

FIG. 135.

If the difference of potential V_{AB} between two points A, B, say, of that series is known, then from that difference and the resistance intercepted between the points the current C flowing can be determined, in absolute measure, since we have $CR_{AB}=V_{AB}$. The difference of potential is generally determined by means of a standard cell, the electromotive force of which has been determined. This cell has its terminals applied at A, B so that its electromotive force is opposed by V_{AB}. The resistance is varied until it is found that no current passes through a sensitive galvanometer in the derived circuit which the cell forms on AB. Care is taken of course by guarding the cell with a high resistance, and only tapping down for a moment the key which brings the cell into action, that no appreciable quantity of electricity is allowed to traverse the standard cell.]

From the resistance R_2, which was in series with the fixed coil, auxiliary leads were carried to a second potentiometer, by means of which and a continuously variable resistance in circuit with F, the current in R_2 and F could be made of any required value and kept constant. C_2 is a commutator which interchanged R_1, R_2 with reference to M and F, r, r_1, r_2 were adjusting resistances, C_1 was a commutator which reversed the current through the coils. A main switch opened the circuit before and opened it after C_1 was altered, so that large deflections of the galvanometer were avoided.

The coils M and F, being wound on metal bobbins, have each when alone a considerable time-constant [see VIII. 11, 16 above]. The field in M in the experiments now being considered was to a great extent neutralized by F; on the other hand, the field outside M and inside F was enhanced by M. Thus a deflection of the magnetometer needle hung at the common centre of the coils occurred whenever the current was started or stopped.

A large inductance L was placed in the moving coil circuit (at a considerable distance from the magnetometer needle to avoid direct effect) to reduce the violent deflection of the magnetometer needle caused by change of induction when the circuit of F is closed through M.

The procedure was as follows. The adjusting resistances r, r_1, r_2 were made to give fields in M and F approximately equal, and of such a strength that the sensibility of the magnetometer was such as to give a change of 1 mm in the reading for reversal with a difference of fields of about 3 parts in a million. The current was left running for an hour with water at the proper temperature circulating through the channels in the fixed coil. The resistances of the coils were measured, the thermometers in the coils read, one observer then kept the fixed coil current at the desired value, another connected a potentiometer across the standard resistance on F, and adjusted for the nearest balance with an even setting of the potentiometer dials, while the want of exact balance was measured by the galvanometer deflection and could be allowed for. This observer then placed his potentiometer across the standard resistance of the moving coil circuit, adjusted the current, " and then allowed it to drift slowly towards his potentiometer balance, while a third observer damped and read the magnetometer." This third observer signalled the instant of reading the magnetometer, and the second noted the galvanometer deflection at that instant, while all this time the first observer had held the current through F at a constant value.

The currents were then reversed on the two coils, and the operation repeated. After several pairs of such sets of operations, the second observer put his potentiometer on the standard resistance of F, and observed the deflection with the same setting of the dials as at first, and the first observer's indicating apparatus is balanced as at first. The slight difference between the new balance and the former one was due to change in the circuits during the time observation and was allowed for in the reduction. Then the thermometers were read, C_1 reversed so as to interchange the standard resistances on the coils M and F, and the operation repeated. The mean of these observations eliminated the values of R_1 and R_2, and it was unnecessary to know these values exactly.

58. Current balance of the National Physical Laboratory. A current balance constructed with great care and accuracy was completed for the National Physical Laboratory in 1907. We give here also a short account of this instrument.

At the Toronto meeting of the British Association in 1897 it was agreed by the Committee on Electrical Standards that it was "a matter of urgent importance that the general question of the absolute measurement of electric currents should be investigated." At the following meeting—at Bristol in 1898—it was reported that preliminary experiments with this object in view had been made during the year by Professor W. E. Ayrton and Professor J. V. Jones, on a form of current-weighing apparatus with single-layer coils, such as had been used by Jones in his Lorenz apparatus, which promised to give results of great accuracy. Accordingly a grant was made in aid of the construction of the proposed balance, and the work put in charge of a committee with Lord Rayleigh as chairman and Mr. R. T. Glazebrook as secretary. The elaborate new current balance now in use at the National Physical Laboratory was the result of the work of the following nine years. Prof. J. V. Jones died in 1901, and the work during the following years was a good deal delayed by the ill health of Professor Ayrton. A full account of the instrument was communicated to the Royal Society in 1908 by Professor Ayrton, Mr. T. Mather and Mr. F. E. Smith, under whose care the instrument had been constructed in the workshops of the National Physical Laboratory [*Phil. Trans. R.S.*, 207, 1908].

59. General description of the balance. The diagram of Fig. 136 shows the arrangement adopted. Below each extremity of the balance, which is a special balance of great strength and delicacy constructed by Oertling, and capable of carrying 5 kilogrammes at each end of the beam, are placed two pairs of fixed coils wound in opposite directions and connected as shown in the diagram. As will be seen, the upper and lower coils on each side are oppositely wound, while the windings in the two pairs are also opposed. Hung from the ends of the balance are two smaller coils, each consisting in the actual instrument of two helices, which in their zero positions are each symmetrically placed with respect to the pair of outer fixed coils. The action was therefore, when the current flowed as shown by the arrows, to lift one coil and depress the other.

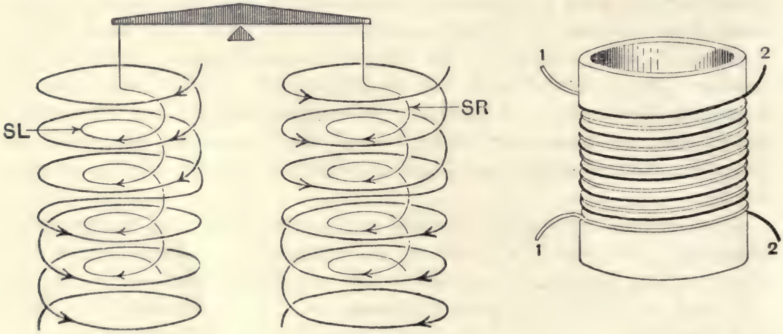

FIG. 136.

The couple deflecting the coils was balanced by the action of weights placed in scale pans independently supported on the balance, and the couple on the movable coils was thus obtained always for the same zero position.

The coils were all wound on hollow cylinders of marble, in double-threaded screw grooves cut on the surface, as shown in the diagram on the right of Fig. 136. In these were wound two helices of wire—one shown by a full line, the other by two thin lines—which were usually connected together in series to act as one coil, but which could be disconnected at any time to enable an insulation test to be made between them. The marble used for the coils had been very carefully tested for the possible presence of magnetizable matter, and found to be

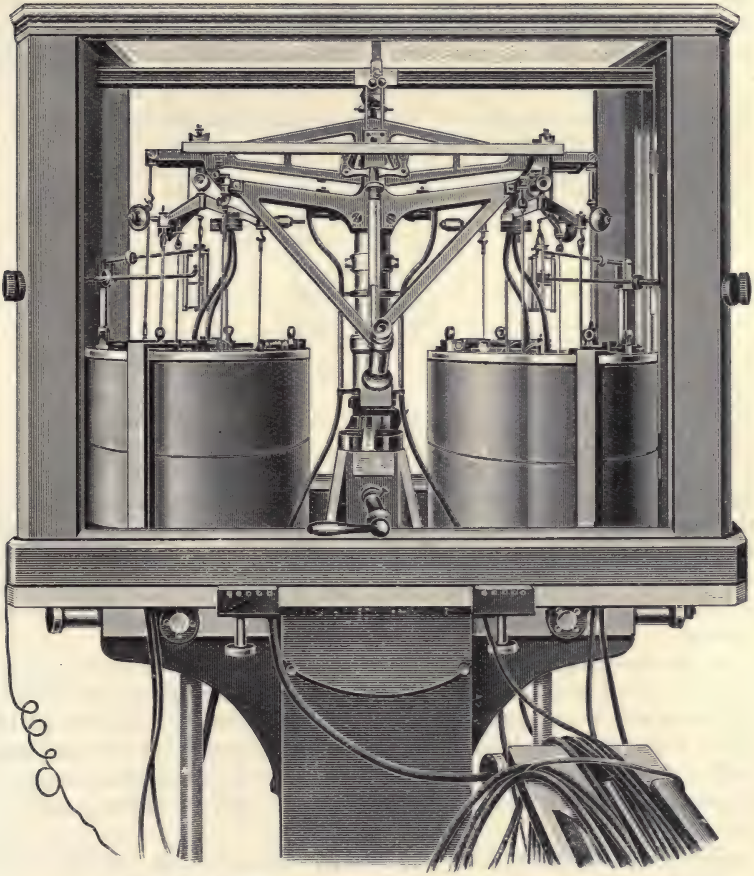

FIG. 137.—Complete current weigher (sides of cases removed).

practically free from any such substance. This was done by quickly inserting the cylinders when received from the marble merchants as cores of the secondary of an induction coil, and observing the deflection of a galvanometer in the secondary circuit.

Since each fixed cylinder carried four helices, two upper and two lower, and each suspended cylinder carried two, there were twelve helices in all. These in the normal use of the instrument were connected in series by concentric cables, carried to a plug board and commutators, arranged outside the balance case. The connections to the suspended coils were made by flexible conductors. It was possible, by means of the commutators, to reverse the current at will in any of the coils. If in the use of the instrument the current is reversed on the fixed coils the forces on the suspended coils are reversed, and a measure of the double forces is obtained, which is proportional to the square of the current.

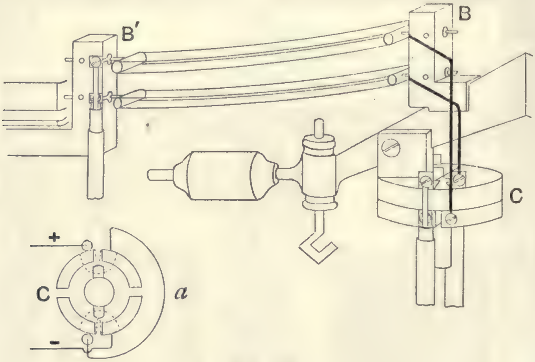

FIG. 138.

To give an idea of the dimensions of the instrument it may be stated that the axial length of the suspended cylinders is about 13 cm, their diameter rather more than 20 cm. Each suspended cylinder had 184 turns. The diameters of the fixed coils are about 33 cm, and each half of these has a length of about 12·7 cm, so that the whole length of a cylinder is about 25·4 cm. Each half of a fixed cylinder contained 163 turns.

The whole apparatus is very solidly supported on an adjustable pedestal of phosphor bronze which can be exactly levelled.

Fig. 137 shows a front view of the instrument, and will give an idea of its appearance. The beam is 20 inches (50·8 cm) long, constructed so as to carry 5 kilos at each end, and turns with $\frac{1}{10}$ of a milligramme. A rider beam divided into 100 parts is carried on each side. All the knife edges and bearing planes are of agate. The scale pans hang from separate planes on the same knife edges as support the cylinders, and weights can be placed on, or removed from the scale pans without disturbing the levelling of the suspended cylinders. The detail is shown in Fig. 138.

60. Calculation of constants of balance. It is not possible to give here any details of the tests made of the materials employed in the construction of the current balance, of the methods of making the various adjustments, or of the measurement and calibration of the cylinders and helices. We shall only indicate how the constants of the balance have been calculated, and give a short account of results obtained in its use.

The positions of the coils are shown diagrammatically in Fig. 139. In the first place it is to be noticed that the use of two pairs of fixed coils with corresponding suspended coils introduce cross actions between the coils. The vertical forces due to these were approximately calculated, and could also be measured, and were allowed for. A careful test was made for the effect of horizontal cross forces, but none was observed.

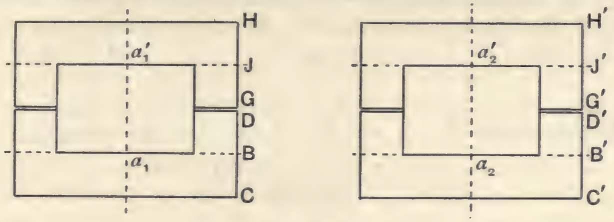

FIG. 139.

The force between a suspended helix and the fixed current sheet within which it hung is given by

$$F=\gamma_h\gamma(M_2-M_1),$$

where γ_h is the current in the helix, γ that in the current sheet, and M_1, M_2 the mutual inductances of the helix and the two ends of the current sheet. This formula has been proved in IV. 17 above. The error introduced by applying the formula, which is true for a helix and a current sheet, to two helices of fine pitch, made of wire of sensible thickness, such as were here used, was considered and found to amount to about 17 parts in 10,000,000.

The value of M_2-M_1, say for the left-hand coils in Fig. 139, was computed as follows. The mutual inductance of one of the two helices, say BC wound on CD, and the circle a_1, was calculated. This was approximately half the inductance between CD and a_1. Then two mutual inductances were found—that between a'_1, and (1) the helix JD, (2) the helix JC, and taking the difference. There were thus three mutual inductances, which were denoted by M_Θ, M_{Θ_1}, M_{Θ_2}, and the value of M_2-M_1 for the two helices on CD was given by

$$M_2-M_1=2\{2M_\Theta-(M_{\Theta_2}-M_{\Theta_1})\}.$$

For the same current sheet a_1, a'_1, and the helices on GH, $M_2 - M_1$ was found from M_Θ, M_{Θ_1}, M_{Θ_2}, by allowing for small differences in dimensions by the equation

$$\frac{dM_\Theta}{M_\Theta} = q\frac{dA}{A} + r\frac{da}{a} + s\frac{dx}{x}, \qquad \text{.......................(71)}$$

where A is the radius of the helix, a that of the circle, x the length of the helix, and q, r, s, coefficients which were determined, as explained above, p. 431. The sum of the two values of $M_2 - M_1$ thus obtained gave the total for the left-hand system, and was denoted by M_L.

The value of M_R (the force for the right-hand coils) was determined from M_L by applying the correcting equation. The whole force therefore, when the two sets assisted each other, was then

$$F = \gamma_h \gamma (M_L + M_R) = mg, \qquad \text{.......................(72)}$$

where m was the balancing mass in the scale pans.

61. Results obtained. It was found that for one ampere

$$m = 0{\cdot}1 \times \frac{0{\cdot}1 \times 184}{12{\cdot}9830} \times \frac{51922{\cdot}47}{981{\cdot}2}, \qquad \text{.................(73)}$$

since there were 184 turns on each suspended cylinder and the axial length of each cylinder was 12·983 cm. The value of g was taken as 981·2: a more exact determination is probably necessary.

The forces between, for example, a_1, a'_1 and CD, GH were called direct forces, the vertical force between a_1, a'_1 and $C'D'$, $G'H'$ was called a secondary force. It will be clear that the coils could be joined so that the electromagnetic force in action was the sum of the direct and secondary forces $(D+S)$, and also so that the secondary forces opposed the direct forces $D-S$. Two sets of observations, a $(D+S)$ and a $(D-S)$, were made to eliminate the secondary forces.

The change of mass in the scale pans on reversal of 1 ampere in both sets of coils was found to be 14·99928 grammes.

By taking the sum of the balancing masses obtained in a $(D+S)$ observation and a $(D-S)$ observation, with the same current, and calling it m', the equation for the number of amperes flowing was

$$\text{amperes} = \sqrt{m'/29{\cdot}99856}.$$

The mutual inductances were calculated from the equation [given and fully explained in VI. 11 above]

$$M = \Theta(A+a)\beta\gamma\left\{\frac{G-H}{\gamma^2} + \frac{1-\beta^2}{\beta^2}\Big(G - \Pi\Big)\right\}. \qquad \text{...........(74)}$$

It is to be remembered that γ here denotes the modulus of the elliptic integrals. The elliptic integrals were calculated in three ways, (1) from Legendre's tables by interpolation, (2) by successive quadric transformation, (3) directly by series. The values of the elliptic integral of the third kind, Π, were obtained from the expression given for it in terms of incomplete elliptic integrals of the first and second kinds, G, H.

The electromotive force of a cadmium cell was found by measurements of current by this balance to be 1·01830 semi-absolute volts, that is 1·01830 times the difference of potential in an international ohm when an absolute ampere is flowing through it. If the international ohm may be taken as $1{\cdot}00041 \times 109$ c.g.s. units, the cadmium cell is to be reckoned as having an e.m.f. of $1{\cdot}0187_1 \times 10^8$ c.g.s. units.

It is reckoned that the value of the ampere is given by this balance to 1 in 300,000. There is (or was) however some uncertainty in the value of g and as to the measurement of the axial lengths of the coils.

62. Lord Kelvin's standard current balances. Lord Kelvin constructed current-weighers or balances for use as standards for current measurement in practice, and as instruments on the principle of the balance have been adopted for the same purpose by the Board of Trade Committee on Electrical Standards (see their Report in Appendix) we give here a short account of the most generally useful form of these balances. They are not instruments for absolute determinations, but have to be calibrated by comparison directly or indirectly with absolute instruments; and for the exact determination of currents it is necessary to have recourse to the use of a standard cell and a potentiometer [57 above].

They are based on the principle, set forth in Chap. V. above, of the mutual action between the fixed and movable portions of a circuit carrying a current. Each of the mutually influencing portions consists in most of the instruments of one or more complete turns or spires of the conductor, but in some cases consists of only half or part of a turn. In all cases in what follows we shall call each portion a *ring*.

In each of the balances, except that for very strong currents (the kilo-ampere balance), the movable portion of the conductor consists of two rings, carried with their planes horizontal at the extremities of a balance beam free to turn in the ordinary way round a horizontal axis. Above and below each ring on the beam is a fixed ring with its plane parallel to that of the movable ring. The rings are (except in what is called the Composite Balance used for measuring power) all joined in series, and the current to be measured is sent through them so that the mutual action between the movable ring at one end and each of the two fixed rings there is to raise that movable ring, while the mutual action of the other group of three rings is to depress the corresponding movable ring. The action is therefore to turn the beam round the horizontal axis on which it is pivoted, with for any given position a couple varying as the square of the current flowing.

Fig. 140 shows diagrammatically the rings and the course of the current through them: a, e, b, f are the two pairs of fixed rings, c, d the movable rings. The current entering by the terminal T passes round all the rings in series, in the two movable rings in opposite directions, and returns to the terminal T_1. Since each movable ring is in general in a magnetic field, terrestrial or artificial, which has a horizontal component, it tends to set itself so that the greatest number of horizontal lines of force may pass through it and therefore is acted on by a couple which tends to turn the beam round its axis. But since the current passes round the movable coils in opposite directions, and these are very approximately equal, the two couples are nearly equal and opposite, and the instrument is practically free from disturbance by horizontal magnetic force.

The turning couple produced by the mutual action of the fixed and movable rings is balanced for the horizontal or "sighted position" of the beam by an equal and opposite couple produced as described below by a stationary weight at the end of the beam, and a sliding weight placed, steelyard fashion, at a suitable point on a graduated bar attached to the beam. The amount of the current flowing in the rings is deduced from the amount of the equilibrating couple thus applied, or rather from a number proportional to it, by means of a table of reckoning.

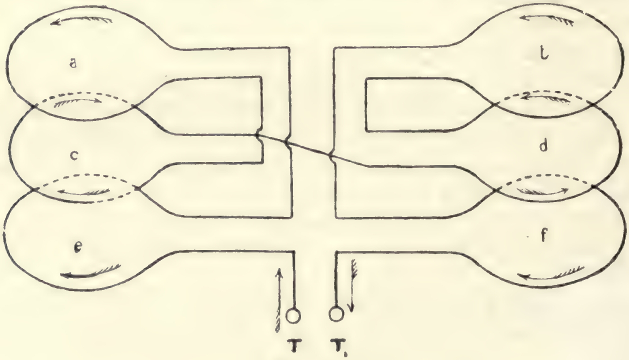

FIG. 140.

63. Centi-ampere balance. Most of the constructive details will be made out from Fig. 140 which shows the Standard Centi-ampere Balance, and illustrates the arrangement of the beam, the graduation, and the mode of applying the equilibrating couple, for all the instruments.

The beam is hung on two trunnions, each supported by a flat elastic ligament made of fine copper wires, through which the current passes to and from the movable rings.

The horizontal or sighted position of the beam is that in which the pointers on the extreme right and left are at the middle divisions of their scales. This position, in all the instruments in which a movable ring is acted on by two fixed rings between which it is placed, is not that midway between these two rings, as that would be a position of minimum force and therefore of instability. For stability it is so chosen that the movable ring is nearer to the repelling fixed ring than to the attracting ring by such an amount as to give about $\frac{1}{5}$ per cent. more than the minimum force.

Fixed to the beam and parallel to it is a finely graduated bar, and above this is a horizontal fixed scale, called the Inspectional Scale, less finely divided. Both graduations begin from zero on the extreme left and have numbers increasing towards the right. A carriage is moved along the graduated bar to any required position by a sliding piece controlled by a cord which can be pulled from either end, and this carriage, by itself or with an additional weight, forms the movable weight referred to above. The position of the carriage is indicated by a pointer which moves along the lower scale. Each additional weight has in it a small hole and slot which pass over conical pins in the carriage. This ensures that the weight is always placed in a definite position. The balancing weight is moved along the beam by means of a self-releasing pendant carried by the sliding piece above referred to. To this pendant is attached a vertical arm (seen in the figure) which passes up through the recess in the front of the weight and carriage and so enables the carriage to be moved with the sliding piece. The stationary weight is placed in the trough shown at the right-hand end of the instrument. The trough is V shaped, and the weight cylindrical, with a cross pin which passes through a hole in the bottom of the trough. The weight is thus placed in a perfectly definite position and always has the same leverage. It is so chosen as just to keep the beam in the sighted position when the sliding weight is at the zero of the scale.

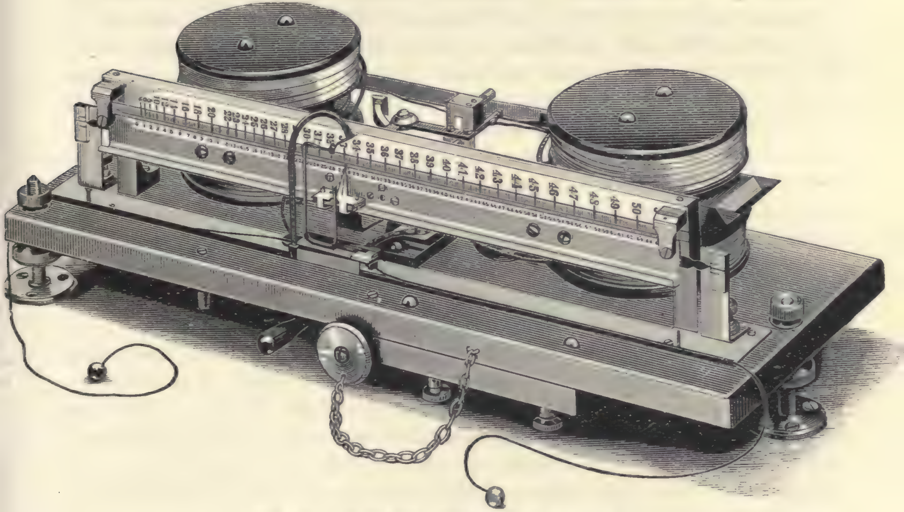

FIG. 141.—Standard Centi-ampere Balance.

Since the mutual action of the rings is to bring the beam towards the sighted position when displaced by the weights, and the equilibrating couple is that due to the displacement of the sliding weight from zero, the latter couple increases as the current increases, and hence motion of the sliding weight towards the right corresponds to increasing currents. The use of the stationary weight gives a scale of double the length which would be obtained without it.

In the top of the lower or finely graduated scale are notches which correspond to the exact integral divisions in the upper fixed scale. Thus the reading in the fixed scale is got when the pointer is at a notch, without error from parallax due to the position of the eye. The reading when the pointer is between two notches is easily obtained by inspection and estimation with sufficient accuracy for most practical purposes. When however the greatest accuracy is required, the reading is taken on the lower scale, with the aid of a lens, and the current strength calculated from a table of doubled square roots.

Four pairs of weights are given with each instrument. Of these one set is for the sliding platform, the other set are the corresponding counterpoises. The weights of each set are in the ratios 1 : 4 : 16 : 64, and are so adjusted that, when the carriage is placed with its index at a division of the inspectional scale, the instrument shows a current of an integral number of amperes, half-amperes, or quarter-amperes, or some decimal subdivision or multiple of one of these units of current.

The accurate adjustment of the zero is effected by a small metal flag as in a chemical balance. This flag is set in any required position by means of a fork moved by a handle beneath and outside the case of the instrument. The sliding weight is brought to zero with the corresponding counterpoise in the trough, and then the flag is turned to one side or the other until the pointer of the beam (seen on the extreme right and left in Fig. 141) is just at zero.

When necessary for transit or otherwise, the beam in the centi-ampere and deci-ampere balances is lifted off its supporting ligament by turning an eccentric by a shaft under the sole-plate of the instrument. In the other balances the beam is fixed for carriage by placing distance pieces between the upper and lower parts of the trunnions and screwing them together by milled headed screws kept always in position for the purpose.

Section II.

MEASUREMENT OF CURRENTS AND GRADUATION OF INSTRUMENTS BY ELECTROLYSIS.

64. Determination of the electro-chemical equivalent of silver. We shall now give a short account of determinations of the electro-chemical equivalent of silver. We take first that made by the late Lord Rayleigh and Mrs. Sidgwick. The arrangement of apparatus is shown in Fig. 142. A circuit was made up of a battery *A* in series with three silver voltameters, a tangent galvanometer *D* (which gave a rough measurement of the current), the current weigher *F*, *G*, described in 45 above.* The voltameters were each composed of a platinum bowl which served as kathode, and an anode of pure silver plate suspended horizontally above the bowl in the electrolytic liquid, which was a solution of pure nitrate of silver. To prevent disintegrated silver from falling from the anode the plate was wrapped round with pure filter paper secured at the back with sealing wax. The electrolyte was in general a neutral solution of 15 parts by weight of pure silver nitrate in 100 parts of water. The area of deposit in two of the basins was about 37 square centimetres, and 75 square centimetres in the other.

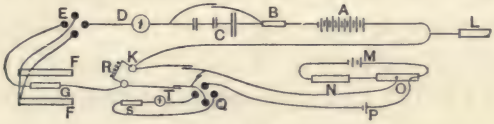

FIG. 142.

After a number of trials of the addition of acetate of silver in small quantity to the pure nitrate solution, it was found that, while the acetate had the desired effect of giving a firmly coherent deposit of close texture, the very closeness of its texture rendered very difficult the after freeing of the deposit from retained salt or other impurity tending to increase its weight. It was therefore decided to use pure nitrate solutions, which it was found after all gave deposits coherent enough for the subsequent treatment.

65. Details of an experiment. The procedure in an experiment was as follows. The current roughly regulated to the desired value was allowed to pass through the current-weighing apparatus for half an hour, but not through the voltameters. The copper conductors of the circuit heated somewhat, and thus the current slightly fell off during this time. The voltameters in the meantime were charged with the solution, and the anodes fixed in position. Then when all had been adjusted the current was, at an instant observed on a chronometer, sent through the voltameters arranged in series; and the weights then required to bring the pointer of the suspended coil to zero were observed. At intervals the current was reversed, and the change of weights observed. For one direction of the current, of course, the electromagnetic action assisted gravity, in the other opposed it.

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* The rest of the arrangements shown in Fig. 142 have no relation to the electro-chemical determination. They were required for the experiments on Clark cells described in 77 below.

The following table gives the result of a series of experiments made on March 10, 1884. The two sets of numbers are the weights which had to be added to give equilibrium according as the current was in one direction or the other.

Time of Weighing.			Weight in Grammes.
H.	M.	S.	
4	19	30	7·694
4	25	0	6·795
4	32	15	7·698
4	40	20	6·791
4	42	50	7·699
4	50	30	6·790
4	53	10	7·699
4	56	30	6·789
5	1	15	6·789

Current sent through voltameters at 4h. 17m., interrupted at 5h. 2m.

Difference of weights = 2 × Force on suspended coil.

The curves, Fig. 143, show these results for each position of the key.

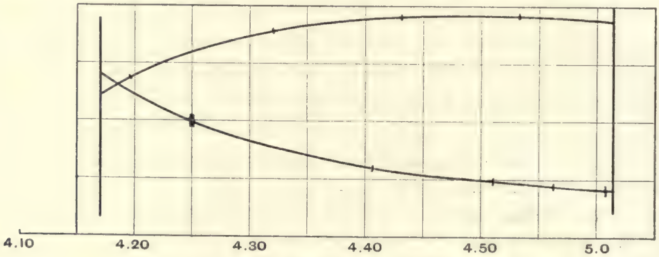

FIG. 143.

The current was integrated by dividing the whole interval of 45 minutes during which the current was flowing into 9 intervals of 5 minutes each, and the magnitude of the current at the middle of each interval was taken to represent its value during the period.

The differences of the ordinates of the curves of Fig. 140, at the middles of these intervals, give the difference of weights, and therefore twice the force exerted by the fixed coils on the suspended one. These differences and their square roots are shown in the following table. The mean of the square roots is the square root of the difference of weights which would have been shown by the mean current.

Time.			Difference of Weights.	Sq. Root of Difference of Weights.
H.	M.	S.		
4	19	30	·897	·9471
4	24	30	·900	·9487
4	29	30	·904	·9508
4	34	30	·906	·9518
4	39	30	·908	·9529
4	44	30	·908	·9529
4	49	30	·909	·9534
4	54	30	·910	·9539
4	59	30	·910	·9539
				Mean ·95171

The 45 minutes' interval during which the experiment lasted was corrected for the time taken to work the reversing key. This was done by carrying the main current, between the battery and the key, round a reflecting galvanometer consisting of a few turns of wire. The momentary stoppage of the current caused the needle to fall back towards zero, and from the observed amount of the corresponding motion of the spot of light, and the period of the needle, the time of duration of the interruption could obviously be found. The correction rendered necessary was ·083 second for each operation. This brought down the whole interval by ·6 second, or to 2699·4 seconds.

The deposits were washed immediately after formation first with alcohol, then with boiling water, and lastly with cold water. They were then left to soak in water overnight, then rinsed and put to dry in an air-bath at 160° C. After cooling over a desiccator the deposits were weighed, then were heated nearly to redness over a spirit lamp to drive off traces of adhering salt, then cooled and weighed again.

66. Results of a series of experiments. The following table gives the results of the weighings for the set of experiments already referred to:

March 10, 1884.

	Large bowl. I. Pure Nitrate. Normal Strength.	Large Bowl. II. Pure Nitrate. Double Strength.	Small bowl. III. Pure Nitrate. Normal Strength.
Before deposit -	80·4490 grms.	17·2958 grms.	21·8789 grms.
After deposit, first weighing -	81·5138 ,,	18·3628 ,,	22·9434 ,,
Gain - -	1·0648 ,,	1·0643 ,,	1·0645 ,,
After strong heating - -	81·5135 ,,	18·3627 ,,	22·9433 ,,
Gain - -	1·0645 ,,	1·0642 ,,	1·0644 ,,
			Mean gain 1·0644 grammes.

Thus the amount of silver deposited per second is 1·0644/2699·4. Dividing the mean square root of the difference of weights by this we get $\sqrt{m}$/(rate of deposition) $= {\cdot}95171 \times 2699{\cdot}4/1{\cdot}0644 = 2413{\cdot}7$.

The mean result of several series of experiments was to give instead of the last found number 2414·45. From this the value of the electro-chemical equivalent of silver was deduced. We have seen that if m is the difference of weights, we have

$$\gamma = {\cdot}0370484\sqrt{m}.$$

67. Electro-chemical equivalent of silver. If w be the electro-chemical equivalent of silver, we have for the rate of deposit $w\gamma$. But

$$\frac{\sqrt{m}}{w\gamma} = 2414{\cdot}45.$$

Hence, as final result,

$$w \frac{1}{2414{\cdot}45}\,\frac{\sqrt{m}}{\gamma} = \frac{1}{2414{\cdot}45 \times {\cdot}0370484} = \mathbf{{\cdot}0111794}, \quad \ldots\ldots\ldots(75)$$

as the weight of silver deposited on a kathode plate by the passage of one c.g.s. unit of electricity.

It is stated in the paper that the strength of the nitrate solution may be considerably varied without affecting the result if the current does not exceed ¼ ampere for the 37 sq. cm area of deposit. In this case a 4 per cent. solution may be used. If the currents are comparatively strong, the solutions should be from 15 to 30 per cent. in strength. Too weak a solution would give a somewhat loose deposit. Currents not exceeding 1½ amperes can be conveniently measured by running them for about a quarter of an hour through a strong solution.

68. Measurement of currents by electrolysis of copper sulphate. The graduation of instruments for use as standards in practical electricity can be carried out with all needful accuracy by means of the electrolysis of copper sulphate. The behaviour of this substance as an electrolyte, and hence the conditions necessary for obtaining consistent results in its use, and the ratio of the electro-chemical equivalent of copper to that of silver, were carefully investigated by the late Prof. T. Gray,* who was for some time in charge of the graduation of Lord Kelvin's standard instruments, and a short account of his results is here given.

A form of cell very convenient for use with solutions whether of nitrate of silver or sulphate of copper, when the current strength is not greater than 10 amperes, is shown in Fig. 144. It consists of three parallel plates of pure silver or pure copper, suspended from spring clips in a glass vessel containing the proper solution. This form of cell has the advantages of giving light plates, which facilitate the accurate weighing of the amount of loss or gain of metal, and allowing, when silver is used, and the size of the plates is properly proportioned, the loss from the anode to be used as a check in estimating the gain on the kathode. There is of course the objection which attends the use of vertical plates that the solution becomes less dense near the kathode, but the only practical effect due to this has been found to be a slightly greater thickness of deposit in the lower part of the plate due to the greater density there.

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* See a paper on the "Electrolysis of Silver and Copper," T. Gray, *Phil. Mag.* Oct. 1886, from which the details here given are mostly taken. See also a paper by A. W. Meikle, *Electrical Engineer*, Mar. 23, 1888.

Lord Rayleigh used, as explained above, as voltameter a platinum bowl as kathode, and a silver plate as anode. This cell, though it had several advantages, was found, according to Prof. T. Gray's experience, more difficult to manipulate than that here described.

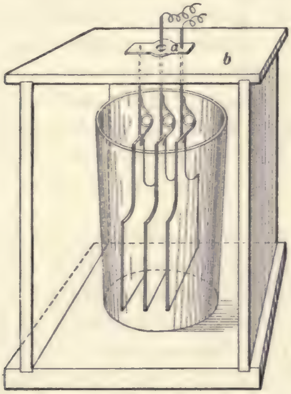

FIG. 144.

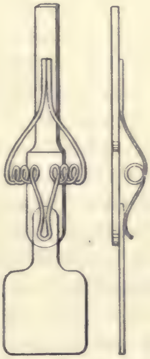

FIG. 145.

The form of clip or plateholder, as illustrated in Fig. 145, almost explains itself. It is made of stiff platinoid or brass wire. A piece is taken of the proper length, bent into a close loop at the middle, then each half wound two or three times round a rod of metal to form springs as shown, and the two ends bent round to meet side by side, and there soldered to a stiff back-piece of brass. The springs when soldered in position should cause the loop to press firmly against the back-piece so as to form a firm clip.

The stems of the two outer clips when in position are connected by a cross-piece *a* of copper. Both are insulated from the inner clip by a block of vulcanite through which its stem passes. This whole arrangement of cross-piece and insulating block is fixed on the top *b* of the wooden framing shown in Fig. 144.

The two plates attached to the outer clips form the anode of the electrolytic cell, and the plate between them the kathode. The kathode thus gains on both sides, and as it is safer to use the gain than the loss of metal in estimating the current, the weight of the plate itself is thus made as small as possible in comparison with the alteration in weight to be determined.

The form of cell shown in Fig. 144 was improved by the substitution for the cover *b* of a rectangle of wood, well soaked in paraffin or varnished, which carried on one side the clips for the anode, and at the middle of the opposite side the single clip for the kathode.

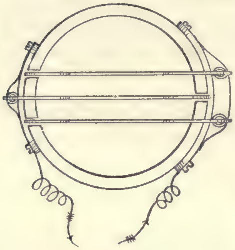
FIG. 146.

When currents of over 10 amperes are to be used the form of cell shown in Fig. 146 is preferable. An insulating rim rests on the top of the cell, which for the larger sizes is conveniently made of earthenware and of rectangular shape. A groove in the rim fits the top of the cell loosely so that the rim with its attachments can be easily removed and cleaned. To the rim are fixed on opposite sides two sets of spring clips, each made as shown in Fig. 147, by soldering flat strips of springy metal to a stiff base-piece which can be screwed to the insulating rim of the cell. To make the effective area of the plates as great as possible in comparison with the ineffective part, the part above the liquid is cut away to two narrow strips connecting the lower part to an upper cross bar *c*, *d*. One end *c* of this cross-bar rests in a clip, the other in a notch in the insulating rim. Anode plates and kathode plates alternate with one another, and there is one more of anodes than of kathodes, so that each kathode is between two anodes. In large cells where the plates are close and liable to touch, they are kept apart by two U-shaped glass tubes hung over each alternate plate.

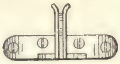
FIG. 147.

69. Preparation of plates. With regard to the size and preparation of plates it was found that in the cases of both silver and copper there is a certain density of current (current strength per unit of area of plate) which gives the most adherent and, in the case of silver, most finely crystalline deposit. When silver is used there is a tendency, if the plate be too large or too small, for the crystals of deposited silver to grow out branch-like from one plate to the other, an effect which is most marked where there is a sharp edge or corner. Hence the plates must have their edges and corners rounded off to prevent the formation of these "trees," which cause great risk of loss of silver from the plate in its treatment before being weighed.

The best deposit has been found to be obtained with a solution made with five parts by weight of nitrate of silver to 95 of water, and a kathode plate giving not more than 600 sq. cm nor less than 200 sq. cm of active face to the ampere of current. If a stronger solution be used, the density of current may be somewhat increased, but the strength should not be less than 4 per cent. nor greater than 10 per cent.

The anode plates should be considerably greater in area than the kathode plates if their surface is to remain bright and moderately hard so as to admit of the plates being weighed if necessary. The density of the current for them should be less than one ampere to 400 sq. cm.

If the anodes are of rolled sheet silver the surface skin should be polished off with fine silver sand, and the plate washed in distilled water before being used; as otherwise the silver would be dissolved away from under the skin, which would hang as a loose sheet ready to break away when the plate was moved. A plate of silver becomes soft and inelastic by repeated use as an anode, owing to solvent action going on below the surface, and to remedy this, after being used each time, should be heated to a red heat in the flame of a spirit lamp.

The following mode of treating silver plates has been found very successful. The plate cut from the new sheet has its corners first rounded and smoothed, then is polished with fine silver sand in water, rubbed on with a soft clean pad of cloth, so as to remove the skin above referred to, and still leave a smooth surface. A gentle stream of clean water is then run over the surface from a tap to remove the sand, next the plate is washed, first with clean soap and water, then with water alone, then immersed for a few minutes in a boiling solution of cyanide of potassium, and finally washed thoroughly in a stream of clean water. The plate is dried in a current of hot air, for example before a clear fire; and great care must be taken in handling it after it has been cleaned not to touch it with the fingers, otherwise the parts which have been in contact with the skin will receive no deposit. Of course the plate must be allowed to cool before it is weighed to obviate risk of disturbance from air currents in the balance case.

When the silver deposit is to be washed and weighed, the plates are gently removed by easing the springs to prevent risk of rubbing off metal by the friction of the clips, then dipped gently in clean, recently distilled water contained in a glass vessel, so that any small crystals which may fall from the plate may be detected. The adherent nitrate solution is thus to a great extent removed; and the plates are then laid in the bottom of a shallow glass tray containing clean distilled water, and washed by gently tilting one side then the other of the tray so as to make the water flow gently over their surfaces. Then they are washed in a second tray in the same way, and allowed to soak for a quarter of an hour before being dried.

To dry the plates one corner is laid on a pad of blotting-paper and the greater part of the water drained off. The plate is then dried by holding the upper end in a spirit flame.

70. Electrolysis of copper sulphate. The electrolysis of copper sulphate with copper anode and kathode gives results which for very high accuracy in standardizing are but little if any inferior to those obtained with silver: for most practical purposes results quite accurate enough can be obtained with much less experimental skill on the part of the operator. Repeated experiments made in the Physical Laboratory of the University of Glasgow,* showed that under certain easily fulfilled conditions the method of standardizing by the electrolysis of copper sulphate is perfectly accurate and trustworthy.

The size of plates is not of so great importance as in the case of silver, but the kathode plate for the best results in long-continued electrolyses should have about 50 cm of active surface or upwards per ampere. When the current is of small density deposits are obtained which are much more solid and adherent than those of silver, and therefore much more easily dealt with. As in the case of silver the anode should be of much greater area than the opposed surface of the kathode. With a density of current of upwards of $\frac{1}{30}$ of an ampere per sq. cm the resistance at the anode becomes variable and very considerable, sometimes almost stopping the current, which after a little, with evolution of gas at the anode, regains nearly its former strength.

It was found by Prof. T. Gray in the experiments above referred to that satisfactory and concordant results could be obtained with a solution of any ordinary pure commercial copper sulphate made with pure water, provided the density did not fall below 1·05, and the solutions were made slightly more acid than in the normal state. An addition for example of $\frac{1}{10}$ per cent. of sulphuric acid to different solutions, which gave results differing among themselves, brought them into complete accordance. The loss of weight which is well known to take place when a copper plate is left standing in a copper sulphate solution, was also carefully investigated. This loss it was found seldom exceeds $\frac{1}{200}$ of a milligramme per sq. cm per hour, or about $\frac{1}{5000}$ of that which would be deposited by a current of one ampere per 50 sq. cm. When the current density is smaller than this the loss is nearly the same as when no current flows. The effect seemed to have a minimum for a density of solution between 1·10 and 1·15, and seemed for this density to be rather retarded than the reverse by the addition of a small percentage of free acid.

71. Treatment of copper plates. The kathode plate having been cut and rounded at the corners is polished with silver sand in the same manner as the silver plate. It is then placed in the cell and a thin coating of copper deposited over it, while the current (if a large current is to be used) is adjusted to its proper strength by placing resistance in the circuit. The plate is then removed, washed in clean water and dried before a clear fire without being sensibly heated. Any defect in the first cleaning will be shown by the deposit, and if no such defect is shown, the plate is weighed and replaced in the cell for the continuation of the electrolysis. If feeble currents are to be used this preliminary adjustment is hardly necessary, as it is preferable then to use a larger number of cells than are absolutely necessary to produce the current, and bring down the current to the necessary strength by adding an amount of resistance which can be easily enough estimated.

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* See the Ref. in 68 above. The remarkable concordance of standardizings made at different times is illustrated by results quoted in Mr. Meikle's paper.

After the electrolysis the plates are carefully removed and at once dipped in ordinary (not necessarily distilled) clean water, containing two or three drops of sulphuric acid per litre, then washed in a tray like the silver plates. The plates are then rinsed in clean water without acid, and dried first in a clean pad of white blotting paper, and then before a fire or over a spirit lamp. If this is carefully done and the deposit be fairly good no copper will be lost and there will be no gain of weight by oxidation. The plates may be weighed after having been allowed to cool down to the ordinary temperature.

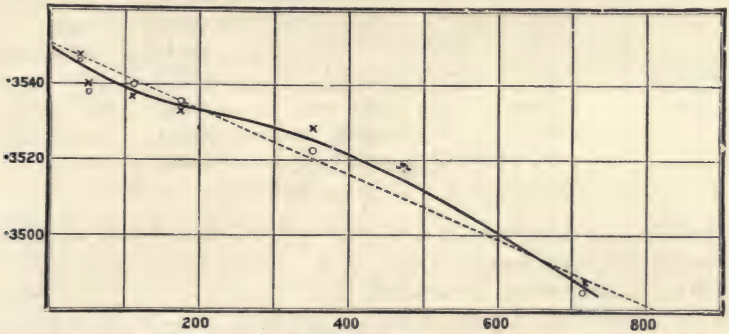

FIG. 148.—Abscissae give area of kathode in square centims. per ampere. Ordinates are electro-chemical equivalent, multiplied by 10,000.

The anode plates are treated in a similar manner (except as regards the drying in a blotting-pad, which might cause loss of silver) without loss of copper, or gain by oxidation, but owing to loss of weight in the solution etc., they give much less satisfactory results than do the kathode plates.

The arrangement of the circuit for electrolytic experiment consists of a battery of large surface Daniells, or other constant cells joined in series with the electrolytic cells to be used, a sensitive galvanometer, and a rheostat (or other readily variable resistance) by which the current is to be regulated. The current is adjusted so that a convenient deflection is obtained, which is restored by slightly turning the rheostat in the proper direction if any alteration takes place. The conduct of an experiment will be understood from the description of the process of standardizing given below.

72. Electro-chemical equivalent of copper. From Lord Rayleigh's result for the electro-chemical equivalent of silver (see 65 above), namely that a coulomb deposits ·0011179 gramme of silver, very nearly, Professor T. Gray has determined by comparison the electro-chemical equivalent of copper, and found it to be very approximately ·0003287 (or for practical purposes ·0003290) at ordinary temperatures, and with a current density of one ampere per 50 sq. cm of active surface of kathode. This number can be corrected for other current densities by the dotted curve given in Fig. 148.

The results from which this curve has been plotted are given in the following table:

AMOUNTS OF COPPER DEPOSITED BY THE SAME QUANTITY OF ELECTRICITY ON KATHODE PLATES OF DIFFERENT AREAS.

Area of plate in sq. cm.	Amount of deposit in grammes (first experiment).	Amount of deposit in grammes (second experiment).
3	·3534	·3534
5	·3530	·3529
11	·3528	·3530
18·5	·3526	·3527
36	·3524	·3521
73	·3503	·3502

The effect of variation of temperature * on the amount of copper deposited has been found by Mr. A. W. Meikle to be very slight at ordinary temperatures; for a change from 12° C. to 28° C. it is a diminution for a given size of plate of only $\frac{1}{16}$ per cent.

At temperatures rising above 20° C. the effect of variation of size of plate becomes more and more important.

73. Graduation of standard instruments by electrolysis. The application of electrolysis to the standardizing of instruments will now be illustrated by a short account of its application to the determination of the proper weights for use in the Kelvin standard current balances described above. The arrangement of apparatus is shown in Fig. 149, which may be taken as a plan of the standardizing table with instruments in position. *C*, *C*, *C*, *C*, *C*, *C* are six of the Electric Power Storage Co.'s secondary cells, shown joined in series, by being connected to a series of mercury cups, *m*, *m*, ... which are connected across by thick copper rods as indicated by the full and dotted lines. (These cups are on a vulcanite base, and have bottoms of thick copper to ensure contact.) When however currents of great strength are required for the graduation of low resistance instruments, these cups are joined in parallel by two rods of copper which have teeth at the proper distance apart to fit into the cups, so as to join all in each row together. The battery fully charged and thus joined in parallel will maintain a current of 200 amperes for 10 hours.

The terminal cups of the commutating board are shown joined to a distributing board provided with cups, 1, 2, ... 12, by which the battery is put in series with a rheostat *R*, in parallel arc with a set of conductance bars *D*, a galvanometer *G*, a pair of large electrolytic cells joined by a movable cup *M*, and finally the balance *B* to be standardized. The conductance bars are constructed as shown in Fig. 150. Rods of platinoid of thickness according to the conductance required are bent into U-shape, as shown, and the limbs held at proper distances apart by wooden blocks at intervals, or by a strip of wood running along their whole length, according as the rods are thick or thin. The length of rod in each U is about 4 metres, and the thickness is chosen such that one or two volts difference of potential produces very little heating of the wire. The troughs, *t*, *t* (Fig. 150), are made with bottoms of thick copper and contain mercury in which the ends of the rods (or thick copper pieces soldered to the wires if thin) rest pressed down by their own weight. The different Us beginning from one side are graduated so as to have conductances nearly in the ratios 1 : 1 : 2 : 4, etc., so that the total conductance in the set may be increased at will by a step equal to the lowest conductance (since each conductance is that amount greater than the sum of all that precede it in the series). When any bar is not in use its lower ends are lifted out of the troughs as shown in the figure. The rheostat, which has a least conductance rather less than that of the smallest bar, furnishes an auxiliary variable bar by which the conductance can be gradually altered. Its wire is of stranded copper and can carry 10 amperes without damage.

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* See Ref. in 68 above.

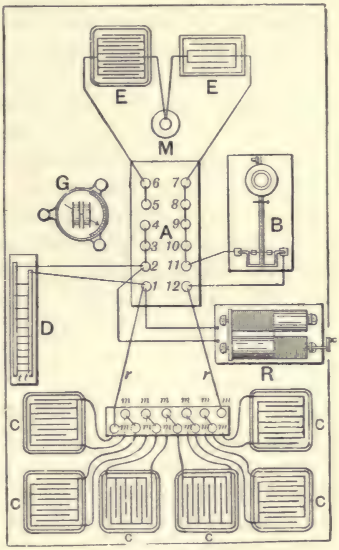

FIG. 149.

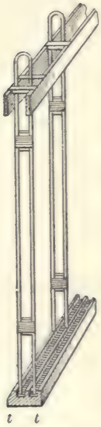

FIG. 150.

The current balance has previously had its scale graduated and attached as described above, and it remains only to show how the constant of the instrument is determined, or in other words the weight which placed on the beam will enable the current to be obtained from its indications in the manner already described (63 above). A chosen arbitrary counterpoise weight is placed in the trough, and another, which then just brings the beam to the sighted position without current when at the zero of the scale, is placed on the beam with the index at some division near the right-hand end so that a current of, say, 10 amperes (more or less according to the instrument) is required to bring the beam to the sighted position. The electrolytic cells are then arranged to give about 500 sq. cm of kathode surface, and are joined up with a conductance sufficient to give nearly the required current. The balance will come nearly to zero, and is brought to zero exactly by adjusting the current by means of the rheostat. These adjustments having been made, the kathode plates are removed, washed, weighed, and replaced. At an instant observed on an accurate time-keeper the circuit is closed, and any deviation of the current corrected by means of the rheostat. The current is brought to its correct value in from five to ten seconds, and hence in an electrolysis of say an hour (the usual duration of an experiment) the error due to its deviation from the final constant value for this short variable period is quite imperceptible. Any variations of the current strength are observed on the instrument itself, or if (which rarely happens) that is not sensitive enough, on a more sensitive galvanometer G (Fig. 149), which is introduced when required, and kept out of circuit at other times. Any sufficiently sensitive instrument which can have its (not necessarily known) constant changed by any required amount by varying the field at the needle, or by using an instrument provided with two parallel coils with the needle midway between them, and arranged to permit the distance of the coils apart to be altered at pleasure, is convenient for this purpose.

The electrolysis having thus been carried on and completed, the circuit is broken, and the plates washed and weighed. The current is calculated from the result by dividing the gain of copper on the kathode expressed in grammes, by the electro-chemical equivalent of copper (·0003287, or, as explained above, the proper value for the density of current), and the result by the number of seconds during which the electrolysis has lasted. Let C be the current for the position of the weight on the beam as given by the table of doubled square roots, w_1, w_2, the corresponding counterpoise and beam weights respectively, C' the current given by the electrolysis, w'_1, w'_2 the counterpoise weight and beam weight applied, then we have

$$\frac{C^2}{C'^2} = \frac{w_1 d_1 + w_2 d_2}{w'_1 d_1 + w'_2 d_2},$$

where d_1, d_2 are constants. But $w_1/w_2 = w'_1/w'_2$; hence this equation gives

$$\frac{C^2}{C'^2} = \frac{w_1}{w'_1} = \frac{w_2}{w'_2}.$$

Thus w_1, w_2 are found by multiplying the ratio C^2/C'^2 by w'_1, w'_2 respectively, and the determination is complete.

74. Arrangement for strong or weak currents. When a very strong or a very weak current is required, as in the graduation of a hektoampere or a centiampere balance, it is desirable in the former case to allow the whole current to flow through the instrument, and only a convenient part through the electrolytic cell, and in the latter case to use a considerably greater current through the electrolytic cell than through the instrument. The current must therefore be divided in both these cases into two parts whose ratio is accurately known, and this may be done by the conductance bridge shown in Fig. 151. A set of parallel straight wires of platinoid are each soldered at one end to a thick terminal bar of copper b, and have soldered to them at the other ends thick terminal pieces of copper by which they can be connected in two groups by means of mercury troughs b_1, b_2. In the figure they are shown in two groups of 10 and 1 respectively.

The wires are adjusted so that when they are in position they have all precisely the same resistance. Between the troughs b_1, b_2, a sensitive reflecting galvanometer [XI. 1] g is joined which indicates no current when b_1, b_2 are at the same potential. The electrolytic cells E, E', and the instrument G to be standardized, are placed as shown in the figure when the standardizing current must be greater than that which the cells can carry, and the positions shown are interchanged when the reverse is the case. The currents are adjusted to balance in both cases by the rheostats R, R'. The currents are of course in the ratio of the conductances of the groups r, r' of the wires of the bridge.

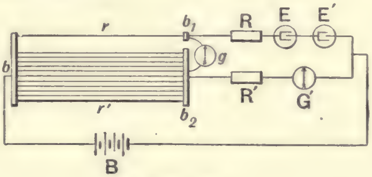

FIG. 151.

Section III.

DETERMINATION OF ELECTROMOTIVE FORCES OF CELLS AND GRADUATION OF VOLTMETERS.

75. Potential measuring instruments or voltmeters. When a current known in absolute measure flows through a known resistance the difference of potential between the terminals of the resistance is also known. By means of this known difference of potential, which may be varied at pleasure, a voltmeter may be graduated. A voltmeter of any type is an instrument, the resistance of which is so high that the attachment of its terminals to two points in a conductor carrying a current does not perceptibly change the difference of potential formerly existing between these points. Of course every absolute galvanometer, electrodynamometer, or standard balance measures differences of potential, for, if its resistance is known, the difference of potential between its terminals can be calculated from Ohm's law; but the convenience of a voltmeter especially made with a high resistance coil is that its terminals may be applied at any two points in a working circuit, and the difference of potential, thus calculated as existing between these two points while the terminals are in contact, may, in most cases, be taken as the actual difference of potential which exists between the same points when nothing but the ordinary conductor connects them. For, let V be this actual difference of potential in volts, let r ohms be the equivalent resistance of the whole circuit between the two points, without the voltmeter, and R ohms the resistance of the voltmeter. Then (VI.) by the application of R, V is diminished in the ratio of R to $R+r$, and therefore the difference of potential between the ends of the coil is now $VR/(R+r)$. Hence the current through the galvanometer has the value $V/R(1+r/R)$. If r be only a small fraction of R, r/R is inappreciable, and the difference of potential calculated from the equation $C=V/R$ will be nearly enough the true value. So far, it is to be observed, r is the equivalent resistance between the two points, and the result stated holds, however the electromotive force may have its seat in the circuit, if only R be great in comparison with r. If, however, either of the two parts of the circuit between the two points in question have a resistance r' small in comparison with R, then, as can be easily proved, the value of the difference of potential between the terminals of r is practically unchanged by the addition of R as a derived circuit.

76. Graduation of a voltmeter. The voltmeter has its terminals attached to those of the resistance through which the current is flowing; or, if the standard measuring instrument is sensitive enough, the measured current is sent through the voltmeter itself; and readings of the needle or other indicator are taken. In either case the readings are proportional to the difference of potential between the terminals of the instrument, but in the former arrangement the difference of potential is equal, in volts, to the current in amperes flowing through the resistance multiplied by the value of the resistance in ohms, in the latter the difference of potential is equal to the measured current through the voltmeter into the resistance between its terminals.

If the scale of the instrument does not follow any known law, it is necessary to determine by direct experiment the electromotive force corresponding to different deflections and thus, so to speak, calibrate the instrument. To do this the most convenient plan is to divide the scale accurately into equal divisions and to number these from zero at the position of equilibrium with no current. Then the current measured by the standard galvanometer is varied conveniently by introducing resistance into the circuit by a rheostat, and the deflection observed for several different values. The corresponding differences of potential are then plotted on squared paper as ordinates for which the number of divisions of the deflections are the corresponding abscissae. A curve is then carefully drawn through the extremities of these ordinates, and the ordinate of this curve drawn for any chosen abscissa will be the difference of potential for that deflection.

77. Clark's standard cell. For verifying the accuracy of the graduation of the potential instruments when performed by either of these methods, or for actually performing the graduation when other methods are not convenient, some form of voltaic cell of known electromotive force may be used.

As the result of many careful experiments made by Lord Rayleigh and others, it has been found that the most reliable standard cell is that proposed by Mr. Latimer Clark. When certain precautions are taken in its preparation the electromotive forces of different specimens are very nearly the same, and remain constant for a long time provided care is taken to prevent more than a very feeble current from ever passing through them.

The cell may be made in a reliable and handy form in the following way, which includes the precautions that Lord Rayleigh's experience * has shown to be necessary. The vessel is a weighing tube, or for small sizes merely a test-tube, with a platinum wire sealed through the bottom, and rests on a suitable stand as shown (Fig. 152). This wire makes contact with mercury, which occupies the bottom of the cell and forms one of the plates. The mercury must be pure, and it is desirable to ensure its being so by redistilling in vacuo good clean commercial mercury. On the mercury rests a paste made by adding to 150 grammes of mercurous sulphate 5 grammes of zinc carbonate, and sufficient saturated zinc sulphate solution to give a stiff pasty consistency.

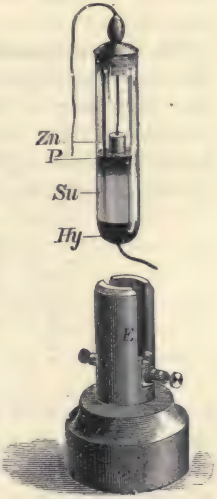

FIG. 152.

The zinc sulphate solution should be made from pure zinc sulphate dissolved under gentle heat in distilled water so as to make a saturated solution, and, after having been allowed to stand for some time to precipitate any iron which may have been present in the sulphate, filtered in a warm place into a stock bottle. When required the solution is gently warmed, and drawn off by a siphon from just above the crystals at the bottom. The paste is made by placing the mercurous sulphate and zinc carbonate in a mortar and rubbing in the zinc sulphate at intervals during two or three days, to give time for all carbonic acid to pass off.

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* See Lord Rayleigh and Mrs. Sidgwick's paper on the "Electro-Chemical Equivalent of Silver" already cited (*Phil. Trans.* part ii. 1884, *Collected Papers*, ii.), also Lord Rayleigh on the "Clark Cell as a Standard of Electro-motive Forces" (*Phil. Trans.* part ii. 1885). These papers contain particulars of the method of determining the electromotive force of Clark cells, and the latter especially details of the mode of constructing them, of which an abstract is given below in the text.

A rod of what is called "redistilled zinc" resting in the paste, and held upright in the vessel by a notched ring of cork, forms the other plate. The zinc is cleaned before putting it in position by dipping it in sulphuric acid and then washing it in distilled water. Connection with it is made by a gutta-percha-covered copper wire soldered to it, and passed up through a cork which closes the cell and nearly fills the upper part of it, so that very little air is included. The cork is flush with the top of the tube, and the edges of the tube and the whole upper surface of the cork are covered with marine glue to seal up the cell.

A cell thus made, if used with only the feeblest currents, never short-circuited, nor exposed to great variations of temperature will have a constant electromotive force E in volts at temperature $t°$ C. given according to Lord Rayleigh and Mrs. Sidgwick's determination (if 1 B.A. unit = ·9866 ohm) by the equation

$$E = 1{\cdot}4345\{1 - {\cdot}00077(t - 15)\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(76)$$

78. Determination of e.m.f. of Clark cell. Potentiometer method. The method employed by Lord Rayleigh and Mrs. Sidgwick in the determination of the electromotive force of the Clark cell, and the method of using the cell for purposes of graduation, will be understood from Fig. 142. (For convenience Fig. 142 is here repeated.) Two Leclanché cells M, and two resistance boxes N, O, were joined in circuit.

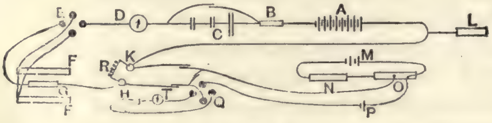

FIG. 142.

At two points in O were attached two wires, in one of which was placed the Clark cell P, which was to be tested. These wires formed with a resistance R a derived branch of the circuit of M including a mercury reversing key Q, a reflecting galvanometer T, and a resistance S of 1000 ohms.

In the earlier experiments the galvanometer had in its coils a resistance of about 200 ohms, but in later determinations it was provided with a coil containing a much greater length of wire, so that a higher sensibility was obtained.

The other arrangements connected with the circuit are the battery A, the voltameters, and the current weigher as described in 50 above. One extremity of A was connected to earth at L.

The main current from *A* after traversing the voltameters and current weigher passes through the resistance *R* back to *A*. To prevent undue heating by the electrolysing current, which was about $\frac{1}{3}$ ampere, the resistance *R* was constructed of bare german silver wire wrapped round a frame of two parallel ebonite rods kept apart by wooden bars, and was provided with stout copper terminals which rested on the copper bottoms of cups *H*, *K* filled with mercury. The resistance was 4·00699 B.A. units at 17°·6 C. This was corrected for the difference between 17°·6 C. and the temperature of the atmosphere, and also for heating produced by the current. It was found that a correcting factor 1·00041 had to be applied to take account of the latter effect.

In the first determinations the battery *M* was not used and the electromotive force of *P* was balanced by the difference of potential existing between its terminals *H* and *K*. The adjustment to balance was made by placing a high-resistance box in parallel with *R*, between *H* and *K*, and unplugging resistance until with the current flowing through the voltameters, no current passed through *T* when the derived circuit was thrown in for a moment.

The difference of potential between *H* and *K* was then obtained from the resistance of the double arc now constituting *R*, and the absolute value of the current given by the electrolysis. The value of the current at the instant when *P* was balanced could be obtained from the curves (Fig. 143) showing the results of the two current weighings; and thus several determinations of electromotive force could be made in a short time.

In later determinations the balance was finally adjusted by including in the derived circuit with *P* a part of the electromotive force of the pair of Leclanché cells. An independent comparison of the electromotive force of the Leclanchés with that of the Clark cell, was made by balancing the Clark cell, in the manner just described, by the difference of potential between two points of a resistance in circuit with the Leclanchés. This enabled the part of the balancing electromotive force supplied by the Leclanchés to be found from the known resistance intercepted between the terminals of the derived circuit and the whole resistance in *N* and *O* together, which was kept at 10,000 ohms.

The following values have been obtained by other experimenters for the electromotive force of a Clark cell at 15° C.:

Carhart	1·434 volts.
Kahle (Zeitschrift für Instrumentenkunde) . .	1·4341 ,,
Glazebrook and Skinner, *Proc. R.S.* 54 (1892) .	1·4342 ,,

79. Graduation of voltmeter by standard cells. Standard cells of known electromotive force being available they may be used for the graduation of voltmeters by the same compensation method. A circuit is made of a battery *A* (Fig. 153) of storage or Daniell's cells, in series with resistances R, S, and the voltmeter G to be graduated. A battery of a suitable number of standard cells has its terminals applied at the extremities of the variable resistance R, and its circuit contains a sensitive galvanometer D, and a key K. Along with D should be included a resistance large enough to ensure that only a small current can ever flow through the standard cells in the process of testing. Clark cells should never have any sensible current passed through them. R or S is adjusted until no current flows through D when the key K is tapped down for an instant. When this is the case the electromotive force of O is balanced by the difference of potential at the two ends of R produced by B. Hence the difference of potential in volts then existing between the terminals of G is given (for a Clark's cell at 15° C.) by the equation

$$V = 1{\cdot}4345\,\frac{G}{R}. \qquad \text{.................................(77)}$$

By this method, which is an application of Poggendorff's method of comparing the electromotive forces of batteries, balance is obtained when no current is flowing through the standard cell, and disturbance from polarization is altogether avoided. It has been found very easy and convenient in practice. Special forms of apparatus with attached resistances and contact-marking devices, and called potentiometers, are now available for all tests of this kind [see 57 above].

80. Standard Daniell cell. Some form of Daniell's cell is easily set up, and though less trustworthy is convenient for use as a standard when Clark cells are not available. A small current through such a cell does no harm. A well-known form is Raoult's, which has the zinc and copper solutions in separate vessels connected by a tube filled with zinc sulphate and tied over the ends with bladder. This, when made with a plate of pure zinc amalgamated with mercury and a plate of electrolytically deposited copper, was found by Lord Rayleigh to have an electromotive force of approximately ·7703 of that of a Clark cell.

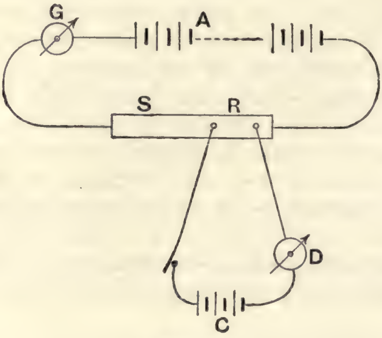

FIG. 153.

A standard Daniell's cell was used by Lord Kelvin, which consists of a zinc plate resting at the bottom of a glass vessel in a stratum of saturated zinc sulphate, and a copper plate in a solution of copper sulphate of density 1·02, which has been so gently placed on the stratum of zinc sulphate as to leave a clear surface of separation. The copper sulphate solution is introduced by means of a glass tube dipping down into the liquid and terminating in a fine point, which is bent into a horizontal direction so as to deliver the liquid with as little disturbance as possible. This tube is connected by a piece of indiarubber tubing with a funnel, by the raising or lowering of which the sulphate of copper can be run into or run out of the cell. By this means the sulphate of copper is run in when the cell is to be used, and at once removed when the cell is no longer wanted. The solutions should be kept in stock bottles and the cell set up fresh when wanted.

The standard Daniell's cell is very conveniently used along with a Daniell's battery in the manner represented in the diagram (Fig. 154). *C* is the standard cell, and *B* a battery of from 30 to 40 small gravity Daniells.* A circuit is formed of a resistance box, the galvanometer *G* to be graduated, and the battery *B* joined in series with the standard cell *C*. A sensitive galvanometer *D*, which may be a reflecting galvanometer, or any very sensitive galvanometer of low resistance, has one terminal attached at a point *M* between the battery and the standard cell, and the other terminal through the key *K* to an intermediate terminal *L* of the resistance box. The resistances in the box, on the two sides of *L*, are adjusted until no current flows through the galvanometer *D*, when the key is depressed.

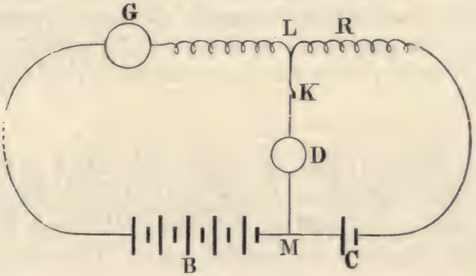

FIG. 154.

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* These can be very easily made by using large preserve-pots as containing vessels, and placing at the bottom of each a copper disc of from three to three and a half inches in diameter, in a stratum of saturated copper sulphate solution, and a grating or plate of zinc a little below the mouth of the vessel immersed in a solution of zinc sulphate, of density 1·2. The copper sulphate may be kept saturated by crystals dropped into a glass tube passing down through a hole in the zinc plate to the copper. A copper wire well covered with guttapercha should be used as the electrode of the copper plate.

Let R be the resistance in the box to the right of L, r the resistance of the cell C, and G the resistance of the galvanometer. Then if V be the difference of potential, in volts, between the terminals of the galvanometer,

$$V = 1{\cdot}072\,\frac{G}{R+r}. \qquad \text{(78)}$$

In practice a resistance of from 300 to 400 ohms is generally required for R. The electromotive force of the standard cell was determined by Prof. T. Gray and found to be 1·072 volts at ordinary temperature. [A determination of the electromotive force of the same cell has also been made by Lord Rayleigh, who found it to be ·743 of a Clark cell, the electromotive force of which was 1·4542 B.A. volts, nearly, at 15°. This would give very approximately 1·068 true volts for the Daniell cell.] It was taken as 1·072 volts, as, notwithstanding the large battery in the circuit, the total resistance is so great that there is very little polarization. This method in fact is peculiarly well adapted for the Daniell's cell, as the slight current flowing through serves to keep its plates in a constant and fresh state. It is known as Lumsden's and also as Bosscha's method of comparing electromotive forces.

The difference of potential, the magnitude of which is thus obtained, is chosen such as to give a convenient deflection on the instrument to be graduated.

Note. The Weston cell, in which the chief difference is that cadmium is used instead of the zinc of the Clark cell, has a lower temperature coefficient, and is now generally used for exact work. Full particulars are given in Appendix VII. [See also 57 above.]

CHAPTER XIII.

CALCULATION OF INDUCTANCES.

1. Geometric mean distance—g.m.d. The calculation of inductances is facilitated in various important cases by a knowledge of what Maxwell called the geometric mean distance of a straight uniform conductor, of given form of cross-section, from a parallel conductor of the same or a different form of cross-section. Let us consider the self-inductance of a circuit consisting of two long parallel wires, A and B, carrying equal currents which flow in opposite directions. Unless the current is alternating and of high frequency, we may suppose that it is uniformly distributed over the cross-section of the wire. If the strength of the current be γ, and S_1, S_2 denote the areas of cross-section of A and B respectively, the current per unit area of cross-section is γ/S_1 in A, and γ/S_2 in B. We may suppose each wire made up of the same number, n, of equal slender uniform filaments, having their lengths parallel to the direction of flow, and each carrying a current of strength γ/n. By taking n large we can make the value of γ/n as small as we please.

2. Theory of circuit of two long parallel conductors. We calculate first the energy of the system per unit length of the conductors. In order to do so, we suppose, what is most frequently the case, that the magnetic permeability is everywhere unity, and then take into account the permeability of the conductors when that has a different value.

In dealing with this circuit we shall consider the parallel conductors as practically infinitely long, that is, such that lengths of the conductors which are great in comparison with the distance between them lie opposite to one another, so that the influence of the cross conductors at the ends may be neglected for any element considered. Thus, we shall calculate only the induction, and corresponding energy, for a portion of the circuit intercepted between two parallel planes perpendicular to the conductors and at unit distance apart.

Let the distance between the conductors be measured in a plane at right angles to the two conductors, from a convenient point in one cross-section made by that plane to a convenient point in the other, and be denoted by b. Then let these points be taken as origins of rectangular coordinates, x, in the direction in which b is measured, and y, at right angles to that direction, by which the position of the cross-section of any particular filament can be specified. We denote the coordinates of a filament in A by x_1, y_1, of a filament in B by x_2, y_2, and shall indicate a particular filament by its coordinates enclosed in brackets, as (x_1, y_1) or (x_2, y_2).

Now let (x'_1, y'_1) denote a second filament in A, dS'_1 its area, r' its distance from any point in the plane of its cross-section, r'_{11} its distance from the filament (x_1, y_1), and r'_{12} its distance from the filament (x_2, y_2) in B. The induction produced by (x'_1, y'_1) through unit length of the circuit formed by the two filaments (x_1, y_1), (x_2, y_2) is by above,

$$2\gamma\frac{dS_1}{S_1}\int_{r'_{11}}^{r'_{12}}\frac{dr'}{r'}=2\gamma\frac{dS'_1}{S_1}(\log r'_{12}-\log r'_{11}). \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Hence the total induction dB_A, per unit of length, through this filamental circuit, produced by the current in A, is given by

$$dB_A=\frac{2\gamma}{S_1}\int_A dS'_1(\log r'_{12}-\log r'_{11}), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where the integral is taken, as indicated, over the cross-section of A. To this must be added the induction through this circuit due to the filaments of B. According to the notation adopted above, we denote the distances of any filament (x'_2, y'_2) of B from (x_2, y_2) and (x_1, y_1) by r'_{22} and r'_{21} respectively. Thus, if B_B denote the numerical value of the induction specified,

$$dB_B=\frac{2\gamma}{S_2}\int_B dS'_2(\log r'_{21}-\log r'_{22}). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

3. Electrokinetic energy of a current. Geometric mean distances. The electrokinetic energy of the circuit is half the product of the induction by the current in the circuit. The value of the current may be written $\gamma\, dS_1/S_1$ or $\gamma\, dS_2/S_2$. Using the second form in the first term of the integral in (2) and the second term of the integral in (3), and the first form in the remaining two terms of these integrals, we get

$$dT=\gamma^2\left\{\frac{dS_2}{S_1S_2}\int_A dS_1\log r'_{12}-\frac{dS_1}{S_1{}^2}\int_A dS'_1\log r'_{11}\right.$$
$$\left.+\frac{dS_1}{S_1S_2}\int_B dS'_2\log r'_{21}-\frac{dS_2}{S_2{}^2}\int_B dS'_2\log r'_{22}\right\}. \quad \ldots\ldots\ldots(4)$$

Hence we get the total electrokinetic energy by finding the values of dT for all the circuits which can be formed. Thus we have only to integrate each of the terms of (4) over S_1, or over S_2, as the case may be. Thus

$$T=\gamma^2\left\{\frac{2}{S_1S_2}\int_{B_1}\int_A\log r_{12}\,dS_1\,dS_2-\frac{1}{S_1{}^2}\int_A\int_A\log r'_{11}\,dS_1\,dS'_1\right.$$
$$\left.-\frac{1}{S_2{}^2}\int_B\int_B\log r'_{22}\,dS_2\,dS'_2. \quad \ldots(5)\right.$$

If we write
$$\left.\begin{aligned} S_1 S_1 \log R_{12} &= \int_B \int_B \log r_{12}\, dS_1\, dS_2, \\ S_1^2 \log R_{11} &= \int_A \int_A \log r'_{11}\, dS_1\, dS'_1, \\ S_2 \log R_{22} &= \int_B \int_B \log r'_{22}\, dS_2\, dS'_2, \end{aligned}\right\} \quad \text{.................. (6)}$$

then R_{12}, R_{11}, R_{22} are called *geometric mean distances*, R_{12} of the area S_1 from S_2, R_{11} of S_1 from itself, and R_{22} of S_2 from itself. We shall denote *geometric mean distance* by g.m.d.

4. Self-inductance of a circuit. The determination of the self-inductance of a circuit composed of two long straight parallel wires, is thus reduced to the calculation of the g.m.d.s of the cross-sectional areas from themselves (each from itself) and from one another. The conductors may have any form of cross-section, and the calculation of their self-inductions is of course theoretically possible. Its evaluation, however, except in a few comparatively simple but important cases, is a tedious and troublesome operation. We shall consider these cases presently; in the meantime we can infer from electrical results already obtained the required g.m.d. for two right circular conductors whether tubular or solid, subject to the conditions stated in 7.

In the first place, the g.m.d. of the conductors from one another is equal to the distance between their axes. As we have seen in V. 12 above, the magnetic force at any point external to either of the conductors, A say, and due to A, is the same as if the whole current in that conductor flowed in a filament along the axis. Thus the induction through any external area may be found by supposing A replaced by an axial filament carrying the same current. The electromagnetic action of the current in A on unit length of an external parallel filament, carrying unit current, is $2\gamma/r$, if r be the distance of the filament from the axis of A. We infer therefore that the action of the filament on A is the same as would be exerted on an axial filament replacing the latter. Thus the total action of the conductor B on A is the same as it would be if the conductors were replaced by filaments coinciding with their axes.

It follows from this that the expression for their mutual electrokinetic energy must be the same as it would be if the conductors were replaced by axial filaments, that is, the g.m.d. between the conductors is equal to the distance between their axes. A direct analytical proof of this theorem will be given presently.

The reader may establish the following results from the formulae given above for the g.m.d.

5. Examples of the use of g.m.d. (1) *The self-inductance of a straight tape of length l and breadth b, and of negligible thickness* [see 14 below]. This is approximately equal to the mutual inductance of two parallel straight lines at distance apart, R, equal to the g.m.d. of the section of the conductor from itself, which, since the thickness is negligible, is given by (6),

$$\log R_1 = \log b - \tfrac{3}{2}, \quad \text{.............................(7)}$$

or

$$R_1 = 0{\cdot}22313b.$$

Thus, for the self-inductance L_1 we obtain the approximate value [see also (9) below]

$$L_1 = 2l\left(\log\frac{2l}{R_1} - 1\right) = 2l\left(\log\frac{2l}{b} + \frac{1}{2}\right). \quad \text{..................(8)}$$

If the thickness is not negligible a more exact formula must be employed. The self-inductance of a straight bar of length l, and rectangular section of breadth b and thickness c, is given by

$$L_1 = 2l\left(\log\frac{2l}{b+c} + \frac{1}{2} + 0{\cdot}2235\frac{b+c}{l}\right). \quad \text{..................(9)}$$

For the mutual induction of two thin tapes of length l laid closely together side by side in the same plane, as in Fig. 155, without touching at the edges, we have, if R be the g.m.d. of one tape from the other, that is $0{\cdot}89252b$,

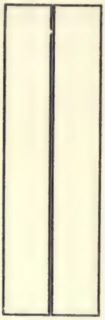
FIG. 155.

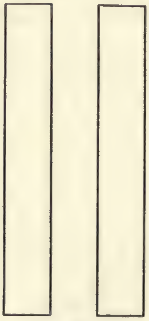
FIG. 156.

FIG. 157.

$$M_2 = 2l\left(\log\frac{2l}{R} - 1\right) = 2l\left(\log\frac{2l}{b} - 0{\cdot}8863\right). \quad \text{...............(10)}$$

If a return circuit is made up of these two tapes the self-inductance is clearly $2L_1 - 2M$, where L_1 is the self-inductance of a single tape and M is the mutual inductance of the two tapes. Thus we get, by (7) and (10),

$$L = 2L_1 - 2M = 4l\left(\log\frac{R_2}{R_1}\right) = 4l\log 4, \quad \text{...................(11)}$$

if R_2 denote the g.m.d. of one tape from the other, and R_1 the g.m.d. of either tape from itself. This self-inductance is thus independent of the breadth of the tapes.

If, as in Fig. 156, the near edges of the coplanar tapes are at a distance b apart equal to the breadth of the tapes,

$$R_2 = 1{\cdot}95653b, \quad L = 8{\cdot}685l.$$

If the tapes are not in the same plane, but are parallel and opposite to one another, as shown in section in Fig. 157, at a distance apart equal to their breadth we have $\log(R_2/R_1) = \frac{1}{2}\pi$, and

$$L = 4l \times \tfrac{1}{2}\pi = 2\pi l. \quad \text{.............................(12)}$$

The self-inductance is thus 2π per unit length of the double conductor, and for this distance apart is independent of the width. One-half of the strip conductor is here a return for the other half, an arrangement often made by doubling a strip conductor on itself to diminish the inductance. We have then an easily remembered rule for the residual inductance.

If the tapes have their planes parallel and opposite and at a distance d apart the g.m.d. between them is given by

$$\log R_2 = \frac{d^2}{b^2}\log d + \frac{1}{2}\left(1 - \frac{d^2}{b^2}\right)\log(b^2 + d^2) + 2\frac{d}{b}\tan^{-1}b - \frac{3}{2}, \quad \text{.....(13)}$$

and when $d = b$, $$\log R = \log b + \tfrac{1}{2}\pi - \tfrac{3}{2}. \quad \text{...........................(13')}$$

From this by the formula used in (11) we obtain, when $d = b$, once more the result stated in (12).

6. Calculation of g.m.d. We can now calculate the g.m.d. in some important cases as a preliminary to the discussion of inductances. Incidentally proofs of the results already stated will appear. It is first to be noticed that if there exist any number of areas of extent A, B, etc., the g.m.d.s of which, R_A, R_B, etc., from another area S are known, it follows from the definition that the g.m.d., R of their sum from S, is given by the equation

$$\log R = \frac{A \log R_A + B \log R_B + \ldots}{A + B + \ldots}. \quad \text{....................(14)}$$

We consider first the g.m.d. of a circular area, annular or complete, from a point P, (1) external to the area and in its plane, (2) in the circular area itself. Let b be the distance of P from the centre of the circular area, a, a' the internal and external radii of the latter, x and $x + dx$ the radii of two intermediate circles very near to one another. Let two radii OR, OS (Fig. 158) be drawn, making the angles θ, $\theta + d\theta$ with OP, so as to intercept the element of area $x\,dx\,d\theta$, on the annulus bounded by the circles of radii x and $x + dx$. The distance of P from the element, or r, is $\sqrt{b^2 + x^2 - 2bx\cos\theta}$.

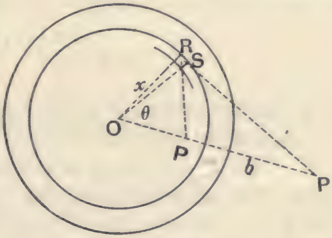

FIG. 158.

Hence the g.m.d. of the annulus from P is given by

$$2\pi x\,dx\log R=\tfrac{1}{2}x\,dx\int_0^{2\pi}\log(b^2+x^2-2bx\cos\theta)\,d\theta.$$

This can be written

$$2\pi x\,dx\log R=\tfrac{1}{2}x\,dx\int_0^{2\pi}\log\left(1+\frac{b^2}{x^2}-2\frac{b}{x}\cos\theta\right)d\theta+x\,dx\log x\int_0^{2\pi}d\theta.\quad\ldots.(15)$$

Now the first integral on the right of (15) is known to have the value $4\pi\log b/x$, if $b>x$, and 0 if $b<x$. Hence in the first case for the annulus

$$2\pi x\,dx\log R=2\pi x\,dx(\log b-\log x)+2\pi x\,dx\log x$$

or

$$R=b.\quad\ldots\ldots\ldots\ldots(16)$$

On the other hand, if P be within the inner boundary of the elementary annulus $b/x<1$, and the first integral of (15) is zero. Hence we have for the annulus in case (2)

$$2\pi x\log R=2\pi x\log x$$

or

$$R=x.\quad\ldots\ldots\ldots\ldots(17)$$

From these results it follows by (14) that the g.m.d. of any finite annulus from an external point P is simply the distance of the point from the centre. For the annulus is made up of elementary annuli, every one of which has the same g.m.d. from P. This includes, of course, as a particular case a complete circular area.

The g.m.d. of a finite annulus from a point within its inner bounding circle is now easily found. The area of the annulus is $\pi(a'^2-a^2)$. Hence by (14), if R be the g.m.d. required,

$$\pi(a'^2-a^2)\log R=2\pi\int_a^{a'}x\log x\,dx=\pi\left(a'^2\log a'-a^2\log a-\frac{a'^2-a^2}{2}\right),$$

that is

$$\log R=\frac{a'^2\log a'-a^2\log a}{a'^2-a^2}-\frac{1}{2}.\quad\ldots\ldots\ldots\ldots(18)$$

Lastly, if P be on the annulus at a distance b from the centre, the annulus divides into two parts, one internal and the other external to the concentric circle through P. Hence by (16), (18) and (14), if R now denote the g.m.d. for the whole area in this last case,

$$\log R=\frac{a'^2\log a'-a^2\log b}{a'^2-a^2}-\frac{1}{2}\frac{a'^2-b^2}{a'^2-a^2}.\quad\ldots\ldots\ldots\ldots(19)$$

7. Values of g.m.d. in various cases. The following corollaries follow at once from these results.

1. The g.m.d. from a circular area (complete or annular) of any area external to the circular area, and in the same plane, is equal to the g.m.d. of the figure from the centre of the circle. For the g.m.d. of every part of the external area is its distance from the centre, and the result follows by (14).

2. The g.m.d. of any figure completely internal to an annular area

from that area is the value of R given by (18). For R is the g.m.d. of every element.

3. The g.m.d. of a circular annulus of infinitesimal breadth from itself is simply its radius. For the g.m.d. of every point on it from the annulus is the radius.

4. The g.m.d. of the finite annulus from itself is given by

$$\log R = \log a' - \frac{a^4}{(a'^2-a^2)^2}\log\frac{a'}{a} + \frac{1}{4}\frac{3a^2-a'^2}{a'^2-a^2}. \quad\text{............(20)}$$

For consider the g.m.d. of the annulus from a point in it distant x from the centre. The g.m.d. of the internal part is x, the logarithm of the g.m.d. of the external part is $(a'^2 \log a' - x^2 \log x)/(a'^2 - x^2) - \frac{1}{2}$. Hence, as found in (19), the g.m.d. of the whole area from the point is given by

$$\log R' = \frac{a'^2 \log a'}{a'^2-a^2} - \frac{a'^2-x^2}{2(a'^2-a^2)} - \frac{a^2}{a'^2-a^2}\log x. \quad\text{............(21)}$$

The g.m.d. of an infinitesimal annulus of breadth dx and radius x from the total area is thus R'. Hence by (14) the g.m.d. of the whole area from itself is to be found from

$$\pi(a'^2-a^2)\log R = \int_a^{a'} 2\pi x\, dx \log R'.$$

Substituting the value of $\log R'$ from (21) and integrating we obtain (20).

If $a=0$, the area is a complete circle, and (20) gives in that case for the g.m.d. of the circle from itself

$$\log R = \log a' - \frac{1}{4},$$

or

$$R = a'e^{-\frac{1}{4}} = \cdot 7788a'. \quad\text{............................(22)}$$

A good example is the cable, consisting of a wire of radius a, within a coaxial tube of radii b and c, discussed in IX. The wire and tube are supposed to be of the same material. It is required to find the self-inductance L, per unit length for steady currents.

For the present we suppose all the inductivities to be unity. By (3), (5) and (6),

$$L = 4\log R_{12} - 2\log R_{11} - 2\log R_{22}, \quad\text{..................(23)}$$

where R_{11}, R_{22}, R_{12} are the g.m.d.s of the cross-sections, R_{11} for the wire, R_{22} for the tube. By the results stated in (18) and (20),

$$\log R_{11} = \log a - \tfrac{1}{4},$$

$$\log R_{22} = \log c - \frac{b^4}{(c^2-b^2)^2}\log\frac{c}{b} + \frac{1}{4}\frac{3b^2-c^2}{c^2-b^2},$$

$$\log R_{12} = \frac{c^2\log c - b^2\log b}{c^2-b^2} - \frac{1}{2}.$$

Assembling these results and reducing we get

$$L=2\log\frac{b}{a}+\frac{2c^4}{(c^2-b^2)^2}\log\frac{c}{b}-\frac{3c^2-b^2}{2(c^2-b^2)}+\frac{1}{2}, \quad \text{............}(23')$$

which agrees with IX. 10 (52).

It is obvious that here the only part of the magnetic induction the lines of which are in the space between the wire and the tube, is that due to the (unit) current in the wire. If the inductivity of the medium in the space is μ', this induction is $2\mu'\log(b/a)$. The remaining lines are in space of inductivity μ. Thus taking account of these inductivities we get

$$L=2\mu'\log\frac{b}{a}+\tfrac{1}{2}\mu+\frac{2\mu c^4}{(c^2-b^2)^2}-\tfrac{1}{2}\mu\frac{3c^2-b^2}{c^2-b^2}. \quad \text{............}(23'')$$

It will be noticed that the symmetry of the arrangement is such that no lines of induction pass from one region to another, so that the magnetization of one part does not affect the magnetic induction in another part.

There is no difficulty in establishing this result by direct integration by considering the current within any coaxial surface of radius x, and the induction which that produces through a further radial distance dx. The product of these gives the energy, and therefore L is obtained by integration. [See 15 below.]

Next consider the g.m.d. of a line from any point P. Let AB (Fig. 159) be the line, p the length of the perpendicular from P on the line, a and a' the lengths of the segments AO, OB into which the line is divided at O. Then the distance from P of any point Q in the line at distance x from O is $\sqrt{p^2+x^2}$. Hence for the line

$$\begin{aligned}(a+a')\log R &= \int_{-a}^{a'}\log\sqrt{p^2+x^2}\,dx\\ &= \tfrac{1}{2}a'\log(a'^2+p^2)+\tfrac{1}{2}a\log(a^2+p^2)-(a+a')\\ &\qquad +p\left(\tan^{-1}\frac{a}{p}+\tan^{-1}\frac{a'}{p}\right). \quad \text{..................}(24)\end{aligned}$$

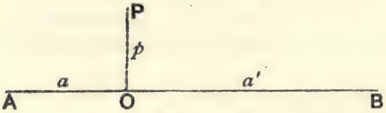

FIG. 159.

If O coincide with the centre of AB, $a=a'=\frac{1}{2}$ the length of AB, and

$$\log R=\tfrac{1}{2}\log(a^2+p^2)-1+\frac{p}{a}\left(\tan^{-1}\frac{a}{p}\right). \quad \text{..............}(25)$$

If O coincide with B, $a=0$, and

$$\log R=\tfrac{1}{2}\log(a'^2+p^2)-1+\frac{p}{a}\tan^{-1}\frac{a}{p}, \text{.................}(25')$$

where a' is now the whole length of AB.

If two lines are of length a, and are collinear, and their centres are at a distance na, the g.m.d. R for the two is given by

$$\log R = \tfrac{1}{2}(n+1)^2 \log\{(n+1)a\} - n^2 \log(na) + \tfrac{1}{2}(n-1)^2 \log\{(n-1)a\} - \tfrac{3}{2}. \quad (26)$$

For this case a very convenient calculation formula is

$$\log R = \log n - \left(\frac{1}{12n^2} + \frac{1}{60n^4} + \frac{1}{168n^6} + \frac{1}{360n^8} + \frac{1}{660n^{10}} + \dots\right), \quad \dots(27)$$

where it is understood that $n > 1$.

8. G.m.d. for lines, rectangles and squares. From (25) we get at once the g.m.d. from the centre for four lines forming a rectangle. For let the length of the rectangle be a and its breadth b. Then, since for the ends $p = \tfrac{1}{2}a$, and for the sides $p = \tfrac{1}{2}b$,

$$2(a+b)\log R = (a+b)\log\frac{a^2+b^2}{4} - 2(a+b) + 2a\tan^{-1}\frac{b}{a} + 2b\tan^{-1}\frac{a}{b}$$

or

$$\log R = \tfrac{1}{2}\log\frac{a^2+b^2}{4} - 1 + \frac{a}{a+b}\tan^{-1}\frac{b}{a} + \frac{b}{a+b}\tan^{-1}\frac{a}{b}. \quad \dots\dots(28)$$

If the rectangle be a square $a = b$, and thus

$$\log R = \log\left(\frac{a}{\sqrt{2}}\right) - 1 + \frac{\pi}{4} = ae^{-\cdot 2146}/\sqrt{2}. \quad \dots\dots\dots\dots(29)$$

The g.m.d. of two parallel lines from one another can now be found. This is an important case, as it enables the self-inductance of a circuit composed of two parallel thin sheets of conducting material to be calculated. Let AB, CD (Fig. 160) be the lines, and E the foot of the perpendicular from A on CD. Let x be the distance of P from E, and p its distance from AB, α and β the distance of C and D from E, taken as positive quantities when measured from E to the right, and negative when measured the other way. The length of CD is thus $\beta - \alpha$, and if a be the length of AB, we have to put in (24) x for a, and $a - x$ for a'. Thus multiplying the expression on the right of (24) by dx, and integrating from $x = \alpha$ to $x = \beta$ we find by (14), if R now be the g.m.d. of CD from AB,

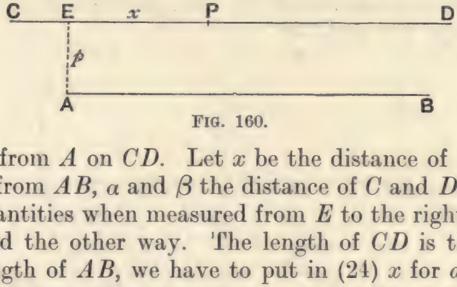

FIG. 160.

$$\begin{aligned} a(\beta-\alpha)\log R = {} & \tfrac{1}{4}(\beta^2 - p^2)\log(\beta^2 + p^2) - \tfrac{1}{4}(\alpha^2 - p^2)\log(\alpha^2 + p^2) \\ & - \tfrac{1}{4}\{(a-\beta)^2 - p^2\}\log\{(a-\beta)^2 + p^2\} \\ & + \tfrac{1}{4}\{(a-\alpha)^2 - p^2\}\log\{(a-\alpha)^2 + p^2\} \\ & + p\beta\tan^{-1}\frac{\beta}{p} - p\alpha\tan^{-1}\frac{\alpha}{p} \\ & - p(a-\beta)\tan^{-1}\frac{a-\beta}{p} + p(a-\alpha)\tan^{-1}\frac{a-\alpha}{p} \\ & - \tfrac{3}{2}a(\beta-\alpha). \quad \dots\dots\dots\dots(30) \end{aligned}$$

The value of R given by this equation may be used for the calculation of the self-induction of a circuit composed of two long thin strips of conducting material arranged with their lengths and planes parallel. The lines AB, CD represent the cross-sections of such an arrangement made by a plane at right angles to the conductors.

The g.m.d. of each line from itself can of course be found from (30) by putting $\alpha = 0$, $\beta = a$ (the length of the line considered), and $p = 0$. We thus obtain

$$\log R = \log a - \tfrac{3}{2}, \qquad \text{(31)}$$

which can be verified at once by calculating directly for this particular case.

If $\alpha = 0$, and the lines have each the same length $2a$, while still situated as in Fig. 160, the equation for the g.m.d. reduces to

$$\log R = \tfrac{1}{2}\left(1 - \frac{p^2}{4a^2}\right)\log(4a^2 + p^2) + \tfrac{1}{4}\frac{p^2}{a^2}\log p + \frac{p}{a}\tan^{-1}\frac{2a}{p} - \tfrac{3}{2};$$

and if p, the distance between the lines, be equal to $2a$, the equation reduces to

$$\log R = \log 2a + \tfrac{1}{2}\pi - \tfrac{3}{2}.$$

We can now find the g.m.d. of a given line from an area in the same plane. We shall consider first a given line and a parallel rectangle, and from the result for this case deduce the g.m.d. of two parallel coplanar rectangles from one another. The practically important arrangements are those in which the line and rectangle, or two rectangles, are symmetrical about the line passing through their centres, as shown in Figs. 157 and 159.

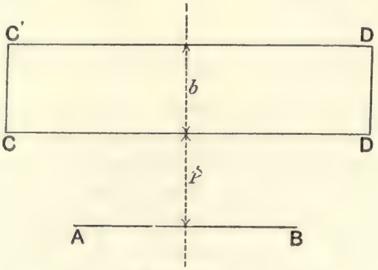

FIG. 161.

Taking first the line and rectangle, as in Fig. 161, and putting a, a' for the lengths AB and CD, we suppose the rectangle to be generated by the motion of CD, at right angles to itself through a distance b, the breadth of the rectangle. We may thus suppose CD made up of parallel strips of area, each of infinitesimal breadth dp, and find the g.m.d. of the rectangle by multiplying the expression for $\log R$ in (30) (modified to suit the circumstances supposed) by dp, and integrating from p to $p + b$. The constant factors on the left will for simplicity be retained.

We have here $\alpha=\frac{1}{2}(a-a')$, $\beta=\frac{1}{2}(a+a')$, so that

$$a-\alpha=\tfrac{1}{2}(a+a')=\beta,$$
$$a-\beta=\tfrac{1}{2}(a-a')=\alpha.$$

Thus (30) becomes

$$aa'\log R=\tfrac{1}{2}(\beta^2-p^2)\log(\beta^2+p^2)-\tfrac{1}{2}(\alpha^2-p^2)\log(\alpha^2+p^2)$$
$$+2p\beta\tan^{-1}\frac{\beta}{p}-2p\alpha\tan^{-1}\frac{\alpha}{p}-\tfrac{3}{2}aa'. \qquad\text{............(32)}$$

Multiplying by dp, integrating as stated above, and putting R now for the g.m.d. of the rectangle CD' from the line AB, we obtain

$$\begin{aligned}aa'b\log R=\tfrac{1}{2}(p+b)&\left\{\beta^2-\frac{(p+b)^2}{3}\right\}\log\{(p+b)^2+\beta^2\}\\&-\tfrac{1}{2}p\left(\beta^2-\frac{p^2}{3}\right)\log(p^2+\beta^2)\\&+\beta(p+b)^2\tan^{-1}\frac{\beta}{p+b}-\beta p^2\tan^{-1}\frac{\beta}{p}\\&+\tfrac{1}{3}\beta^3\tan^{-1}\frac{p+b}{\beta}-\tfrac{1}{3}\beta^3\tan^{-1}\frac{p}{\beta}\\&-(\text{the same series of terms with }\beta\text{ replaced by }\alpha)\\&-\tfrac{11}{6}aa'b. \qquad\text{..............................(33)}\end{aligned}$$

A rectangle of breadth b, might have been generated by moving the line AB away from CD (Fig. 162). We should have obtained the same expression for the g.m.d. of the latter rectangle from the line CD, as is given in (33) for the other case. That these two g.m.d.s are equal is easily seen from (32). Each rectangle may be divided into the same number of strips of equal breadth, and the g.m.d. of each strip in the rectangle CD' from AB is the same as the g.m.d. of each strip of AB' from CD, so that the result follows by (14).

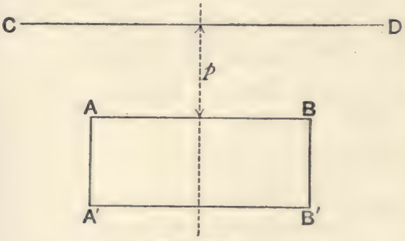

FIG. 162.

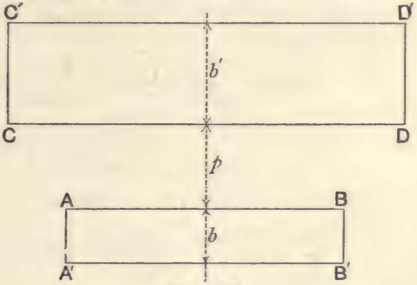

FIG. 163.

We can now find the g.m.d. in the important case of the two rectangles shown in Fig. 163. Multiplying the expression on the right of (33) by dp, integrating from p to $p+b'$ (so that p is the distance of AB from CD) and arranging the results, we find for R the g.m.d. of the rectangles from one another.

$$\begin{aligned}4aa'\,bb'\log R = {} & [(p+b+b')^2\{\beta^2 - \tfrac{1}{6}(p+b+b')^2\} - \tfrac{1}{6}\beta^4]\log\{(p+b+b')^2+\beta\}^2 \\ & - [(p+b')^2\{\beta^2 - \tfrac{1}{6}(p+b')^2\} - \tfrac{1}{6}\beta^4]\log\{(p+b')^2+\beta^2\} \\ & - [(p+b)^2\{\beta^2 - \tfrac{1}{6}(p+b)^2\} - \tfrac{1}{6}\beta^4]\log\{(p+b)^2+\beta^2\} \\ & + \{p^2(\beta^2 - \tfrac{1}{6}p^2) - \tfrac{1}{6}\beta^4\}\log(p^2+\beta^2) \\ & - (\text{the same series of terms with } \beta \text{ replaced by } \alpha) \\ & + \tfrac{4}{3}\beta(p+b+b')\left\{(p+b+b')^2\tan^{-1}\frac{\beta}{p+b+b'}\right. \\ & \qquad \left. + \beta^2\tan^{-1}\frac{p+b+b'}{\beta}\right\} \\ & - \tfrac{4}{3}\beta(p+b')\left\{(p+b')^2\tan^{-1}\frac{\beta}{p+b'} + \beta^2\tan^{-1}\frac{p+b'}{\beta}\right\} \\ & - \tfrac{4}{3}\beta(p+b)\left\{(p+b)^2\tan^{-1}\frac{\beta}{p+b} + \beta^2\tan^{-1}\frac{p+b}{\beta}\right\} \\ & + \tfrac{4}{3}\beta p\left\{p^2\tan^{-1}\frac{\beta}{p} + \beta^2\tan^{-1}\frac{p}{\beta}\right\}\end{aligned}$$

$$\begin{aligned}& - (\text{the same series of trigonometrical terms with } \beta \text{ replaced by } \alpha) \\ & - \tfrac{1}{2}(\beta^2 - \alpha^2)\{(p+b+b')^2 - (p+b')^2 - (p+b)^2 + p^2\} - \tfrac{22}{3}aa'\,bb'. \quad (34)\end{aligned}$$

Here it is to be remembered that $\beta = \frac{1}{2}(a+a')$, $\alpha = \frac{1}{2}(a-a')$.

The g.m.d. of either rectangle from itself can be found from (34) by putting $\alpha = 0$, $a = a' = \beta$, $b = b'$, $p+b = p+b' = 0$. Hence for the g.m.d. from itself of a rectangle of length a and breadth b, we have the equation

$$\begin{aligned}\log R = {} & \frac{1}{2}\log(a^2+b^2) - \frac{1}{12}\frac{b^2}{a^2}\log\left(1+\frac{a^2}{b^2}\right) - \frac{1}{12}\frac{a^2}{b^2}\log\left(1+\frac{b^2}{a^2}\right) \\ & + \frac{2}{3}\frac{b}{a}\tan^{-1}\frac{a}{b} + \frac{2}{3}\frac{a}{b}\tan^{-1}\frac{b}{a} - \frac{25}{12}. \quad \text{..................(35)}\end{aligned}$$

If the rectangle is a square $a = b$, and

$$\log R = \log a + \frac{1}{3}\log 2 + \frac{\pi}{3} - \frac{25}{12}$$

or

$$R = \cdot 44705a. \quad \text{..................................(36)}$$

9. G.m.d. for adjacent squares in different relative positions. For two adjacent squares placed as in Fig. 164 we get, with a as the length

FIG. 164.

of side, from this result for the g.m.d. between them,

$$\log R = \log a + \tfrac{7}{24}\log 5 + \log 2 + 2\tan^{-1}\tfrac{1}{2} - \tfrac{25}{12}, \quad \text{............(37)}$$

or, as the reader may verify,

$$R = 1{\cdot}0065498a. \qquad\qquad (37')$$

Thus the g.m.d. is not very different from the distance between the centres.

The g.m.d. between any two squares in a row can easily be found by (34). For example, between any square and the next but one it is given by

$$\log R = \log a - \tfrac{7}{4}\log 5 + \tfrac{27}{4}\log 3 - \tfrac{11}{3}\log 2 + 8\tan^{-1}\tfrac{1}{3} - 4\tan^{-1}\tfrac{1}{2} - \tfrac{25}{12},$$

or

$$R = 1{\cdot}0005106 \times 2a,$$

still more nearly equal to the distance $2a$ between centres.

Consider now two adjacent rows of equal squares and the g.m.d. between a square in one row and a square in the other row. An expression for the g.m.d. is easily obtained by an easy double integration first for one square, then for the other. We record here only the following results. For the two squares which are corner to corner (Fig 165) we have

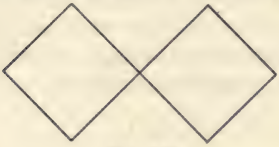

FIG. 165.

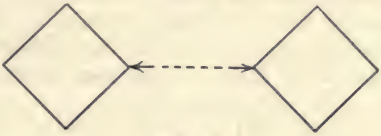

FIG. 166.

$$\log R = \log a + 3\log 2 - \tfrac{7}{12}\log 5 + \pi - 4\tan^{-1}\tfrac{1}{2} - \tfrac{25}{12} = \log(1{\cdot}410962a).$$

The distance between the two centres is $a\sqrt{2} = 1{\cdot}4142135a$.

The g.m.d. for the two squares situated as in Fig. 166 is

$$\begin{aligned}\log R = \log a &- \tfrac{17}{3}\log 2 - \tfrac{27}{8}\log 3 + \tfrac{91}{48}\log 5 + \tfrac{119}{48}\log 13 \\ &- \tfrac{1}{2}\pi - 8\tan^{-1}\tfrac{1}{3} + 5\tan^{-1}\tfrac{1}{2} + 5\tan^{-1}\tfrac{2}{3} - \tfrac{25}{12} \\ = \log a &+ 0{\cdot}8046295. \qquad\qquad (38)\end{aligned}$$

The reader will find a large number of other results completely worked out in a valuable paper by Mr. E. B. Rosa (*B.B.S.W.* 3, p. 1). The results in such cases as those quoted here are important, as giving a means of estimating the mutual inductance between a turn of wire in a coil and neighbouring turns.

10. Self-inductance of circular coil of large radius. The determination of the g.m.d. of the cross-section of a conductor is important in other cases than that of a long straight conductor. The following example may serve to show its importance. If we have a circular coil of n turns each of radius great in comparison with any dimension of cross-section, it is easy to see that the coefficient of self-induction of the coil is very approximately equal to n^2 times the coefficient of mutual induction of two parallel coaxial circles, each of radius equal to the mean radius of the section, and at a distance apart equal to the g.m.d. of the cross-section from itself. For the coefficient of self-induction of a circuit is equal to the total magnetic induction through the circuit produced by unit current, and the coefficient of mutual induction of two circuits is the total induction through either produced by unit current in the other. Consider then the induction through a circle of reference A coaxial with the given circuit, and at a distance from the latter small in comparison with the radius. Let it be supposed as before that the current is of uniform density over the cross-section, so that the cross-section may be supposed divided into a very large number of parallel thin filaments each of cross-section dS. If S be the whole area of cross-section, and unit current flow in the conductor, the current in each filament is $n\,dS/S$. Let a cross-section of the whole system, including A, by a plane through the axis, be taken, and let r_a be the distance of the section of A from that of any one of the system of equal filaments, and r_m the distance between the section of the latter filament and any other of cross-section dS_m. The difference between the total induction produced by the assemblage of filaments through the circuit of this latter filament, and that which they produce through A is

$$2n\int\frac{dS_m}{S}(\log r_a - \log r_m) = 2n(\log R_a - \log R_m),$$

where R_a is the g.m.d of the cross-section of the given conductor from that of A, and R_m is its g.m.d. from dS_m.

Now let A be composed of as many coincident filaments as there are imagined to be in the given circuit. Thus the induction through each filament of the conductor may be compared with that through a corresponding filament of A. Since the number of filaments is S/dS, we have for the total difference between the induction through A, and the sum of the inductions through each filament of the conductor due to the whole assemblage,

$$2n\frac{S}{dS}\left(\log R_a - \int\frac{dS}{S}\log R_m\right) = 2n\frac{S}{dS}(\log R_a - \log R), \quad\text{........}(39)$$

where R is now the g.m.d. of the cross-section of the conductor from itself.

The energy of the given system corresponding to this induction is half the product of the current $n\,dS/S$, in each filament into the expression just found, that is, it is $n^2(\log R_a - \log R)$. This vanishes when $R_a = R$, that is, when the g.m.d. of the cross-section of A from that of the conductor is equal to the g.m.d. of the latter from itself. The energy of the given system is then equal to half the product of the total current into the induction through A, that is, in other words, the self-induction coefficient of the given circuit is equal to that of mutual induction between the given circuit and A.

That the coefficient of mutual induction in the latter case is equal to that between A and an equal circuit B at a distance apart equal to R, if not evident, may be seen as follows. The induction through

A due to the given circuit is for equal currents equal to that produced by A through the given circuit, and by the reasoning above, this is equal to the induction due to A through a circuit B replacing the given circuit at the distance R.

11. Mutual induction of two close coils of large radius. The expression found above (34) for the g.m.d. between two symmetrically placed rectangles is applicable to the approximate calculation of the coefficient of mutual induction of two coils, A and B, of which the cross-sections by a plane passing through the common axis are rectangles, provided the radius of either coil is great in comparison with every dimension of the sections, and with the distance between them. Clearly to find the total induction through coil B due to unit current in A, we may proceed by calculating (a) the total induction through each turn of A due to unit current in that turn; (b) that part of each of these total inductions which does not pass through B. The difference between the sum of the results in (a) and the sum of those in (b) is the coefficient M of mutual induction. First we suppose the current in the coil A to be uniformly distributed over the cross-section, so that if S_1 be the area of the section, and there be n_1 turns each carrying unit current, the current per unit area will be n/S_1. Thus the current across an element of the cross-section dS_1 is $n_1 dS_1/S_1$.

Now consider, as before, the difference between that part of the total induction due to a filament of section dS_1, which escapes a filament of the other coil of section dS_2, and that part which escapes a near coaxial circular circuit of reference. Let r_{12}, r, be the distances from dS_1 to dS_2, and from dS_1 to the cross-section of the circle of reference. The difference of total inductions specified is then

$$n_1 dS_1/S_1 \,.\, (\log\, r_{12} - \log r).$$

Integrating over the whole area S_1 we get for the difference due to all the filaments into which S_1 can be divided the value

$$\frac{n_1}{S_1}\int dS_1 (\log r_{12} - \log r).$$

Now let the other circuit be divided into any convenient number n of circuits, each of the same small area dS_2. It is the difference between the total induction through one of these, and that through the circle of reference that has just been found. We have then $dS_2 = S_2/n$. Hence the result just obtained may be written

$$\frac{nn_1}{S_1 S_2}\int dS_1 dS_2 \log r_{12} - \frac{n_1}{S_1}\int dS_1 \log r.$$

Integrating now over both cross-sections we get for the total difference

$$nn_1(\log R_{12} - \log R),$$

where R_{12} is the g.m.d. between the cross-sections, and R is that of the section S_1 from the circuit of reference.

If the number of turns in the second coil be n_2 instead of n, this result must be reduced in the ratio of n_2 to n, by multiplying it by n_2/n. For accuracy of course n_2 must be large. Hence for the final value of the difference of the total inductions we have

$$n_1 n_2(\log R_{12} - \log R).$$

If $R = R_{12}$, that is, if the g.m.d. of the cross-section of the conductor of reference from A be equal to that of the cross-sections of A and B from one another, the total induction which escapes the conductor of reference is equal to that which escapes the coil; in other words, the coefficient of mutual induction of the two coils is equal to that of the coil A and a coaxial circular conductor, the cross-section of which by any plane through the axis is at a g.m.d. from that of the coil A, equal to that of the cross-sections of A and B from one another.

It must be possible to replace the coil A by a conductor of proper mean radius carrying the whole current of n units which flows in the coil, so that the total induction through it is equal to the sum of those through the coaxial filamental conductors into which the coil has been supposed divided. If the radius of any part of the coil be large in comparison with the dimensions of cross-section, this proper mean radius may be taken as the simple mean radius of the coil. The other coil can then be also supposed replaced by a coaxial circular conductor at a distance from the other equal to R_{12}. Thus the determination of the coefficient of mutual induction of two coaxial coils is reduced to the determination of that of two coaxial circles.

The relative positions of these two circles is not definite. If we consider the lines of force through a coil due to the current in it, we see that these are closed round the coil, and any closed circuit placed in its field will pass through certain lines of force. The circuit may be placed in any position or have any size consistent with passing through the same lines of force, and the coefficient of mutual induction of the coil and circuit will be the same for all. If, in the present case we suppose the primary circular conductor fixed, the other may be situated anywhere on the toroidal surface marked out by the closed nearly circular lines of force, the radius of which is the g.m.d. of the cross-sections.

12. Mutual inductance of two coaxial circles. We proceed now to calculate the coefficients of mutual induction of coaxial circular circuits and coils. Taking first the case of two coaxial circles of nearly equal radii, we see that if we can find their coefficient of mutual induction when the circles are in one plane, we can find that for the actual arrangement by calculating, in the manner described above, the portion of the total induction due to one which escapes passing through the other owing to the deviation from coplanarity.

Consider first two coaxial circles in the same plane. Let the radius of the outer circle be $a + c$, and of the inner a. Then if we take any element ds of the outer circle at A (Fig. 167), and let θ be the angle OAE between the diameter through ds and a line of length r drawn to an element E_1 of area $r\,d\theta\,dr$ in the inner circle, we have for the magnetic induction through that area the value $ds\cos\theta/r^2\,.\,r\,d\theta\,dr$. Hence for the total induction B through the inner circle we get

$$\mathsf{B}=2\int ds\int_{r_1}^{r_2}\int_0^{\theta_1}\frac{\cos\theta\,d\theta\,dr}{r},\qquad\qquad(40)$$

where $r_1=AB$, $r_2=AC$, $\theta_1=\sin^{-1}a/(a+c)$, and the final integral is taken round the outer circle. The distances r_1, r_2 are evidently the roots of the equation

$$r^2-2r(a+c)\cos\theta+(a+c)^2-a^2=0.$$

These roots are

$$\left.\begin{matrix}r_1\\ r_2\end{matrix}\right\}=(a+c)\cos\theta\mp\sqrt{(a+c)^2\cos^2\theta-c^2-2ac}.\qquad\qquad(41)$$

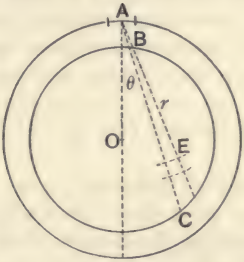

FIG. 167.

If c be very small, then approximately

$$r_2=2a\cos\theta,\quad r_1=c/\cos\theta.$$

Integrating then with respect to r we find

$$B=2\int ds\int_0^{\theta_1}\cos\theta\log\frac{r_2}{r_1}\,d\theta=2\int ds\int_0^{\theta_1}\cos\theta\log\left(\frac{2a}{c}\cos^2\theta\right)d\theta.$$

Now

$$\int_0^{\theta_1}\cos\theta\log\left(\frac{2a}{c}\cos^2\theta\right)d\theta=\sin\theta_1\left\{\log\left(\frac{2a}{c}\text{ccs}^2\,\theta_1\right)-2\right\}$$
$$+\log\frac{1+\sin\theta_1}{1-\sin\theta_1},$$

which reduces approximately to

$$\log 4-2+2\log\sqrt{\frac{2a}{c}}.$$

Hence, to the same degree of approximation,

$$M=4\pi a\left(\log\frac{8a}{c}-2\right).\qquad\qquad(42)$$

Now let the circle of radius a be carried out of the plane of the other circle a distance b while still remaining coaxial with the latter. The difference between the total inductions which escape from passing through the smaller circle in its two positions may be calculated as if the circles were straight. Putting now r for $\sqrt{b^2+c^2}$, the shortest distance between the circular arcs, the difference of inductions per unit of length is approximately $2(\log r - \log c)$, and for the whole circle $4\pi a(\log r - \log c)$. Hence the coefficient M of mutual induction between the circles in the specified configuration is approximately given by

$$M = 4\pi a\left(\log\frac{8a}{r} - 2\right). \qquad \text{(43)}$$

13. Coil of maximum self-inductance. From this result we can find approximately the relative dimensions of a coil of large radius, which for a given length and gauge of wire has a maximum coefficient of self-induction. By the theorem proved in 10 above the self-induction coefficient is equal to the coefficient of mutual induction between two equal coils each of the given mean radius, and at a distance apart equal to the g.m.d. of the cross-section from itself. Let the g.m.d. be R. Then by the preceding result, if the number of turns be n,

$$L = 4\pi n^2 a\left(\log\frac{8a}{R} - 2\right). \qquad \text{(43')}$$

Now for similar sections of different linear dimensions R varies as the dimensions, and since for a given thickness of wire the number of turns varies as the cross-section, we have $n = CR^2$, where C is a constant. Again the total length of wire l is $2\pi na$, so that we have the two conditions, $2\pi na = l$, $2\pi CR^2 a = l$, which give $dn/da = -n/a$, $dR/da = -R/2a$. Hence taking a as independent variable, differentiating the value of L, and substituting these values of dn/da and dR/da, we find

$$\frac{dL}{da} = 4\pi n^2\left(\frac{7}{2} - \log\frac{8a}{R}\right),$$

which for a maximum gives

$$\log R = \log 8a - \frac{7}{2}. \qquad \text{(44)}$$

If the section of the coil is circular of radius ρ, then by (22) above $\log R = \log \rho - \frac{1}{4}$, and (44) gives

$$a = \frac{1}{8}\rho e^{\frac{13}{4}} = 3{\cdot}224\rho. \qquad \text{(45)}$$

For a coil of circular section of the relative dimensions stated in (45) a more exact value of L is given in XV. 22 below. This gives, if the mean diameter of the toroid be D, and the total number of turns be n, $L = 9{\cdot}69 n^2 D$. This is of course for uniformly distributed current.

If the section of the coil is square, the value of R, from (36) substituted in (44), gives

$$a = 1{\cdot}850 s \qquad \text{(46)}$$

if s is the side of the square. These dimensions (a and s) are, however, too nearly equal to enable the approximate formula by which the relation is found to apply with accuracy, and the result can only be regarded as a rough rule to guide the experimenter in the construction of coils.

14. Meaning of self-inductance of part of a circuit. A large amount of valuable and accurate work has been done at the Bureau of Standards at Washington on the inductance of conductors and of coils of different forms. To that work what follows in the present chapter is a good deal indebted. Some remarks are necessary here as to what is meant by the self-inductance of a part (such as a length l of straight uniform wire) of a circuit. The self-inductance of a complete circuit has a perfectly definite value whatever tested and approved law of the action of elements is used for computing it, but that of a part of a circuit is a matter of definition, and this fact must be recalled when the inductance in question is to be used for any practical purpose. For example, the self-inductance of a length l of straight wire of radius a may be defined either as the result obtained by applying the law $ds \,.\, \sin\theta/r^2$ to compute for this wire the total magnetic flux through the plane strip of area extending from the conductor AB in Fig. 168 towards the right to $A'B'$ at infinity, or as the same thing *plus* the flux through the infinite areas above BB' and below AA'. The former would be finite, the latter would be infinite. Unless it is otherwise stated we shall use the former definition.

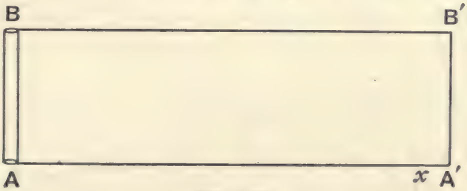

FIG. 168.

The inductance might be defined as that due to the finite conductor supposed provided with a displacement return part, so that it is obtained for a complete circuit. For the present we shall take it as measured by the magnetic flux through the plane area referred to above. It may be remarked that this is exactly the result given by applying to the finite conductor Neumann's mutual inductance formula

$$M=\iint\frac{\cos\epsilon\, ds\, ds'}{r}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(47)$$

where ds, ds' are the lengths of narrow longitudinal filaments of equal cross-section of which the conductor is supposed to be composed, r is the distance apart of the elements, and ϵ the angle between them.

15. Self-inductance of straight conductor of given length. As a first example we take the calculation of the self-inductance of a straight conductor of length l, and of uniform circular cross-section, radius a. The current is we suppose symmetrically, in general uniformly, distributed round the axis. Its action external to the conductor, and to the prolongation of the conductor either way, is exactly the same as if the whole current were concentrated in an infinitely thin filament along the axis. We take first the external action. Let the axis of the conductor be the z-coordinate axis drawn from A as origin, and the x-axis be drawn from A as in the diagram. Consider then an external point P the coordinates of which are x, z, and an element E of length dz' of the axial filament, in which for the present the whole current is supposed to flow. If a line through P parallel to the axis of x meet the axial filament in p, the distance pE is $z'-z$. Then by the law of magnetic action assumed, the magnetic force at P due to the element dz' carrying a current γ is

$$\gamma \frac{dz'}{x^2+(z'-z)^2}\sin\theta = \gamma \frac{x\,dz'}{\{x^2+(z'-z)^2\}^{\frac{3}{2}}}.$$

The induction per unit area of the x, z plane at P is

$$\gamma\int_0^l \frac{x\,dz'}{\{x^2+(z'-z)^2\}^{\frac{3}{2}}} = \frac{\gamma}{x}\left[\frac{l-z}{\{x^2-(z'-z)^2\}^{\frac{1}{2}}} + \frac{z}{(x^2+z^2)^{\frac{1}{2}}}\right].$$

Multiplying this result by an area $dx\,dz$ taken at P, integrating with respect to z from 0 to l, and with respect to x from $x=a$ to $x=\infty$, we obtain

$$\gamma\int_0^l\int_a^\infty \frac{1}{x}\left[\frac{l-z}{\{x^2+(l-z)^2\}^{\frac{1}{2}}} + \frac{z}{(x^2+z^2)^{\frac{1}{2}}}\right]dx\,dz. \quad\ldots\ldots\ldots\ldots\ldots(48)$$

Here it will be observed the integration is not extended above BB' or below AA'.

Calling the induction outside the conductor N, for unit current, we obtain by integration with respect to z,

$$N = 2\int_a^\infty \frac{(x^2+l^2)^{\frac{1}{2}}-x}{x}\,dx = 2\left\{l\log\frac{l+(l^2+a^2)^{\frac{1}{2}}}{a} - (l^2+a^2)^{\frac{1}{2}} + a\right\}$$

$$= 2l\left(\log\frac{2l}{a} - 1\right), \text{ nearly.} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(49)$$

The induction inside the conductor has still to be reckoned. If the diameter be small enough in comparison with the length l we may take the magnetic force per unit of the current γ as $2x/a^2$. Hence the induction through a strip of breadth dx and length l of the conductor is $2xl\,dx/a^2$ if the permeability is unity or $2\mu xl\,dx/a^2$, if the permeability is μ. Thus the whole magnetic induction inside the conductor is μl.

But to find the effect on the self-inductance we must take account of the fact that the induction through any strip of breadth dx and length l at distance x from the axis is put round the current in the cylinder of radius x. The work done in creating this inductance is therefore $\frac{1}{2}\gamma x^2/a^2 . 2\mu\gamma l x^2\,dx/a^2 x$, integrating from $x=0$ to $x=a$, for the total work done in the conductor $\frac{1}{4}\gamma^2\mu l$. But if L be the corresponding part of the inductance we have $\frac{1}{2}L\gamma^2 = \frac{1}{4}\gamma^2\mu l$, and so

$$L = \tfrac{1}{2}\mu l. \qquad\qquad (50)$$

The whole energy due to the induction calculated is thus given by

$$T = \tfrac{1}{2}\gamma^2 2l\left(\log\frac{2l}{a} - 1 + \frac{\mu}{4}\right). \qquad\qquad (51)$$

If L be the self-inductance we have

$$L = 2l\left(\log\frac{2l}{a} - 1 + \frac{\mu}{4}\right). \qquad\qquad (51')$$

This of course is, as already stated, the inductance calculated from the application of Laplace's law to the portion of the circuit specified, and takes no account of the rest of the circuit.

16. Mutual inductance of two parallel wires of equal length. We may now find the mutual inductance in a similar sense to that in which we have just determined self-inductance, for two parallel wires of length l and radius a, opposite one another, with axes at distance d apart. The mutual inductance is in this case measured by the total induction which must pass inwards across one of the wires, A say, when unit current in the other, B, is annulled. Any passage of induction lines across A in consequence of the annulment of the current in any other part of the circuit B is here left out of account. By (49) we have simply to find the induction outside the distance d from B. It is

$$N = 2\left\{(d^2+l^2)^{\frac{1}{2}} - d - l\log\frac{l+(d^2+l^2)^{\frac{1}{2}}}{d}\right\}_d^{\infty}$$

or

$$N = 2l\left(\log\frac{2l}{d} - 1 + \frac{d}{l}\right) \qquad\qquad (52)$$

if the length l is great in comparison with d. The value of N is the mutual inductance sought.

If one of the two wires in this arrangement is part of the return conductors for the other, then neglecting the end connections, and the magnetization of one wire by the current in the other, we get for the self-inductance, from the values just found,

$$L = 2(L_1 - M), \qquad\qquad (53)$$

where L_1 is the self-inductance for either wire, supposing that both have the same permeability. This gives, if d/l is small,

$$L = 4l\left(\log\frac{d}{a} + \frac{\mu}{4}\right). \qquad\qquad (54)$$

The signs of the parts of L on the right of (53) are here reversed from (5) above. This is not very material, but it should be noticed.

The mutual inductance of two straight infinitely thin conductors of lengths l, l' in the same line is also easily found, on suppositions similar to those already stated. By this is meant the magnetic induction which (as calculated by Biot and Savart's law) surrounds the linear conductor BC. From the result already obtained in (52) and the relation in (53) we easily obtain for this mutual inductance $M_{ll'}$ the result

$$M_{ll'} = l \log \frac{l+l'}{l} + l' \log \frac{l+l'}{l'}. \qquad \text{(55)}$$

This may also be obtained by direct integration.

If $l = l'$ this becomes

$$M = 2l \log 2 = 2l \times 0{\cdot}693147, \qquad \text{(55')}$$

in centimetres of course if l is measured in centimetres.

17. Self-inductance of a straight bar and any form of section. The self-inductance of a straight bar, of length l and of any form of cross-section, may be deduced from the expression already found above for the mutual inductance of two parallel conductors. For by the theory of the geometric mean distance, and the definition of self-inductance here employed, the self-inductance of the bar is equal to the mutual inductance of two parallel straight filaments of length l, and separated by a distance equal to the g.m.d. of the section from itself. Thus, if R be this g.m.d. we have by (52) above, approximately,

$$L = 2l\left(\log \frac{2l}{R} - 1 + \frac{R}{l}\right). \qquad \text{(56)}$$

For example, if the section be a circle of radius a,

$$R = ae^{-\frac{1}{4}} = 0{\cdot}7788a,$$

and so $$\log R = \log a - \tfrac{1}{4}.$$

Thus $$L = 2l\left(\log \frac{2l}{a} - \frac{3}{4} + \frac{ae^{-\frac{1}{4}}}{l}\right). \qquad \text{(57)}$$

Again, if the section be a square, of length of side a, $R = 0{\cdot}447a$, and

$$L = 2l\left\{\log \frac{2l}{a} - (1 - \log 0{\cdot}447) + \frac{0{\cdot}447a}{l}\right\}. \qquad \text{(58)}$$

If the section is a rectangle of lengths of sides a and b, R is very approximately $0{\cdot}2235(a+b)$, even if $a = 20b$, and therefore, since $\log(1/0{\cdot}2235) = 1{\cdot}500$, we get

$$L = 2l\left\{\log \frac{2l}{a+b} + \frac{1}{2} + \frac{0{\cdot}2235(a+b)}{l}\right\}. \qquad \text{(59)}$$

18. Mutual inductance of parallel conductors of square section. The mutual inductance of two parallel conductors of square section may now be found. It is equal to the mutual inductance of two filaments of the same length and of distance apart equal to the g.m.d. of the two areas apart. This, as the reader may verify (see 9 above), is very nearly the distance between the centres, whether the squares are situated as in Fig. 165 or as in Fig. 166. In the former case the g.m.d. is slightly greater, in the latter case slightly smaller, than the distance between the centres. This holds whatever the distance between the centres. Thus very approximately we have, if d be the latter distance,

$$M = 2l\left(\log\frac{2l}{d} - 1 + \frac{d}{l}\right). \quad \text{......................(60)}$$

As a numerical example take $l = 10$ metres and $d = 2$ cm. We get

$$M = 2000\,(\log 1000 - 1) + 4 = 2000 \times 5{\cdot}9078 + 4 = 11820, \quad \text{......(60')}$$

that is 11·820 in microhenrys.

The self-inductance of a return circuit composed of two parallel bars of square section, and of the same length, is given as in (53) by

$$L = 2\,(L_1 - M),$$

where L_1 is the self-inductance of one bar and M the mutual inductance of the two.

19. Inductances in various cases. We now state a number of results of calculation of inductances for different cases, leaving to the reader the verifications by the methods indicated.

(1) Two equal parallel rectangles composed of filamental conductors the thickness of which may be neglected: sides of lengths a and b and distance of planes apart d. [Calculate mutual inductance by Neumann's equation.]

$$\begin{aligned} M = 4\Bigg[\,a\log\left\{\frac{a + (a^2+d^2)^{\frac{1}{2}}}{a + (a^2+b^2+d^2)^{\frac{1}{2}}}\,\frac{(b^2+d^2)^{\frac{1}{2}}}{d}\right\} \\ + b\log\left\{\frac{b + (b^2+d^2)^{\frac{1}{2}}}{b + (a^2+b^2+d^2)^{\frac{1}{2}}}\,\frac{(a^2+d^2)^{\frac{1}{2}}}{d}\right\}\Bigg] \\ + 8\left\{(a^2+b^2+d^2)^{\frac{1}{2}} - (a^2+d^2)^{\frac{1}{2}} - (b^2+d^2)^{\frac{1}{2}} + d\right\}. \quad \text{...(61)} \end{aligned}$$

Putting in this $a = b$, we get for two squares at distance d apart

$$\begin{aligned} M = 8\Bigg[\,a\log\left\{\frac{a + (a^2+d^2)^{\frac{1}{2}}}{a + (2a^2+d^2)^{\frac{1}{2}}}\,\frac{(a^2+d^2)^{\frac{1}{2}}}{d}\right\} \\ + (2a^2+d^2)^{\frac{1}{2}} - 2\,(a^2+d^2)^{\frac{1}{2}} + d\Bigg]. \quad \text{...............(61')} \end{aligned}$$

(2) The self-inductance of a square formed of four equal conductors each of length a and diameter of cross-section 2ρ small in comparison with a. We have

$$L = 4L_1 - 4M,$$

where L_1 is the self-inductance for each side, and M that of a pair of opposite sides.

The mutual inductance of a pair of adjacent sides is zero. According to results obtained above we have, neglecting ρ^2/a^2,

$$L = 8a\left\{\log\frac{a}{\rho} - \log\frac{1 + 2^{\frac{1}{2}}}{2} + \frac{\rho}{a} - 1{\cdot}75 + 2^{\frac{1}{2}}\right\}$$

or

$$L = 8a\left(\log\frac{a}{\rho} + \frac{\rho}{a} - 0{\cdot}524\right). \quad \text{..................................(62)}$$

(3) The self-inductance of a rectangle of sides a and b in length made of round wire of radius of cross-section ρ. Putting $d=(a^2+b^2)^{\frac{1}{2}}$, we have

$$\begin{aligned} L &= 2(L_a+L_b-M_a-M_b) \\ &= 4\left\{(a+b)\log\frac{2ab}{\rho}-a\log(a+d)-b\log(b+d)\right. \\ &\qquad\left.-\frac{7}{4}(a+b)+2(d+\rho)\right\}. \qquad \text{.................(63)} \end{aligned}$$

If the conductor have a rectangular section $\alpha\times\beta$,

$$L_a=2\left\{a\log\frac{2a}{\alpha+\beta}+\frac{1}{2}a+0{\cdot}2235(\alpha+\beta)\right\},$$

$$M_a=2\left\{a\log\frac{a+(a^2+b^2)^{\frac{1}{2}}}{b}-(a^2+b^2)^{\frac{1}{2}}+b\right\}.$$

With, as before, $d=(a^2+b^2)^{\frac{1}{2}}$, these values, with those corresponding for the sides of length b, we have

$$\begin{aligned} L=4\left\{(a+b)\log\frac{2ab}{\alpha+\beta}-a\log(a+d)-b\log(b+d)\right. \\ \left.-\frac{a+b}{2}+2d+0{\cdot}447(\alpha+\beta)\right\}. \qquad \text{.........(64)} \end{aligned}$$

For a square this becomes

$$L=8a\left\{\log\frac{a}{\alpha+\beta}+0{\cdot}2235\frac{\alpha+\beta}{a}+0{\cdot}726\right\}. \qquad \text{..........(64')}$$

20. Self-inductance of a non-inductive shunt and of a thin tape. (4) The self-inductance of two equal rectangular sheets placed opposite to one another, and connected at two adjacent opposite edges, as when a piece of conducting strip is doubled on itself to form a nearly non-inductive shunt. In this case

$$L=4l(\log R_2-\log R_1), \qquad \text{..........................(65)}$$

where R_1 is the g.m.d. of each part from itself and R_2 is the g.m.d. between the parts.

(5) When the more accurate formula

$$M=2[l\log\{l+(l^2+d^2)^{\frac{1}{2}}\}-l\log d-(l^2+d^2)^{\frac{1}{2}}+d] \qquad \text{.........(66)}$$

for the mutual inductance between two parallel conductors of length l and distance d apart is applied to obtain the self-inductance of a straight thin tape, arithmetical mean distances as well as geometric mean distances have to be taken account of. For a discussion of these we have no space, and the reader must be referred to Mr. Rosa's papers in the *B.B.S.W.* [see, for example, 4, p. 325, *et seq.*]. The following approximate result is obtained for the self-inductance of a straight thin strip of length l and breadth b:

$$L=2l\left(\log\frac{2l}{b}+\frac{1}{2}+\frac{b}{3l}-\frac{b^2}{24l^2}\right). \qquad \text{.....................(67)}$$

21. Solid conductor in coaxial tube. Multiple conductors. (6) A right cylindrical straight solid conductor of length l and radius a_1 with return supplied by a thin coaxial tube of radius a_2. The self-inductance is given by

$$L=2l\left(\log\frac{a_2}{a_1}+\frac{1}{4}\right). \quad \text{.........................(67')}$$

If the outer tube have sensible thickness, so that its external radius is a_3 and its inner a_2, then

$$L=2l\left(\log\frac{a'}{a_1}+\frac{1}{4}\right),$$

where
$$\log a'=\frac{a_3{}^2\log a_3-a_2{}^2\log a_2}{a_3{}^2-a_2{}^2}-\frac{1}{2}. \quad \text{..................(68)}$$

(7) A current is divided equally between two equal wires of length l and radius ρ and at distance d apart. Then

$$\begin{aligned}L&=L_1+L_2+2M\\&=2l\left(\log\frac{2l}{(rd)^{\frac{1}{2}}}-1\right),\end{aligned} \quad \text{.........................(69)}$$

where r is the g.m.d. of the section of a wire from itself $=0{\cdot}7788\rho$.

For three straight conductors in parallel so arranged that in a cross-section of the arrangement the axes of the three wires are at the vertices of an equilateral triangle of side d,

$$L=2l\left(\log\frac{2l}{(rd^2)^{\frac{1}{3}}}-1\right). \quad \text{........................(69')}$$

(8) A to-and-fro "non-inductive" arrangement of $2n$ wires in a plane each of length l and distance d between adjacent extremities.

$$L=2l\left(\log\frac{d}{\rho}+\frac{1}{4}-A\right), \text{..................................(70)}$$

where
$$A=2\log\left(\frac{2\,.\,4\,.\,6\,...\,(n-1)}{1\,.\,3\,.\,5\,...\,(n-2)\,n^{\frac{1}{2}}}\right).$$

This result is due to Rosa (*B.B.S.W.* 4, p. 339). The result for n infinite had been previously given by G. A. Campbell, *Elec. World*, 44 (1904). It is

$$L=2l\left(\log\frac{2d}{\pi\rho}+\frac{1}{4}\right). \quad \text{.............................(71)}$$

(9) A "non-inductive" winding on a circular cylinder of radius a and distance d between adjacent turns,

$$L=4\pi a\left(\log\frac{d}{\rho}+\frac{1}{4}-\mathrm{A}\right), \quad \text{.........................(72)}$$

where A has the value just given. [It is understood that the axial length of the coil is small compared with the radius, otherwise this value would not be exact.]

22. Inductances of coils. Gray's formulae. We now consider the calculation of the self- and mutual inductances of coils of different forms and dimensions.

The expression given in Chapter VII. above was obtained by the author of this work in 1892 * for the mutual kinetic energy of two solenoidal coils with intersecting axes included at any angle θ. This enables the mutual inductance of the two coils, or by making them coincident, the self-inductance of one to be obtained by formulae, instantly derivable, which are convenient and accurate in a large number of cases. The expression was obtained by the special process of integration of Maxwell's expression in zonal harmonics for the mutual inductance of two circles the axes of which intersect, which is set forth in Chapter VII.

For this case precisely the same formulae were given † in 1905 by Messrs. Searle and Airey, no doubt independently. The reader may refer to the chapter cited for the general expressions; we shall here deal with cases of importance in practice.

First, we suppose the coils to be coaxial. This renders the zonal harmonic multipliers all unity. Further, if the origin be taken at the centre of one of the coils, we see at once that all the terms with even suffixes in the series are zero. Thus for the mutual inductance of the two coils we have

$$M=\pi^2 a^2 A^2 n_1 n_2 (K_1 k_1 + K_3 k_3 + K_5 k_5 + \ldots), \quad \ldots\ldots\ldots\ldots(73)$$

where A, n_1 are the radius and number of turns per cm of length for one coil and a, n_2 for the other. If x_1, x_2, ξ_1, ξ_2 be the distances of the nearer and farther ends of the two coils from the origin, and r_1, r_2 be the diagonal distances as shown, we have here $\xi_2 = -\xi_1\ (=\xi)$, and so

$$\left.\begin{aligned}
K_1 &= \frac{2}{A^2}\left(\frac{x_2}{r_2}-\frac{x_1}{r_1}\right), \quad K_3 = -\tfrac{1}{2}\left(\frac{x_2}{r_2^{\,5}}-\frac{x_1}{r_1^{\,5}}\right),\\
K_5 &= -\tfrac{1}{8}\left\{\frac{x_2}{r_2^{\,9}}(4x_2^2-3A^2)-\frac{x_1}{r_1^{\,9}}(4x_1^2-3A^2)\right\},\\
K_7 &= -\tfrac{1}{8}\left\{\frac{x_2}{r_2^{\,13}}(4x_2^4-10x_2^2A^2+\tfrac{5}{2}A^4)\right.\\
&\qquad\qquad \left.-\frac{x_1}{r_1^{\,13}}(4x_1^4-10x_1^2A^2+\tfrac{5}{2}A^4)\right\},\\
K_9 &= -\tfrac{1}{8}\left\{\frac{x_2}{r_2^{\,17}}(x_2^6-21x_2^4A^2+\tfrac{35}{2}x_2^2A^4-\tfrac{35}{16}A^6)\right.\\
&\qquad\qquad \left.-\frac{x_1}{r_1^{\,17}}(x_1^6-21x_1^4A^2+\tfrac{35}{2}x_1^2A^4-\tfrac{35}{16}A^6)\right\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \quad \ldots\ldots(74)$$

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* *Phil. Mag.* 33, 5. S. (1892).

† *The Electrician*, 56, p. 318, 1905.

$$\left.\begin{aligned}
&k_1=2\xi, \quad k_3=-a^2\xi\left(3-4\frac{\xi^2}{a^2}\right),\\
&k_5=a^4\xi\left(\tfrac{5}{2}-10\frac{\xi^2}{a^2}+4\frac{\xi^4}{a^4}\right),\\
&k_7=-a^6\xi\left(\tfrac{35}{16}-\tfrac{35}{2}\frac{\xi^2}{a^2}+21\frac{\xi^4}{a^4}-4\frac{\xi^6}{a^6}\right),\\
&k_9=-a^8\xi\left(\tfrac{63}{32}-\tfrac{105}{4}\frac{\xi^2}{a^2}+63\frac{\xi^4}{a^4}-36\frac{\xi^6}{a^6}+4\frac{\xi^8}{a^8}\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \quad \ldots\ldots(74')$$

Values of the multipliers K and k of higher orders will be found in the Appendix on Spherical Harmonics. A table calculated at the Bureau of Standards at Washington for the use of this formula is given in an Appendix.

23. Gray's formula for coaxial coils. To adapt this formula to two coaxial coils which are also concentric, it is only necessary to remember that, in that case, $x_1=-x_2\ (=-x)$ and $r_1=r_2$, so that we get the following special values from those set down above,

$$\left.\begin{aligned}
&K_1=\frac{4}{A^2}\frac{x}{r}, \quad K_3=-\frac{x}{r^5}, \quad K_5=-\frac{2x}{8r^9}(4x^2-3A^2),\\
&K_7=-\frac{2x}{8r^{13}}(4x^2-10x^2A^2+\tfrac{5}{2}A^4),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(75)$$

Thus if N_1, N_2 be the whole numbers of turns on the coils, so that $N_1N_2=4x\xi n_1n_2$, we get for M the equation

$$\begin{aligned}
M=\frac{2\pi^2a^2N_1N_2}{r}\Big\{1&-\frac{A^2}{2r^4}\,\frac{4\xi^2-3a^2}{4}-\frac{A^2(4x^2-3A^2)}{8r^8}\,\frac{8\xi^4-20\xi^2a^2+5a^4}{8}\\
&-\frac{A^2(8x^4-20x^2A^2+5A^4)}{16r^{12}}\,\frac{64\xi^6-336\xi^4a^2+280\xi^2a^4-35a^6}{64}\\
&-\text{etc.}\Big\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(76)
\end{aligned}$$

This is referred to by Rosa as the formula of Searle and Airey. Clearly it is also only a particular case of the general formula stated above.

24. Mutual inductance of a solenoid and a coaxial circle. If we put $N_2=1$ and $\xi=0$, we get a convenient equation for the mutual inductance of a solenoidal coil of length x, and containing N turns, and a coaxial circle of radius a in the plane of one end of the coil,

$$\begin{aligned}
M=\frac{2\pi^2a^2N}{r}\Big\{1&+\tfrac{3}{8}\frac{a^2A^2}{r^4}+\tfrac{5}{64}\frac{a^4A^4}{r^8}X_2+\tfrac{35}{512}\frac{a^6A^6}{r^{12}}X_4+\tfrac{63}{1024}\frac{a^8A^8}{r^{16}}X_6\\
&+\tfrac{231}{4096}\frac{a^{10}A^{10}}{r^{20}}X_8+\tfrac{429}{8192}\frac{a^{12}A^{12}}{r^{24}}X_{10}+\ldots\Big\}, \quad \ldots\ldots\ldots\ldots(77)
\end{aligned}$$

where

$$\left.\begin{aligned}
X_2 &= 3 - 4\frac{x^2}{A^2}, \\
X_4 &= \tfrac{5}{2} - 10\frac{x^2}{A^2} + 4\frac{x^4}{A^4}, \\
X_6 &= \tfrac{35}{16} - \tfrac{35}{2}\frac{x^2}{A^2} + 21\frac{x^4}{A^4} - 4\frac{x^6}{A^6}, \\
X_8 &= \tfrac{63}{32} - \tfrac{105}{4}\frac{x^2}{A^2} + 63\frac{x^4}{A^4} - 36\frac{x^6}{A^6} + 4\frac{x^8}{A^8}, \\
X_{10} &= \tfrac{231}{128} - \tfrac{1155}{32}\frac{x^2}{A^2} + \tfrac{1155}{8}\frac{x^4}{A^4} - 165\frac{x^6}{A^6} + 55\frac{x^8}{A^8} - 4\frac{x^{10}}{A^{10}}.
\end{aligned}\right\} \quad \ldots(78)$$

If the length of the coil be $2x$, and N be the whole number of turns, the formula will give the mutual inductance between the solenoid and a circle of radius a coincident with the median cross-section of the coil.

This formula was proposed by Rosa (*B.B.S.W.* 3, p. 224). The derivation from equation (76) shows that it also is a particular case of the author's general formula.

As an example of calculations carried out by these formulae we take a coil specified by Viriamu Jones to illustrate a formula which he used to calculate the mutual induction of the coil and circle of the Lorenz apparatus. The data given were $A=10$, $a=5$, $x=2$, in inches, $r^2=104$. The results of calculation are those found by Rosa (*B.B.S.W.* vol. 8, No. 1) :

$X_2=2{\cdot}8400$	1st term	$=1{\cdot}0000000$
$X_4=2{\cdot}1064$	2 ,,	$=0{\cdot}0866771$
$X_6=1{\cdot}5208$	3 ,,	$=0{\cdot}0118537$
$X_8=1{\cdot}0173$	4 ,,	$=0{\cdot}0017781$
$X_{10}=0{\cdot}5818$	5 ,,	$=0{\cdot}0002670$
	6 ,,	$=0{\cdot}0000379$
$\dfrac{2\pi^2a^2}{r}=48{\cdot}38972$	7 ,,	$=0{\cdot}0000046$
	Sum	$1{\cdot}1006184$

Multiplying these two results together, we get

$$M=53{\cdot}25861N,$$

where N is the number of turns in the coil. The result obtained by Jones was $M=53{\cdot}25879M$, which is about 18 parts in 5 million greater.

25. Calculation of current-weigher constant. Rosa has also used this formula to obtain the constant of a current-weigher of the Ayrton-Jones pattern which is indicated in the diagram. The constant is proportional to the difference between the mutual inductances of the outer coil on the terminal circles of the inner. The calculation was carried out by taking the outer coil as consisting of two terminating in the plane of one of the ends of the inner coil, which in the normal position was symmetrically placed in the outer. These two coils O_1A, O_2A had respectively 80 turns and 240 turns. In centimetres A, a, x were 16, 10, 8 respectively for the former coil and 16, 10, 24 for the latter, while r^2 was 320 and 832 in the two cases. The mutual inductance, M_S, of the coil O_1A on the circle S, and that, M_R, of the coil O_2A on the circle R were calculated. The following are the X multipliers in the two cases:

	For M_S.	For M_R.
X_2	+2·000	− 6·00
X_4	+0·250	+ 0·25
X_6	−0·9375	+23·5
X_8	−1·203	−45·7
X_{10}	−0·562	−49·0

Terms of Series.

1st term	1·0000000	1·0000000
2 ,,	+0·0937500	+0·0138683
3 ,,	+0·0097656	−0·0006411
4 ,,	+0·0002670	+0·0000009
5 ,,	−0·0002253	+0·0000027
6 ,,	−0·0000662	−0·0000002
7 ,,	−0·0000072	0·0000000
Sum	1·1034839	1·0132306

The outside multipliers, $2\pi^2a^2N/r$, of the series gave $M_S=9741{\cdot}16$, $M_R=16641{\cdot}32$. Jones's elliptic integral formula gave $M_S=9741{\cdot}17$, $M_R=16641{\cdot}32$, in centimetres, practically the same result.

26. Coils of lengths $\sqrt{3}$ times the radius. The following example of formula (74) shows the effect of making the lengths of the coils equal to $\sqrt{3}$ times the radius in abbreviating the calculation of their mutual induction. We take the data

$$\left.\begin{array}{l} A=25\text{ cm},\ N_1=n_1A\sqrt{3}, \\ a=10\text{ cm},\ N_2=n_2a\sqrt{3}, \end{array}\right\}\ \begin{array}{l}\text{so that } N_1N_2=3n_1n_2Aa, \\ \qquad n_1=n_2=20, \end{array}$$

$$d=(x^2+A^2)^{\frac{1}{2}}=\tfrac{1}{2}A\sqrt{7}.$$

Hence
$$M=\frac{2\pi^2a^2N_1N_2}{r}=4\pi^2a^2n_1n_2\frac{3a}{\sqrt{7}}$$
$$=0{\cdot}0179057\text{ henry}.$$

Here the number of turns is not integral, which of course it must be in any actual case. Altering the lengths to 43·3 and 17·3, we make the values of N_1, N_2 866, 346. For this one correction term in (74) must be calculated so that the value of M comes out ·0178854 henry. This example is given by Rosa; and a much longer calculation by a formula due to Roiti (not here given) used as a check gives a result only differing by 1 part in 178,000.

27. Practical example of Gray's formula. The use of the author's formula is illustrated by the following practical example worked out by Rosa. The data are

$$2x = 20{\cdot}55, \quad A = 6{\cdot}44, \quad N_1 = 15,$$
$$2\xi = 27{\cdot}38, \quad a = 4{\cdot}435, \quad N_2 = 75.$$

Distance between adjacent ends of solenoids 7·2.

$$\text{Here} \qquad \begin{array}{ll} n_1 = 0{\cdot}7296 & K_1 k_1 = 0{\cdot}042937 \\ n_2 = 2{\cdot}737 & K_3 k_3 = 0{\cdot}018274 \\ x_1 = 20{\cdot}89 & K_5 k_5 = 0{\cdot}005193 \\ x_2 = 41{\cdot}44 & K_7 k_7 = 0{\cdot}001423 \\ & K_9 k_9 = \underline{0{\cdot}000116} \\ & \qquad\;\; \underline{0{\cdot}067943} \\ & M = \underline{\underline{1092{\cdot}3}} \end{array}$$

This is a case in which the convergence is not sufficiently rapid. The longer coil was therefore considered as in two sections, C and D, of axial lengths 13·51 and 13·87 cm, of which C was the nearer to the other coil, R say. The coil R was considered divided into two sections A and B, of which B was the nearer to S. Then C was divided into two sections F and G, and M_{BF} and M_{BG} calculated. The results obtained, without going beyond $K_5 k_5$, were

$$M_{AC} = 207{\cdot}2, \quad M_{BC} = 683{\cdot}8, \quad M_{AD} = 64{\cdot}9, \quad M_{BD} = 130{\cdot}7,$$

giving $$M = 1087{\cdot}2 \text{ cm}.$$

Higher terms would have given a more exact value. As it is, a check carefully applied by the method of Nagaoka, mentioned below, gave $M = 1086{\cdot}55$, of which the last figure was very uncertain. These calculations illustrate how accurate the results obtained by these formulae, without excessive computation, may be, and how by supposing the coils subdivided the convergence of the series may be made all that may be desired.

28. Coils with multiple layers. In the last chapter the mutual inductance of two circles has been dealt with. We shall only here supplement that discussion by an account of a few more complicated and at the same time practical cases, in which the coils contain a number of layers of wire, and it is necessary to take into consideration the cross-section of the windings.

The first method of taking account of the cross-sections of two coils of which the mutual inductance is to be calculated is that given by the late H. A. Rowland of Baltimore. The method consists in an obvious application of Taylor's theorem of the expansion of functions, and is convenient only when the cross-sectional dimensions are not large, that is, are not comparable with the radius of either coil or the distance between them. It is to be remembered also that it is in general necessary, for exact work, to take account of the fact that the total conducting cross-section of the conductor is different from the cross-section of the coil, both in extent and in distribution.

29. Correction for cross-sections of coils. Let the cross-sections of the coils be situated as in the diagram, and let there be two central turns, of radii A and a, one in each coil about which the other turns are symmetrically arranged. The mutual induction between these central turns we shall denote by M_0. Then if n_1, n_2 be the total numbers of turns in the two coils a first approximation to the mutual induction required is

$$M = n_1 n_2 M_0.$$

If the coils are of finite axial breadth $2b_1$ and $2b_2$, and radial depth $2c_1, 2c_2$ we have for a second approximation

$$\frac{M}{n_1 n_2} = M_0 + \frac{1}{6}\left\{(b_1{}^2 + b_2{}^2)\frac{d^2 M_0}{dx^2} + c_1{}^2 \frac{d^2 M_0}{dA^2} + c_2{}^2 \frac{d^2 M_0}{da^2}\right\}, \quad \text{.......(79)}$$

for it is clear that there can be no terms involving $\partial M_0/\partial A$, $\partial M_0/\partial a$, or $\partial^2 M_0/\partial A\, da$. For coils of equal radii and cross-section this becomes

$$\frac{M}{n_1 n_2} = M_0 + \frac{1}{3}\left(b^2 \frac{\partial M_0}{\partial x^2} + c^2 \frac{\partial^2 M_0}{\partial A^2}\right). \quad \text{..................(80)}$$

It has been shown in the last chapter that for circles of radii A and a,

$$M_0 = 4\pi\sqrt{Aa}\left\{\left(\frac{2}{\gamma} - \gamma\right)G - \frac{2}{\gamma}H\right\}, \quad \text{..................(81)}$$

where $\gamma^2 = 4Aa/\{(A+a)^2 + x^2\}$, the square of the modulus of the first and second elliptic integrals G, H. From this the correction terms can be calculated. By differentiation we find (having regard to the modular relations) that when $A = a$,

$$\left.\begin{aligned} \frac{\partial^2 M_0}{\partial A^2} &= \pi\frac{\gamma}{A}\left\{(2-\gamma^2)G - \left(2 - \gamma^2\frac{1-2\gamma^2}{1-\gamma^2}\right)H\right\}, \\ \frac{\partial^2 M_0}{\partial x^2} &= \pi\frac{\gamma^2}{A}\left(G - \frac{1-2\gamma^2}{1-\gamma^2}H\right). \end{aligned}\right\} \quad \text{............(82)}$$

30. Correction of elliptic integral formula for cross-section. The following correction of the value of M for the effect of section applied to the elliptic integral expression for two *equal* coaxial circles, gives a very exact result, except when the coils are very close together. It was obtained by Rosa * in a revision and correction of the expression given by Weinstein and Stefan † for the effect of cross-section in equal coils. The correction is made by adding to M_0, as given in (81) above, the expression

$$\Delta M = 4\pi a \sin\alpha\left\{(G-H)\left(A + \frac{c^2}{24a^2}\right) + HB\right\}, \quad \text{................(83)}$$

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* *B.B.S.W.* 2, p. 341.

† *Wied. Ann.* 21 and 22, 1884.

where G, H, are the complete elliptic integrals I. and II. to modulus $\gamma = \sin a$, and A and B have the following values:

$$\left.\begin{aligned} A &= \frac{\cos^2 a}{12x^2}\{a_1 - a_2 - a_3 + (2a_2 - 3a_3)\cos^2 a + 8a_3 \cos^4 a\}, \\ B &= \frac{\sin^2 a}{12x^2}\{a_1 + \tfrac{1}{2}a_2 + 2a_3 + (2a_2 + 3a_3)\cos^2 a + 8a_3 \cos^4 a\}, \end{aligned}\right\} \quad \text{...(84)}$$

in which

$a_1 = b^2 - c^2 + c^4/30a^2$, $a_2 = (5b^2c^2 - 4c^2)/60a^2$, $a_3 = (2b^4 + 2c^4 - 5b^2c^2)/20x^2$.

Hence, for coils of equal square section,

$$a_1 = b^4/30a^2, \quad a_2 = b^4/60a^2, \quad a_3 = -b^4/20x^2. \quad \text{...............(85)}$$

A pair of coils of equal radius has been used by Rowland, Glazebrook, Rayleigh, and others, in absolute determinations of resistance (see Chapter XV. below).

31. Lord Rayleigh's correction for cross-section. Another method of obtaining a second approximation to the value of the mutual inductance of two coils of finite cross-section has been used by Lord Rayleigh. Using the notation already explained, but putting $f(A, a, x)$ for the mutual inductance of the central windings, we get approximately, as the reader may verify,

$$M = \tfrac{1}{6} n_1 n_2 \left\{\begin{array}{c} f(A + c_1,\ a,\ x) + f(A - c_1,\ a,\ x), \\ + f(A,\ a + c_2,\ x) + f(A,\ a - c_2,\ x), \\ + f(A,\ a,\ x + b_1) + f(A,\ a,\ x - b_1), \\ + f(A,\ a,\ x + b_2) + f(A,\ a,\ x - b_2), \\ -2f(A,\ a,\ x). \end{array}\right\} \quad \text{...............(86)}$$

This is a general formula of approximation, which is applicable to any function of A, a, x. We have already employed it in considering the force on the suspended coil of a current weigher. The amount by which this approximation is in error may be computed approximately by expanding M by Taylor's theorem as far as differential coefficients of the fourth order and comparing with this formula, which really amounts to

$$\left.\begin{aligned} M = M_0 &+ \frac{1}{24}\left\{(b_1{}^2 + b_2{}^2)\frac{\partial^2 M_0}{\partial x^2} + c_1{}^2\frac{\partial^2 M_0}{\partial A^2} + c_2{}^2\frac{\partial^2 M_0}{\partial a^2}\right\} \\ &+ \frac{1}{1152}\left\{(b_1{}^4 + b_2{}^4)\frac{\partial^4 M_0}{\partial x^4} + c_1{}^4\frac{\partial^4 M_0}{\partial A^4} + c_2{}^4\frac{\partial^4 M_0}{\partial a^4}\right\} + \dots . \end{aligned}\right\} \quad \text{...(87)}$$

It has been shown by Rosa (*B.B.S.W.* 2, p. 373) that the difference ϵ_1 thus obtained is given by

$$\epsilon_1 = 4\pi A \frac{3b^4 + 3c^4 - 20b^2c^2}{480x^4} \quad \text{...........................(88)}$$

for two equal coils. If the coil is of square section this is

$$\epsilon_1 = -4\pi A \frac{14b^4}{480x^4} = -7\pi A \frac{b^4}{60x^4}, \quad \text{..................(88')}$$

so that the error is negative. In any case, whether of equal coils or not, it is clear that the error is smaller the greater the distance of the coils apart. Moreover, the value of M may be too large or too small according to the shape of the cross-section. The error is very approximately zero (for $ab=\text{constant}$) if $b=3{\cdot}65$ and $c=4/3{\cdot}65=1{\cdot}096$, nearly.

32. Lyle's equivalent mean radius for distant coils. For two coaxial and distant coils of square section it has been shown by Professor Lyle * of Melbourne that a closely approximate value of M is obtained by using for each instead of the mean radius a, a radius r given by

$$r=a\left(1+\frac{b^2}{24a^2}\right), \qquad (89)$$

where b is the side of the square section. Of course such a value of r is obtained for each coil.

If the coil section is not square but rectangular, the coil is to be replaced by two filaments according to the following specification. Let Fig. 169 represent the two cross-sections and their relative situation. Let the breadth in each case be denoted by b and the radial depth by c. In A we suppose that $b>c$, and in B that $c>b$. For coil A let $\beta^2=\frac{1}{12}(b^2-c^2)$ be determined. Then A is to be replaced by two filaments (1 and 2) of radius given by $r=a(1+c^2/24a^2)$ and situated at a distance 2β apart symmetrically on the two sides of the mean plane of the coil.

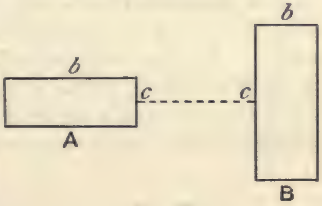

Fig 169.

In coil B, on the other hand, two filaments (3 and 4) in the mean plane and of radii $r+\delta$, $r-\delta$ are to be taken, where

$$r^2=a(1+b^2/24a^2) \quad \text{and} \quad \delta^2=\tfrac{1}{12}(c^2-b^2).$$

The mutual inductance of the two coils is now the sum of the inductances of the filaments of A on the two filaments of B, each filament containing half the number of windings of the coil to which it belongs. Or we have

$$4M=n_1n_2(M_{13}+M_{14}+M_{23}+M_{24}), \qquad (90)$$

where, for example, M_{13} is the mutual inductance of the filament 1 in A on the filament 3 in B.

33. Formulae for inductances of thick coils. Singly infinite solid coils. There remain thick coils and flat coils, which are of importance in wireless telegraphic work. We now give some account of calculations for such cases, using, for the most part, methods due to Butterworth (*Phil. Mag.* 29, 1915. See also *Note* on p. 527).

Consider a magnetic field symmetrical about an axis. Take an origin on the axis, and consider a point P of the field. Let the distance of

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* *Phil. Mag.* 3 (1902).

P from the axis be y, and its axial distance from the origin be z. Denoting the magnetic potential at P by Ω, we have for the differential equation satisfied by Ω,

$$\frac{\partial^2\Omega}{\partial y^2}+\frac{1}{y}\frac{\partial\Omega}{\partial y}+\frac{\partial^2\Omega}{\partial z^2}=0. \quad \text{.........................(91)}$$

Of this a solution (see Gray and Mathews' *Treatise on Bessel Functions*) is

$$\Omega=\int_0^\infty \phi(\lambda)e^{-\lambda z}J_0(\lambda y)\,d\lambda, \quad \text{..........................(92)}$$

where $J_0(\lambda y)$ is the Bessel function of zero order for the argument λy. For points on the axis Ω reduces to

$$\Omega_0=\int_0^\infty \phi(\lambda)e^{-\lambda z}d\lambda. \quad \text{...............................(93)}$$

It is unnecessary to enter into any discussion here as to the meaning of $\phi(\lambda)$. Ω_0 is the potential at the axial point at distance z from the origin. The magnetic induction I in the direction of z, at the distance y from the axis, is for a medium of unit inductivity. The flux of induction through a circle of radius r, with its centre on and its plane at right angles to the axis, is given by

$$I=-2\pi\int_0^r y\frac{\partial\Omega}{\partial z}dy. \quad \text{...............................(94)}$$

By (92) this may be written

$$I=2\pi\int_0^\infty \lambda\phi(\lambda)e^{-\lambda z}\left\{\int_0^r y\,dy\,J_0(\lambda y)\right\}d\lambda. \quad \text{...............(95)}$$

It is proved in the *Treatise on Bessel Functions* referred to above that

$$\int_0^r y\,dy\,J_0(\lambda y)=\frac{r}{\lambda}J_1(\lambda r).$$

Hence we have

$$I=2\pi r\int_0^\infty \phi(\lambda)e^{-\lambda z}J_1(\lambda r)\,d\lambda. \quad \text{.........................(96)}$$

Expanding $J_1(\lambda r)$ in ascending powers of r, and using (93) we get the value of I, the stream function for the circle from the values of the potential at points on the axis. Thus

$$I=-\pi r^2\sum_{n=0}^{n=\infty}(-1)^n\left(\frac{r}{2}\right)^{2n}\frac{1}{n!\,(n+1)!}\,\frac{d^{2n+1}\Omega_0}{dz^{2n+1}}. \quad \text{............(97)}$$

Now consider a solid coil of radius a with its axis along the axis of z, and extending from $z=z$, to $z=\infty$. The total linkage of this coil with the field is N, and we have

$$N=\int_0^r dr\int_z^\infty I\,dz=\pi\sum\left\{\left(-\frac{1}{4}\right)^n\frac{r^{2n+3}}{(2n+3)n!\,(n+1)!}\left[\frac{d^{2n}\Omega_0}{dz_{2n}}\right]_z^\infty\right\}. \quad \text{...(98)}$$

34. Axial potential due to semi-infinite solid coil. Mutual inductance of two such coils. Now let us suppose that the potential Ω_0 is due to a second semi-infinite coil, of radius a and extending from $z=0$ to $z=-\infty$. The value of Ω_0 may be taken as that due to a disk of matter of surface density which, at the centre, is proportional to a, and tapering off to zero at the edge. If σa be this density at zero distance from the centre we have at distance x, $\sigma a(a-x)/a$. The potential at the point 0, z, due to a circular strip of radius x and breadth dx, is

$$2\pi\sigma x(a-x)/\sqrt{x^2+z^2}.$$

Hence we get by integration for the whole potential

$$\Omega_0=2\pi\sigma\int\frac{ax-x^2}{\sqrt{x^2+z^2}}\,dx,$$

or

$$\Omega_0=\pi\sigma a^2\left\{\sqrt{1+\frac{z^2}{a^2}}-\frac{2z}{a}+\frac{z^2}{a^2}\log\frac{1+\sqrt{1+\frac{z^2}{a^2}}}{\frac{z}{a}}\right\}. \qquad \text{............(99)}$$

Expanded in inverse powers of z/a this is

$$\Omega_0=a^2\pi\sigma\frac{a}{z}\sum\frac{(-1)^s(2s)!}{(2s+3)!\,s!\,(s+1)!}\left(\frac{1}{2\frac{z}{a}}\right)^{2s}. \qquad \text{..................(100)}$$

Hence

$$\frac{d^{2n}\Omega_0}{dz^{2n}}=\frac{\pi\sigma}{z^{2n+1}}\sum_0^\infty\frac{(-1)^s(2n+2s)!\,a^{2s+3}}{(2s+3)s!\,(s+1)!}\left(\frac{1}{2z}\right)^{2s}. \qquad \text{.............(101)}$$

Using this in the value of N given in (98) we get, for the total linkage of the field and coil between z and ∞,

$$N=\pi^2\sigma\sum_{s=0}^{s=\infty}\frac{(-1)^s(2n+2s)!}{(2s+3)s!\,(s+1)!}\,\frac{a^{2s+3}}{(2z)^{2(n+s)}}\sum_0^\infty\frac{(-1)^n}{2^{2n}}\,\frac{r^{2n+3}}{(2n+3)n!\,(n+1)!}. \qquad (102)$$

If we write p for $n+s$ we can put this in the form

$$N=\sigma\pi^2\sum_{p=0}^{p=\varpi}\frac{(-1)^p(2p)!}{(2z)^{2p}}$$

$$\times\sum_{n=0}^{n=p}\frac{a^{2(p-n)+3}\,.\,r^{2n+3}}{(2n+3)(2p-2n+3)n!\,(n+1)!\,(p-n)!\,(p-n+1)!}, \qquad \text{...(102')}$$

where ϖ is some sufficiently high value of p according to the degree of accuracy required.

35. Practical formulae for solid singly infinite coils. From this we can obtain an expansion in a form suitable for calculation. We first choose $p=0$, and therefore $n=0$, and get $\frac{1}{9}$ as the value of the first term of the expression on the right of (102'). Then we put $p=1$, and take the values 0, 1 for n. Next we take $p=2$, $p=3$, $p=4$, ... in succession, and in these give to n the values 0, 1, 2, 0, 1, 2, 3, 0, 1, 2, 3, 4, ... for n respectively. Thus we obtain

$$N=\sigma\pi^2\frac{a^3r^3}{z}\left\{\frac{1}{9}-\left(\frac{1}{2z}\right)^2\left(\frac{a^2}{15}+\frac{r^2}{15}\right)+\left(\frac{1}{2z}\right)^4\left(\frac{2}{21}a^4+\frac{6}{25}a^2r^2+\frac{2}{21}r^4\right)\right.$$
$$-\left(\frac{1}{2z}\right)^6\left(\frac{5}{27}a^6+\frac{6}{7}a^4r^2+\frac{6}{7}a^2r^4+\frac{5}{27}r^6\right)$$
$$+\left(\frac{1}{2z}\right)^8\left(\frac{14}{34}a^8+\frac{28}{9}a^6r^2+\frac{40}{7}a^4r^4+\frac{28}{9}a^2r^6+\frac{14}{33}r^8\right)$$
$$\left.-\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\right\},\ \ldots\ldots\ldots\ldots\text{(103), }\mathbf{A}$$

which is convergent for sufficiently large values of z.

From this the mutual inductance between the two solid coils, one extending from $z=0$ to $z=\infty$, can be found with exceeding accuracy if $a \ngtr 1$ and $z>3r$.

36. Mutual inductance of two non-overlapping finite solid coils. From this result we can find by an obvious process the mutual inductance of a pair of coaxial solid coils of finite length which do not overlap. Denote by h_1 the separation distance of the adjacent ends of the coils, by h_2 that of the remote ends, and by h_3, h_4 the separations of the other two pairs of ends, one of one coil, the other of the other coil. Then we have

$$N=\pi^2\sigma a^3r^3\left\{\left(\frac{1}{h_1}-\frac{1}{h_2}-\frac{1}{h_3}+\frac{1}{h_4}\right)\frac{1}{9}\right.$$
$$-\frac{1}{2^2}\left(\frac{1}{h_1{}^3}-\frac{1}{h_2{}^3}-\frac{1}{h_3{}^3}+\frac{1}{h_4{}^3}\right)\left(\frac{1}{15}a^2+\frac{1}{15}r^2\right)$$
$$+\frac{1}{2^4}\left(\frac{1}{h_1{}^5}-\frac{1}{h_2{}^5}-\frac{1}{h_3{}^5}+\frac{1}{h_4{}^5}\right)\left(\frac{2}{21}a^4+\frac{6}{25}a^2r^2+\frac{2}{21}r^4\right)$$
$$-\frac{1}{2^6}\left(\frac{1}{h_1{}^7}-\frac{1}{h_2{}^7}-\frac{1}{h_3{}^7}+\frac{1}{h_4{}^7}\right)\left(\frac{5}{27}a^6+\frac{6}{7}a^4r^2+\frac{6}{7}a^2r^4+\frac{5}{27}r^6\right)$$
$$+\frac{1}{2^8}\left(\frac{1}{h_1{}^9}-\frac{1}{h_2{}^9}-\frac{1}{h_3{}^9}+\frac{1}{h_4{}^9}\right)\left(\frac{14}{33}a^8+\frac{28}{9}a^6r^2+\frac{40}{7}a^4r^4+\frac{28}{9}a^2r^6+\frac{14}{33}r^8\right)$$
$$\left.-\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\right\}.\ \ldots\ldots\ldots\ldots\ldots\text{(104)}$$

This gives at once the mutual inductance of a pair of coaxial flat coils wound full from the centre to the circumference in each case. We have only to write $h+dh_1$ for h_2, $h+dh_2$ for h_3, and $h+dh_1+dh_2$ for h_4. We then get, omitting the factor σ,

$$N=\pi^2a^3r^3\,.\,2dh_1dh_2\left\{\frac{1}{h^3}\,\frac{1}{9}-\frac{1}{2^2}\,\frac{3}{h^5}\left(\frac{1}{15}a^2+\frac{1}{15}r^2\right)\right.$$
$$+\frac{1}{2^4}\,\frac{2\,.\,5}{h^7}\left(\frac{2}{21}a^4+\frac{6}{25}a^2r^2+\frac{2}{21}r^4\right)$$

$$-\frac{1}{2^6}\frac{3.7}{h^9}\left(\frac{5}{27}a^6+\frac{6}{7}a^4r^2+\frac{6}{7}a^2r^4+\frac{5}{27}r^6\right)$$
$$+\frac{1}{2^8}\frac{4.9}{h^{11}}\left(\frac{14}{33}a^8+\frac{28}{9}a^6r^2+\frac{40}{7}a^4r^4+\frac{28}{9}a^2r^6+\frac{14}{33}r^8\right)$$
$$-\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;\Big\}. \quad \text{.................(105)}$$

The quantities dh_1, dh_2 may be taken as the thicknesses of the coils and disappear when the strengths of the shells are substituted.

If in (102′) and (103) we write z/a in place of z, and r/a in place of r, we see that N for the actual infinite solid coils is equal to that for two of radii respectively r/a and unity, at distance apart z/a, multiplied by a^5. That is

$$N=a^5N\left(\frac{z}{a}\,\frac{r}{a}\right).$$

To complete the expressions for the inductances we have to insert the winding densities, that is the number of turns which in each case pass through unit area of a plane containing the axis of symmetry. We put n_1, n_2 for these densities, and therefore multiply by n_1n_2. [No account is here, or later, taken of space occupied by insulating material.] Hence we have, using N in the sense in which it is calculated above for the actual coils,

$$M=n_1n_2N. \quad \text{......................................(106)}$$

For the two coaxial parallel flat coils we have by (105)

$$M=2\pi^2a^2r^3n_1n_2S, \quad \text{..................................(107)}$$

where S denotes the quantity within the brackets { }.

37. Working formula for two semi-infinite coils. Butterworth gives the following working formula (for the case of $a=1$) for two semi-infinite coils with their adjacent ends at a distance z apart,

$$N=2\pi^2r^3[A-(B-\alpha)r^2+Cr^4-Dr^6], \quad \text{................(108)}, \mathbf{B}$$

where

$$\left.\begin{aligned}
3A&=\frac{1}{2}z^2\log\frac{1+\sqrt{1+z^2}}{z}+\frac{1}{2}(\sqrt{1+z^2}-z)=\frac{\Omega_0}{2\pi},\\
40B&=\log\frac{1+\sqrt{1+z^2}}{z}-\frac{1}{\sqrt{1+z^2}} \qquad\qquad =\frac{1}{2\pi}\frac{d^2\Omega_0}{dz^2},\\
1344C&=-\frac{1}{\sqrt{1+z^2}}\left\{\frac{1}{1+\sqrt{1+z^2}}+\frac{1}{1+z^2}-\frac{3}{(1+z^2)^2}\right\},\\
82944D&=\frac{3}{\sqrt{1+z^2}}\left\{\frac{1}{(1+\sqrt{1+z^2})^2}-\frac{1}{1+z^2}-\frac{3}{(1+z^2)^2}\right.\\
&\qquad\qquad\left.+\frac{35}{(1+z^2)^3}-\frac{35}{(1+z^2)^4}\right\},\\
\alpha&=\frac{z}{r}\sum_{n=2}^{n=\infty}\frac{(-1)^n(2n-3)!}{(2n+3)n!(n+1)!}\left(\frac{r}{2z}\right)^{2n}.
\end{aligned}\right\} \quad \text{. ...(109)}$$

This holds for $r<z$, and having regard to the series for $3A$ and $40B$ we see that it holds for $z<4$.

38. Case of distance between ends of coils small. When $z<r$ we can write

$$\Omega_0=2\pi\left[\frac{1}{2}-z+\frac{1}{2}z^2\left(\log\frac{2}{z}+\frac{1}{2}\right)+\sum_{s=2}^{s=\infty}\frac{(-1)^s(2s-3)!}{2^{2s-1}s!\,(s+1)!}z^{2s}\right]. \quad\text{.....(110)}$$

For small values of z the term $-\pi z^2\log z$ causes difficulty, and special treatment is necessary for this term.

If we denote this term by ω we have

$$\frac{d^2\omega}{dz^2}=-\pi\left(\log z+\frac{3}{2}\right),\quad \frac{d^{2n}\omega}{dz^{2n}}=\pi\frac{(2n-3)!}{z^{2n-2}}. \quad\text{...........(111)}$$

If $z>r$, we have

$$n=2\pi^2r^3\left\{-\frac{1}{2}z^2\log z+\frac{r^2}{40}\left(\log z+\frac{3}{2}\right)\right.$$
$$\left.+z^2\sum_{n=2}^{n=\infty}\frac{(-1)^n(2n-3)!}{(2n+3)\,n!\,(n+1)!}\left(\frac{r}{2z}\right)^{2n}\right\}, \quad\text{.......(112)}$$

which is convergent if $z>r$.

The idea of the method adopted is that of finding a simple magnetic distribution which gives rise to an axial potential containing this term, with other terms, to which the preceding method is applicable. Mr. Butterworth considers a linear distribution of poles along the axis of z from $z=0$ to $z=-c$, with a density πz^2. To avoid confusion in finding the potential at a point z on the axis, we write $\pi(x-z)^2$ for the density of the distribution, where x now denotes the distance of the point considered from this latter axial point. The potential at the point z, ω' say, is thus given by

$$\omega'=\int_z^{c+z}\frac{(x-z)^2}{x}\,dx=\omega+\pi\left\{z^2\log(c+z)-cz+\frac{c^2}{2}\right\}, \quad\text{........(113)}$$

in which as before $\omega=-\pi z^2\log z$.

We denote the linkage due to ω which we wish to discuss by n, and that due to ω' by n', and obtain

$$\frac{n'-n}{2\pi^2r^3}=\tfrac{1}{3}\left\{\tfrac{1}{2}z^2\log(c+z)-\tfrac{1}{2}cz+\frac{c^2}{4}\right\}-\frac{r^2}{40}\log(c+z)$$
$$+\text{terms which vanish for } c=\infty. \quad\text{....(114)}$$

To find the linkage n' the work done in removing the linear distribution of poles from the field of the solid coil of radius r, the free (south) pole of which is at the distance z from the origin, has to be calculated. The axial potential at distance x from this pole is

$$-\Omega_0(x,r)=-\pi\left(x^2\log\frac{r+\sqrt{x^2+r^2}}{x}+r\sqrt{x^2+r^2}-2rx\right),$$

and since the density of distribution of poles is $\pi(x-z)^2$,

$$n' = -\pi\int_z^{c+z} (x-z)^2\Omega_0(x_1 r)\,dx. \quad\text{......................(115)}$$

Evaluation of this integral and substitution in (114), give

$$\begin{aligned} n = 2\pi^2 r^3 \Bigg[& \frac{1}{3}\left(\frac{1}{2}z^2\log\frac{2}{r} - \frac{1}{3}z^2\right) - \frac{r^2}{40}\left(\log\frac{2}{r} - \frac{1}{20}\right) \\ & + \frac{r^2}{2}\left\{\frac{1}{3}\frac{z}{r} - \frac{1}{3}\frac{z^3}{r^3} + \frac{1}{6}\frac{z^4}{r^4} - \frac{1}{30}\frac{z^5}{r^5}\left(\log\frac{2r}{z} + \frac{77}{60}\right)\right\} \\ & - \frac{z^3}{r^3}\sum_{n=2}^{n=\infty}\frac{(-1)^n(2n-3)!}{2^{2n-2}(n-1)!\,(n+1)!\,(2n+1)(2n+3)}\left(\frac{z}{r}\right)^{2n}\Bigg]. \quad\text{..(116)} \end{aligned}$$

Using this to replace (112), we get

$$N = 2\pi^2 r^3\left[\left(\frac{z^2}{6} - \frac{r^2}{40}\right)\log\frac{1}{r} + A' - (B'-\beta)r^2 + Cr^4 - Dr^6\right], \quad\text{...(117), }\mathbf{C}$$

in which

$$\left.\begin{aligned} 3A' &= \tfrac{1}{2}z^2\log\{2(1+\sqrt{1+z^2})\} + \tfrac{1}{2}\sqrt{1+z^2} - z - \tfrac{1}{3}z^2, \\ 40B' &= \log\{2(1+\sqrt{1+z^2})\} - \frac{1}{\sqrt{1+z^2}} + \frac{29}{20}, \\ \beta &= \frac{z}{2r}\left\{\frac{1}{3} - \frac{1}{3}\frac{z^2}{r^2} + \frac{1}{6}\frac{z^3}{r^3} - \frac{1}{30}\frac{z^4}{r^4}\left(\log\frac{2r}{z} + \frac{77}{60}\right)\right. \\ &\quad \left. - \frac{z^2}{r^2}\sum_{n=2}^{n=\infty}\frac{(-1)^n}{2^{2n-2}}\,\frac{(2n-3)!}{(n-1)!\,(n+1)!}\,\frac{1}{(2n+1)(2n+3)}\left(\frac{z}{r}\right)^{2n}\right\}. \end{aligned}\right\} \quad (118)$$

39. Solid coils with adjacent ends in contact. When $z=0$ (116) becomes

$$n = -\frac{2\pi^2 r^5}{40}\left(\log\frac{2}{r} - \frac{1}{20}\right), \quad\text{..........................(119)}$$

and hence, by (98) and (110),

$$N = 2\pi^2 r^3\left\{\frac{1}{6} - \frac{r^2}{40}\left(\log\frac{4}{r} + \frac{9}{20}\right) + \frac{1}{2}r^4\sigma\right\}, \quad\text{.............(120), }\mathbf{D}$$

$$\text{where }\ \sigma = \sum_0^\infty\left(\frac{1.3.5\ldots(2n+3)}{2.4.6\ldots(2n+4)}\right)^2\frac{r^{2n}}{(2n+7)(2n+3)(n+3)(n+1)}. \quad (121)$$

40. Derivation of formulae by elliptic integrals. This result was verified by Butterworth by means of formulae developed from the elliptic integral solution of the mutual inductance of two coaxial solenoids. This solution was given by Maxwell, and is set forth in (119) above.

The following formulae of derivation were employed. The mutual induction M of two coaxial circles, radii A, a, and axial distance b, is given by the equation

$$M = 4\pi\sqrt{Aa}\left\{\left(\frac{2}{\gamma} - \gamma\right)G - \frac{2}{\gamma}H\right\}. \qquad \ldots\ldots\ldots\ldots(122)$$

G, H (according to the notation adopted in VI. 10) are the complete elliptic integrals, of the first and second kinds, to modulus γ, where

$$\gamma = \frac{2\sqrt{Aa}}{\{(A+a)^2 + b^2\}^{\frac{1}{2}}}. \qquad \ldots\ldots\ldots\ldots(123)$$

We have also $$M = 8\pi\sqrt{Aa}\frac{1}{\sqrt{\gamma_1}}\{G(\gamma_1) - H(\gamma_1)\}, \qquad \ldots\ldots\ldots\ldots(124)$$

where $\gamma_1 = (r_1 - r_2)/(r_1 + r_2)$, $r_1{}^2 = (A+a)^2 + b^2$, $r_2{}^2 = (A-a)^2 + b^2$.

With these are the modular differential relations

$$\frac{dH}{d\gamma} = \frac{H-G}{\gamma}, \quad \frac{dG}{d\gamma} = \frac{H}{\gamma(1-\gamma^2)} - \frac{G}{\gamma}, \qquad \ldots\ldots\ldots\ldots(125)$$

and the following reduction formulae, which are easily established,

$$\left.\begin{aligned} (n+2)\int\gamma^n H\,d\gamma &= \gamma^{n+1}H + \int\gamma^n G\,d\gamma, \\ (n+2)^2\int\gamma^{n+2}G\,d\gamma &= \gamma^{n+1}H - (n+2)\gamma^{n+1}(1-\gamma^2)G \\ &\qquad + (n+1)^2\int\gamma^n G\,d\gamma, \\ \int\frac{H}{\gamma^2}d\gamma &= \left(\frac{1}{\gamma} - \gamma\right)G - \frac{2}{\gamma}H. \end{aligned}\right\} \qquad \ldots\ldots(126)$$

The integrals $$u = \int_0^\gamma G\,d\gamma, \quad v = \int_0^1 \frac{1}{\gamma}G\,d\gamma \qquad \ldots\ldots\ldots\ldots(127)$$

are important. Values of G and H and some related quantities are given in the Appendix. From these we have

$$u = \tfrac{1}{2}\pi\gamma\left\{1 + \sum_1^\infty\left(\frac{1\,.\,3\,.\,5\ldots(2n-1)}{2\,.\,4\,.\,6\ldots 2n}\right)^2\frac{\gamma^{2n}}{2n+1}\right\}, \qquad \ldots\ldots\ldots\ldots(128)$$

$$u_1 = \int_0^{\frac{1}{2}\pi}d\theta\int_0^1\frac{d\gamma}{(1-\gamma^2\sin^2\theta)^{\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta} = 2\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \ldots\right)$$

$$= 1{\cdot}83193, \qquad \ldots\ldots\ldots\ldots(128')$$

and $$v = \int_0^{\frac{1}{2}\pi}d\theta\int_\gamma^1\frac{d\gamma}{\gamma(1-\gamma^2\sin^2\theta)^{\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi}d\theta\int_{k\sin\theta}^{\sin\theta}\frac{dy}{y(1-y^2)^{\frac{1}{2}}}$$

$$= -u_1 + \tfrac{1}{2}\pi\int_\gamma^1\frac{dy}{y(1-y^2)^{\frac{1}{2}}} + \int_0^{\frac{1}{2}\pi}\frac{\theta\cot\theta\,.\,d\theta}{(1-\gamma^2\sin^2\theta)^{\frac{1}{2}}}, \qquad \ldots\ldots\ldots\ldots(129)$$

by integration by parts. The last integral is obtained by expanding $(1-\gamma^2\sin^2\theta)^{-\frac{1}{2}}$ and integrating term by term. Thus we find

$$v+u_1=\tfrac{1}{2}\pi\left\{\log\frac{4}{\gamma}-\sum_1^\infty\left(\frac{1\,.\,3\,.\,5\ldots(2n-1)}{2\,.\,4\,.\,6\ldots 2n}\right)^2\frac{\gamma^{2n}}{2n}\right\}. \quad\text{.......(130)}$$

The following theorem is of use in this connection.* The potential energy P exhausted in adding to a uniform thin circular disk, of radius A and areal density σ, a coaxial thin disk of radius a and of uniform areal density σ', is given by

$$P=\tfrac{8}{3}\pi\sigma\sigma' A\{(A^2+a^2)H-(A^2-a^2)G\},$$

where G, H are complete elliptic integrals to modulus a/A $[a<A]$. Hence for the work done in placing a solenoid of radius A, extending from $z=0$ to $z=\infty$, in the field of a solid coil of radius a, extending from $z=0$ to $z=-\infty$, we have, to a constant factor, the expression

$$m_1=\tfrac{8}{3}\pi A\int_0^a\{(A^2+a^2)H-(A^2-a^2)G\}da\,; \quad\text{...............(131)}$$

for the solenoid and the solid coil act on one another by their adjacent end disks, so as to lead to (131).

By the reduction formulae (126), (131) becomes

$$m_1=\frac{1}{3}\pi A^4\left\{\left(\frac{13}{2}+2\frac{a^2}{A^2}\right)\frac{a}{A}H-5\left(1-\frac{a^2}{A^2}\right)\frac{a}{A}G-\frac{3}{2}u\right\}. \quad\text{.....(132)}$$

When $A=a$,

$$m_1=m_2=\frac{1}{3}a^4\left(\frac{17}{2}-\frac{3}{2}u_1\right). \quad\text{..........................(133)}$$

The mutual inductance between a solenoid of radius a and a solid coil of radius A $(A>a)$, with their ends in contact, is similarly

$$m_3=m_2+\int m\,dA=\frac{1}{3}\pi A^4\left\{\left(2+\frac{13}{2}\frac{a^2}{A^2}\right)H-\frac{1}{2}\left(4-\frac{a^2}{A^2}-3\frac{a^4}{A^4}\right)G\right.$$
$$\left.-\frac{3}{2}\frac{a^4}{A^4}(v+u_1)\right\}. \quad\text{.....(134)}$$

Finally, the mutual inductance between two solid coils of radii A and a, with their ends in contact, is (with $r=a/A$)

$$N=\int m_3\,dv=\tfrac{1}{30}\pi A^5\{17r(1+r^2)H-r(14+3r^2)(1-r^2)G$$
$$-3u-3r^5(v+u_1)\}. \quad\text{........(135)}$$

When $r=1$, this is

$$N=\tfrac{1}{15}\pi A^5(17-3u_1)=2{\cdot}4094A^5. \quad\text{......................(136)}$$

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* A. Gray, *Phil. Mag.* August, 1919.

When the series for G, H, u, v are inserted in (135), we get

$$N=\frac{1}{3}\pi^2 A^5 r^3\left\{1-\frac{3}{20}r^2\left(\log\frac{4}{r}+\frac{9}{20}\right)+3r^4\sigma\right\}, \qquad (137)$$

where $$\sigma=\sum_0^\infty\left(\frac{1\,.\,3\,.\,5\ldots(2n+3)}{2\,.\,4\,.\,6\ldots(2n+4)}\right)^2\frac{r^{2n}}{(2n+7)(2n+3)(n+3)(n+1)}. \qquad (138)$$

This result is identical with (120). When $r=1$, (138) gives

$$N=2{\cdot}4094A^5,$$

the value stated in (136).

41. Tables for calculation of mutual inductances. Equations (103), (108), (117), (120) (marked also A, B, C, D), enable N to be found for all values of z *and all values of* r *up to unity*. The range of values of z for these are

$$A,\ z>3\,;\quad B,\ 4>z>r\,;\quad C,\ r>z>0\,;\quad D,\ z=0.$$

The following tables have been given by Butterworth, and facilitate greatly numerical computation. Table I. gives the values of $N/2\pi^2r^3$, and shows the agreement of A and B for $z=4$; Table II. the agreement of B and C for $z=r$; Table III. gives the values of $\xi=N/2\pi^2r^3$ for $z<4$ and different values of r; Table IV. gives the values of

$$\eta=\frac{1}{9}-\frac{Nz}{\pi^2r^3}.$$

These forms ξ and η are suitable for graphical interpolation; it is to be noticed that η is almost linear in r^2 and in $1/z^2$.

TABLE I.

Values of $\xi=N/2\pi^2r^3$ for $z=4$.

$r=$	0·2	0·4	0·6	0·8	1·0
Formula A	0·0137566	0·0137419	0·0137172	0·0136836	0·0136406
,, B	0·0137564	0·0137416	0·0137172	0·0136834	0·0136405

TABLE II.

Values of $\xi=N/2\pi^2r^3$ for $z=r$.

$r=$	0·2	0·4	0·6	0·8	1·0
Formula B	0·117413	0·087350	0·067835	0·054688	0·045477
,, C	0·117417	0·087348	0·067834	0·054686	0·045476

TABLE III. $\xi = \frac{N}{2\pi^2 r^3}$.

Values of $\xi = N/2\pi^2 r^3$ for $z < 4$ and r as specified.

r	$z = 0$	0·2	0·4	0·6	$z = 0·8$	1·0	2·0	4·0
1·0	0·122063	0·09469	0·075971	0·062649	0·052866	0·045477	0·026022	0·013641
0·8	0·134215	0·10182	0·080307	0·065405	0·054687	0·046722	0·026297	0·013684
0·6	0·145694	0·10835	0·084106	0·067835	0·056260	0·047778	0·026520	0·013717
0·4	0·155685	0·11378	0·087350	0·069746	0·057477	0·048585	0·026685	0·013742
0·2	0·163223	0·11741	0·089389	0·070970	0·058249	0·049093	0·026785	0·013756

TABLE IV.

Values of $\eta = 1/9 - Nz/\pi^2 r^3$ for r and z as specified.

r	$z = 4$	5	6	8	10
1·0	0·001986	1292	0906	0514	0331
0·8	1642	1066	0746	0423	0271
0·6	1373	0888	0620	0351	0225
0·4	1176	0760	0530	0300	0192
0·2	1058	0683	0476	0269	0173

As we have seen above we may, when the radii of the semi-infinite coils are different, a, b say, and the coils do not overlap, regard N as a function of two variables c/a, b/a, where c is the separation of the adjacent ends. Hence, if n_1, n_2 be the winding densities as defined above, we may write for the mutual inductance M the equation

$$M = n_1 n_2 a^5 N\left(\frac{c}{a}, \frac{b}{a}\right) \quad \text{or} \quad M = n_1 n_2 b^5 N\left(\frac{c}{b}, \frac{a}{b}\right), \quad \text{.........(139)}$$

according as $a > b$ or $b > a$.

42. Case of non-overlapping hollow coils. If the coils are hollow and have inner and outer radii, a_1, a_2, b_1, b_2, we have only to subtract from the mutual inductance for two solid semi-infinite, non-overlapping coils of radii a_2, b_2 that for two such coils of radii a_2, b_1 and radii a_1, b_2, and *add* the mutual inductance of two solid semi-infinite non-overlapping coils, of radii a_1 b_1, so that

$$M=n_1n_2\left[a_2^5\left\{N\left(\frac{c}{a_2},\ \frac{b_2}{a_2}\right)-N\left(\frac{c}{a_2},\ \frac{b_1}{a_2}\right)\right\}\right.$$
$$\left.-a_1^5\left\{N\left(\frac{c}{a_1},\ \frac{b_2}{a_1}\right)-N\left(\frac{c}{a_1},\ \frac{b_1}{a_1}\right)\right\}\right]. \quad \text{.............(140)}$$

For two coils of the same radius the last equation becomes

$$M=n_1n_2\left\{a_2^5N\left(\frac{c}{a_2},\ 1\right)-2a_2^5N\left(\frac{c}{a_2},\ \frac{a_1}{a_2}\right)+a_1^5N\left(\frac{c}{a_1},\ 1\right)\right\}; \text{...(141)}$$

or if the coils have their ends in contact,

$$M=n_1n_2\left\{(a_2^5+a_1^5)N(0,\ 1)-2a_2^5N\left(0,\ \frac{a_1}{a_2}\right)\right\}. \quad \text{...................(142)}$$

The inductance is reduced to that for two coils of finite lengths $2l_1$, $2l_2$, with their mid-points at a distance h apart, by subtracting the inductances due to semi-infinite coils superimposed on an original pair, which have separation $h-l_1-l_2$, so that, indicating coils by their separations (c), we have

$$M=M(h-l_1-l_2)+M(h+l_1+l_2)-M(h-l_1+l_2)-M(h+l_1-l_2), \quad (143)$$

where $M(c)$ is given by (140).

43. Overlapping coils. If the coils overlap, we proceed as follows: find (1) the uniform field which would exist within the outer coil if it were part of an infinite coil; (2) the field due to the polarity of the ends of the outer coil. Thus, if the inductions through the second coil due to these two fields be M_1, M_2, we have $M=M_1+M_2$. For M we have, with overlap l,

$$M_1=\tfrac{4}{3}\pi^2n_1n_2l(a_2-a_1)(b_2^3-b_1^3),$$

where a_2, a_1 are the outer and inner radii of the outer coil, and b_2, b_1 the outer and inner radii of the inner coil. The number of turns in unit length of the outer coil is $n_1(a_2-a_1)$ if n_1 be the winding density in that coil. The field within the outer coil would therefore be $4\pi n_1(a_2-a_1)$ for unit current, if the coil were doubly infinite in length. Hence the induction through a turn of radius x of the inner coil, due to this field, is $\pi x^2\times 4\pi n_1(a_2-a_1)$, and if the overlap is l the whole induction through the inner coil is therefore

$$M_1=4\pi^2n_1n_2l(a_2-a_1)\int_{b_1}^{b_2}x^2dx=\tfrac{4}{3}\pi^2n_1n_2l(a_2-a_1)(b_2^3-b_1^3). \quad \text{...(144)}$$

As regards M_2, the formulae developed above are applicable, and we have by (140), if the lengths of the coils are $2l_1$, $2l_2$, and they have a common centre,

$$M_2=2\{M(l_1+l_2)-M(l_1-l_2)\}, \quad \text{........................(145)}$$

or, if the coils have the same length $2l$,

$$M_2=2\{M(2l)-M(0)\}. \quad \text{..................................(145')}$$

As an example, let the coils have the following dimensions (in cm in each case):

$$\begin{array}{ll} \text{outer radii } a_2 = 10,\ b_2 = 4, & \text{inner radii } a_1 = 5,\ b_1 = 2; \\ \text{lengths}\quad 2l_1 = 6,\ 2l_2 = 44, & \text{distance of centres} = 21. \end{array}$$

Thus the overlap is 4, and we get

$$M_1 = \tfrac{4}{3}\pi^2 n_1 n_2 \times 4 \times 5(4^3 - 2^3) = 2\pi^2 746{\cdot}7 n_1 n_2.$$

Also here

$$\begin{array}{lll} h - l_1 - l_2 = c_1 = 4, & c_1/a_2 = z_1 = 0{\cdot}4, & c_1/a_1 = z'_1 = 0{\cdot}8, \\ h + l_1 + l_2 = c_2 = 46, & c_2/a_2 = z_2 = 4{\cdot}6, & c_2/a_1 = z'_2 = 9{\cdot}2, \\ h - l_1 + l_2 = c_3 = 40, & c_3/a_2 = z_3 = 4{\cdot}0, & c_3/a_1 = z'_3 = 8{\cdot}0, \\ h + l_1 + l_2 = c_4 = 2, & c_4/a_2 = z_4 = 0{\cdot}2, & c_4/a_1 = z'_4 = 0{\cdot}4, \end{array}$$

$$\begin{array}{ll} b_2/a_2 = r_1 = 0{\cdot}4, & b_1/a_2 = r_2 = 0{\cdot}2, \\ b_2/a_1 = r'_1 = 0{\cdot}8, & b_1/a_1 = r'_2 = 0{\cdot}4. \end{array}$$

Hence, by (140) and (143), and using the notation of the tables, putting ξ_{pq} for $\xi(z_p, r_q)$, ξ'_{pq} for $\xi(z'_p, r'_q)$,

$$\begin{aligned} M_2 = 2\pi^2 n_1 n_2 [a_2^5 \{ & r_1^3(\xi_{11} + \xi_{21} - \xi_{31} - \xi_{41}) - r_2^3(\xi_{12} + \xi_{22} - \xi_{32} - \xi_{42})\} \\ & - a_1^5 \{ r_1'^3(\xi'_{11} + \xi'_{21} - \xi'_{31} - \xi'_{41}) \\ & \qquad - r_2'^3(\xi'_{12} + \xi'_{22} - \xi'_{32} - \xi'_{42})\}] \quad \text{..........(146)} \end{aligned}$$

$$\begin{aligned} = 2\pi^2 n_1 n_2 \Big[& 10^5\left(-\frac{4^3}{10^3}0{\cdot}02820 + \frac{2^3}{10^3}0{\cdot}02978\right) \\ & - 5^5\left(-\frac{8^3}{10^3}0{\cdot}03652 + \frac{4^3}{10^3}0{\cdot}03077\right)\Big] \end{aligned}$$

$$= -2\pi^2 n_1 n_2 104{\cdot}4. \quad \text{..........(146')}$$

Thus

$$M = M_1 + M_2 = 2\pi^2 n_1 n_2 (746{\cdot}7 - 104{\cdot}4) = 12680 n_1 n_2. \quad \text{..........(147)}$$

44. Calculation of self-inductances. We now apply this method to the calculation of self-inductances. The self-inductance of a coil is the sum of the following inductions through it (for unit current): (1) L_1, that calculated for the coil regarded as part of a doubly infinite coil; and (2) L_2, that calculated from the polarities due to its ends. Let the length of the coil be c, its outer and inner radii a and b, and its winding density n, and write z for c/a, r for b/a, then, neglecting as before any allowance for insulation, we get

$$L_1 = \tfrac{2}{3}\pi^2 n^2 a^5 z (1 - r)^2 (1 + 2r + 3r^2). \quad \text{..........(148)}$$

Also, by (145'),

$$L_2 = 2\{M(c) - M(0)\}. \quad \text{..........(149)}$$

But

$$\left.\begin{aligned} M(c) &= n^2 a^5 \left\{ N(z, 1) - 2N(z, r) + r^5 N\left(\frac{z}{r}, 1\right)\right\} \\ M(0) &= n^2 a^5 \{(1 + r^5) N(0, 1) - 2N(0, r)\}; \end{aligned}\right\} \quad \text{..........(150)}$$

or in the ξ, η notation,

$$M(c)=2\pi^2n^2a^5\left\{\xi(z,\ 1)-2r^3\xi(z,\ r)+r^5\xi\left(\frac{z}{r},\ 1\right)\right\},\ \text{..............}(151)$$

when $z<4$, and

$$M(c)=\pi^2n^2a^5\{\tfrac{1}{9}(1-r^3)^2-\eta(z,\ 1)+2r^3\eta(z,\ r)-r^6\eta(z/r,\ 1),\}\ (152)$$

when $z>4$. Also

$$M_0=2\pi^2n^2a^5\{(1+r^2)\xi(0,\ 1)-2r^3\xi(0,\ r)\}.\ \text{......................}(153)$$

If $z>4$ the formula $L=L_1\left(1-\frac{\alpha}{z}+\frac{\beta}{z^2}-\frac{\gamma}{z^4}\right)$, ...(154)

where α, β, γ are functions of r, gives an accuracy of 1 in 10,000. Values of α, β, γ calculated from the formulae

$$q=\tfrac{1}{3}(1-r^2)(1+2r+3r^2),$$

$$\left.\begin{aligned}&\alpha=\frac{2}{q}\{(1+r^5)\xi(0,\ 1)-2r^3\xi(0,\ r)\},\quad \beta=\frac{1}{9q}(1-r^3)^2,\\&\gamma=\frac{z^2}{q}\left\{\eta(z,\ 1)-2r^3\eta(z,\ r)+r^6\eta\left(\frac{z}{r},\ 1\right)\right\},\end{aligned}\right\}\text{.......}(155)$$

with $z=4$ in the expression for γ, are given in Table V. This last condition makes γ strictly correct for $z=4$, but for large values of z, the final result is in error only to 1 part in 10,000.

TABLE V.

r	α	β	γ
0·0	0·73238	0·33333	0·0953
0·2	0·73699	0·33719	0·0973
0·4	0·75574	0·35579	0·1071
0·6	0·78447	0·39042	0·1306
0·8	0·81718	0·43906	0·1701
1·0	0·84883	0·50000	0·2306

When $r=1$, the formulae fail, but then the coil becomes a thin cylinder, and the self-inductance is [see (148)]

$$L=L_1\left(1-\frac{8}{3\pi}\frac{1}{z}+\frac{1}{2}\frac{1}{z^2}-\frac{1}{4}\frac{1}{z^4}+\frac{5}{16}\frac{1}{z^6}-\frac{35}{64}\frac{1}{z^8}+\dots\right),\ \text{..........}(156)$$

so that for $r=1$,

$$\alpha=\frac{8}{3\pi},\quad \beta=\frac{1}{2},\quad \gamma=\frac{1}{4}-\frac{5}{16}\frac{1}{z^2}+\frac{35}{64}\frac{1}{z^4}-\dots.$$

45. Calculations for short coils. Range of applicability of formulae. As an illustration of the method of working for short coils, we take a coil of the following dimensions:

outer rad. $a=4$, inner rad. $b=2$, length $c=4$ (in cm in all cases).

Then $z=c/a=1$, $r=b/a=0{\cdot}5$, $L_1=0{\cdot}229165\times 2\pi^2n^2a^5$, by (155).

$$M(c)=2\pi^2n^2a^5\{\xi(1,\ 1)-2(0{\cdot}5)^3\xi(1,\ 0{\cdot}5)+(0{\cdot}5)^5\xi(2,\ 1)\}$$

$$=2\pi^2n^2a^5(0{\cdot}045477-\tfrac{1}{4}0{\cdot}048216+\tfrac{1}{32}0{\cdot}026022)$$

$$=2\pi^2n^2a^5\times 0{\cdot}034236, \quad\text{..........}(157)$$

$$M(0)=2\pi^2n^2a^5\{(1+\tfrac{1}{32})0{\cdot}122062-\tfrac{1}{4}0{\cdot}150930\}$$

$$=2\pi^2n^2a^5\times 0{\cdot}088144. \quad\text{..........}(158)$$

Therefore

$$L=L_1-2M(0)+2M(c)=2\pi^2n^2a^5\times 0{\cdot}121351=2453{\cdot}9n^2. \quad\text{.....}(159)$$

The Weinstein formula gives for this case

$$L=2459{\cdot}5n^2,$$

which is in error to the extent of nearly $\frac{1}{4}$ per cent.

As to the applicability of the formulae given above, Mr. Butterworth, to whom they are due, gives the following caution. They are to be used only when the inner and outer diameters of the coils differ appreciably, and the lengths are not too small ($b/a<0{\cdot}8$, $c/a>0{\cdot}2$). Table V., however, holds (with graphical interpolation) for all values of b/a. For coils whose dimensions are outside these limits the usual solenoid or circular filament formulae are more suitable, with allowance for the section of the channel, in which the wire is wound, made by the method of the geometric mean distance.

46. Inductances of flat coils. Making use of the formulae of integration for elliptic integral expressions which he gave in his *Phil. Mag.* paper (*loc. cit.* 33 above), Butterworth has obtained some valuable results for flat coils. We terminate this discussion with a statement of these.

Let the mean radius of the flat coil be R, and its depth (difference of radius) $2X$, and consider two coaxial circles of the coil which differ in radius by $2x$, and have a mean radius r. Denote the mutual inductance of these by $m(x,\ r)$. The annular strip of breadth $2x$, which these circles bound, may have any position between that in which the radius of the mid-circle is $R-(X-x)$ and that for which this radius is $R+X-x$. The total inductance for all such pairs of circles and a chosen value of $2x$ is

$$2n^2dx\int_{R-X+x}^{R+X-x} m(x,\ r)\,dr, \quad\text{..........}(160)$$

where we regard the current carrying circle as an annulus of breadth dx, and the other circle as an annulus of breadth dr. The factor 2 is due to the fact that the inductance for each pair of circles must be taken twice, since the inductance between them is mutual.

The value of x varies from zero to X, and we have finally

$$L=4n^2\int_0^X dx\int_{R-X+x}^{R+X-x} m(x,\ r)\,dr. \quad\text{..........}(161)$$

The integral for L is evaluated by developing, in a series of ascending powers of $k'[=(1-k^2)^{\frac{1}{2}}=(A-a)/(A+a)=x/r]$, the elliptic integral expression for the mutual inductance of two coplanar circles, at a distance $2x\ [=2(A-a)]$ apart, and of mean radius r. (For the process of expansion see Appendix, *Notes.*) The series obtained is

$$M=4\pi r\left(\Phi_0+\frac{1}{2^2}k'^2\Phi_1+\frac{1^2.\,1^2}{2^2.\,4^2}k'^4\Phi_2+\frac{1^2.\,1^2.\,3^2}{2^2.\,4^2.\,6^2}k'^6\Phi_3+\ldots\right), \quad \text{............(162)}$$

where

$$\Phi_0=\log\frac{4}{k'}-2, \quad \Phi_1-\Phi_0=\frac{1}{1}+\frac{2}{1}, \quad \Phi_2-\Phi_1=\frac{1}{2}-\frac{2}{1}, \quad \Phi_3-\Phi_2=\frac{1}{3}-\frac{2}{3}, \ldots,$$

according to the general relation

$$\Phi_n-\Phi_{n-1}=\frac{1}{n}-\frac{2}{2n-3}, \quad \text{.........................(163)}$$

which holds for $n>1$. Applying the four terms of the expansion exhibited in (162) to (161) we find, writing λ for $\log(4R/X)$ and z for $X/4R$,

$$L=16\pi n^2RX\left\{\lambda-\frac{1}{2}+\frac{2}{3}z^2\left(\lambda+\frac{43}{12}\right)+\frac{44}{45}z^4\left(\lambda+\frac{96}{55}\right)\right.$$
$$\left.+\frac{412}{105}z^6\left(\lambda+\frac{98579}{86520}\right)+\ldots\right\}. \quad \text{.....(164)}$$

This gives a modified version of Weinstein's formula, which is applicable to coils of small inner radius. The first two terms of the series in (164) gives a formula suitable for practical purposes. The series in (164) converges for all possible values of X and R. The worst case is that in which the inner radius is zero, and then $X/R=1$. The terms in the brackets { } in (164) are then

0·886, 0·207, 0·012, 0·002.

47. Self-inductance derived from elliptic integral formulae. Using the second elliptic integral formula (124) above, with G_2 written for $G-H$, we get for the induction, Φ say, through a circle of radius x concentric with the flat coil,

$$\Phi=8\pi n\left\{x\int_{r_1}^{x}G_2(a/x)\,da+\int_{x}^{r_2}aG_2(x/a)\,da\right\},$$

or, if we change the variable from a to $\mu=a/x$, in the first integral, and to $\mu=x/a$ in the second,

$$\Phi=8\pi nx^2\left\{\int_{r_1/x}^{1}G_2(\mu)\,d\mu+\int_{x/r_2}^{1}G_2(\mu)\,d\mu/\mu^3\right\}. \quad \text{.........(165)}$$

The self-inductance of the flat coil is thus

$$L=n\int_{r_2}^{r_1}\Phi\,dx. \quad \text{................ ....(166)}$$

Equation (165) integrated by parts gives

$$L=\frac{16}{3}\pi n^2 r_2^3\int_{r_1/r_2}^{1}\left(1-\frac{r_1^3}{\mu^3 r_2^3}\right)G_2(\mu)\,d\mu. \quad\ldots\ldots\ldots\ldots\ldots(167)$$

By the formulae for integration with respect to the modulus given in 40 above (167) reduces to

$$L=\frac{8}{3}\pi n^2 r_2^3\left\{u_1-1-\left(\frac{r_1}{r_2}\right)^3(1+v)-u+2\frac{r_1}{r_2}H-\frac{r_1}{r_2}\left(1-\frac{r_1^2}{r_2^2}\right)G\right\}, \quad (168)$$

where $u=\int_0^{r_1/r_2}G\,d\mu,\quad v=\int_{r_1/r_2}^{1}G\,\frac{d\mu}{\mu},\quad u_1=\int_0^1 G\,d\mu=1{\cdot}831931248.$ (169)

The series for G, H, u, v enable (168) to be put in the form

$$L=\frac{4}{3}\pi^2 n^2 r_2^3\left[\frac{2}{\pi}(u_1-1)+\frac{r_1^3}{r_2^3}\left\{\frac{2}{\pi}(u_1-1)+\frac{1}{6}-\log\frac{4r_2}{r_1}+\sigma\right\}\right], \quad ..(170)$$

where
$$\sigma=6\sum_1^\infty\left(\frac{1\,.\,3\,.\,5\ldots(2n-1)}{2\,.\,4\,.\,6\ldots 2n}\right)^2\frac{2n+1}{2n(2n+1)(2n+3)}\left(\frac{r_1}{r_2}\right)^{2n}. \quad \ldots..(171)$$

When r_1/r_2 is small this formula is very convergent, and is therefore very suitable for coils of small inner radius. For coils of zero inner radius (170) gives

$$L=6{\cdot}96957n^2r_2^3, \quad\ldots\ldots\ldots\ldots\ldots\ldots(172)$$

while (164) gives in this, for it, the most unfavourable case,

$$L=6{\cdot}956n^2r_2^3,$$

a result which is about 0·2 per cent. too low. The first two terms of (164), the modified Weinstein formula, give

$$L=6{\cdot}87n^2r_2^3,$$

which is 1·5 per cent. below the true value.

For $r_1/r_2=0{\cdot}5$, (164) and (170) give respectively

$$L=4{\cdot}743502n^2r_2^3 \quad\text{and}\quad L=4{\cdot}743500n^2r_2^3, \quad\ldots\ldots\ldots(173)$$

very nearly coincident values. Formula (164) is generally the easier to work with, while (170) affords a check when one is needed.

The following table is given by Butterworth for the use of the equation

$$L=Qn^2r_2^3. \quad\ldots\ldots\ldots\ldots\ldots\ldots(174)$$

TABLE VI.

r_1/r_2	Q	r_1/r_2	Q	r_1/r_2	Q
0·00	6·970	0·35	5·996	0·70	2·528
0·05	6·964	0·40	5·632	0·75	1·946
0·10	6·930	0·45	5·213	0·80	1·397
0·15	6·845	0·50	4·743	0·85	0·8892
0·20	6·728	0·55	4·231	0·90	0·4574
0·25	6·544	0·60	3·682	0·95	0·1394
0·30	6·300	0·65	3·105	1·00	0·0000

Here Qn^2r_2 is the self-inductance, for r_1 = inner radius, r_2 = outer radius (in cm in each case), n = turns per radial cm.

48. Time-constants of coils. The time-constant of a coil is (VIII. 11 above) L/R, where R is the resistance of the coil. Now the resistance of a flat coil wound with wire of circular section is $4\rho n^3(r_2^2 - r_1^2)$, where ρ is the specific resistance (resistivity) of the wire. If the length and section of the wire are given, $r_2^3 - r_1^3$ and n are fixed, and the time constant is proportional to $Q/(1 - r_1^2/r_2^2)^{\frac{3}{2}}$. The following table gives the time constant for various values of r_1/r_2:

TABLE VII.

r_1/r_2	0·0	0·1	0·2	0·3	0·4	0·5
$Q/(1 - r_1^2/r_2^2)^{\frac{3}{2}}$	6·97	7·03	7·15	7·25	7·32	7·30
r_1/r_2	0·6	0·7	0·8	0·9	1·0	
$Q/(1 - r_1^2/r_2^2)^{\frac{3}{2}}$	7·18	6·94	6·46	5·50	0·00	

This table discloses the fact that the flat coil of best time-constant has $r_1/r_2 = 2/5$, though for variation of r_1/r_2 the variation of time-constant is very slow. For such a coil

$$L = 5{\cdot}632 n^2 r_2^3. \quad \text{......................} (175)$$

The time-constant of a cylindrical coil of a single layer is a maximum when the length of the coil is 4/5 of the radius a, and then the value of L is $14{\cdot}90n^2a^3$, where n is again the number of turns per cm. The radius of the cylindrical coil, which can be wound with the length of wire used in the flat coil, is $a = 0{\cdot}725r_2$ with $r_1 = 0{\cdot}4r_2$. For this cylindrical coil

$$L = 5{\cdot}666 n^2 r_2^3,$$

so that the cylindrical coil has a rather greater self-inductance for the same wire.

49. Mutual inductances of coaxial and coplanar flat coils. Finally, the mutual inductance between two coaxial flat coils in the same plane can be inferred from the self-inductances given in Table VI. Let r_1, r_2 be the inner and outer radii of the inner coil and r_3, r_4 these radii for the outer coil, and denote the self-inductances of the coils by L_α, L_γ, the self-inductance of the coil which would fill the interspace by L_β, and the mutual inductances of the three coils by $M_{\alpha\beta}$, $M_{\beta\gamma}$, $M_{\gamma\alpha}$. Further, let the coils indicated by α, β have self-inductance

L_A in series, the coils β, γ have inductance L_B in series, and the coils γ, α have inductance L_C in series. Then

$$\left.\begin{array}{l} L_A = L_\alpha + L_\beta + 2M_{\alpha\beta}, \quad L_B = L_\beta + L_\gamma + 2M_{\beta\gamma}, \\ L_C = L_\alpha + L_\beta + L_\gamma + 2M_{\alpha\beta} + 2M_{\beta\gamma} + 2M_{\gamma\alpha}, \end{array}\right\} \qquad \text{.........(176)}$$

from which the Ms can be found when the Ls are known.

Thus $$M_{\alpha\gamma} = \tfrac{1}{2}(L_C - L_A - L_B - L_\beta)$$

or $$M_{\alpha\gamma} = \tfrac{1}{2}n^2\{r_4{}^3(Q_{r_1/r_4} - Q_{r_2/r_4}) - r_2{}^3(Q_{r_1/r_3} + Q_{r_2/r_3})\}. \qquad \text{.........(177)}$$

When there is no interspace, and $r_1 = 0$,

$$M = \tfrac{1}{2}n^2R^3\{Q_0(1 - r^3/R^3) - Q_{r/R}\}, \qquad \text{.........(178)}$$

where R is the outer radius, r the dividing radius.

The following table illustrating the relations of the various inductances is given. The symbols used have the following significations:

L_1 = self-inductance of inner coil ; k = coefficient of coupling, $M/\sqrt{L_1L_2}$;
L_2 = ,, ,, outer ,, ;
M = mutual inductance between coils ; n = turns per cm.

TABLE VIII.

r/R	L_1/n^2R^3	L_2/n^2R^3	M/n^2R^3	k
0·1	0·00697	6·93	0·0162	0·074
0·2	0·0557	6·73	0·0930	0·152
0·3	0·1880	6·30	0·240	0·220
0·4	0·446	5·63	0·446	0·280
0·5	0·871	4·74	0·678	0·333
0·6	1·503	3·68	0·892	0·379
0·7	2·39	2·53	1·025	0·421
0·8	3·56	1·397	1·005	0·451
0·9	5·08	0·457	0·715	0·470
0·92	5·23	0·317	0·611	0·474
0·94	5·79	0·1910	0·495	0·470
0·96	6·17	0·0942	0·356	0·465
0·98	6·56	0·0272	0·192	0·454
1·00	6·97	0·0000	0·000	0·000

50. Formulae for self-inductance. We collect here in the first place a number of particular results for self- and mutual inductances.

1. For a circle of radius a and circular cross-section of radius ρ,

$$L = 4\pi a\left(\log\frac{8a}{\rho} - 1{\cdot}75\right) \quad \text{(Kirchhoff)},$$

$$L = 4\pi a\left\{\left(1 + 0{\cdot}1137\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho} - 0{\cdot}0095\frac{\rho^2}{a^2} - 1{\cdot}75 \quad \text{(Maxwell)}.\right.$$

For the latter formula ρ/a is supposed very small. It is derived from the g.m.d. of the cross-section from itself, in the computation of which the wire is taken as straight.

2. For a circular coil of n turns and circular section,

$$L = 4\pi n^2 a\left\{\left(1+\frac{\rho^2}{8a^2}\right)\log\frac{8a}{\rho}+\frac{\rho^2}{24a^2}-1\cdot 75\right\} \quad \text{(Rayleigh and Niven).}$$

[See also XV. 22, below.]

3. Mutual inductance of a short secondary outside a long primary. Let $2x$ be the length of the primary, 2ξ that of the secondary, A the radius of the former, a that of the latter. Apply Gray's formula (VI. 22...24) for the case of two coaxial concentric coils. A very few terms will suffice.

4. Self-inductance of a long solenoid [radius a, length b],

$$L = 4\pi a n^2\left\{\log\frac{8a}{b}-\frac{1}{2}+\frac{b^2}{32a^2}\left(\log\frac{8a}{b}+\frac{1}{4}\right)-\frac{1}{1024}\frac{b^4}{a^4}\left(\log\frac{8a}{b}-\frac{2}{3}\right)\right.$$
$$\left.+\frac{10}{131072}\frac{b^6}{a^6}\left(\log\frac{8a}{b}-\frac{109}{120}\right)-\frac{35}{4194304}\frac{b^8}{a^8}\left(\log\frac{8a}{b}-\frac{431}{420}\right)\right\}.$$

This formula is due to Coffin (*B.B.S.W.* 2, p. 113). It is accurate enough for most purposes for coils considerably longer than the radius.

The same investigator gave in the paper just cited the formula, applicable to a solenoid of any length,

$$L = \frac{8\pi}{3}an^2\left\{\left(1+\frac{b^2}{4a^2}\right)^{\frac{1}{2}}\left(\frac{4a^2}{b^2}-1\right)H+\left(1+\frac{b^2}{4a^2}\right)^{\frac{1}{2}}G-\frac{4a^2}{b^2}\right\}.$$

Here a and b have the same meaning as before, and G and H are the elliptic integrals I. and II. for the modulus $2a/(4a^2+b^2)^{\frac{1}{2}}$. This is a very useful formula. The following table for its use is given in *B.B.S.W.* 8, No. 1. It is supposed written in the form $L=n^2aQ$. Then for different values of $2a/b$, the table is

$2\frac{a}{b}$	Q	$2\frac{a}{b}$	Q
0·20	3·63240	1·80	19·57938
0·30	5·23368	2·00	20·74631
0·40	6·71017	2·20	21·82049
0·50	8·07470	2·40	22·81496
0·60	9·33892	2·60	23·74013
0·70	10·51349	2·80	24·60482
0·80	11·60790	3·00	25·41613
0·90	12·63059	3·20	26·18009
1·00	13·58892	3·40	26·90177
1·20	15·33799	3·60	27·58548
1·40	16·89840	3·80	28·23494
1·60	18·30345	4·00	28·85335

51. Correction for deviation of flow from that in a current sheet. The self-inductance formulae given above are current-sheet formulae, and require a correction depending on the ratio of the diameter of the wire to the pitch of the winding. We have

$$L = L_s - 4\pi a n(A + B), \qquad \text{.........................(179)}$$

where L_s is the uncorrected value and A and B are quantities given in the correction table in the Appendix. They are respectively

$$A = \log\left(1{\cdot}7452\frac{d}{D}\right), \quad B = \frac{2}{n}\sum_{1}^{n-1} m \log\frac{m}{R_m}, \qquad \text{..............(180)}$$

where d/D is the ratio of the diameter of the bare wire to the pitch D, and R_m is the g.m.d. of the sections of the current sheet whose centres coincide with those of the wires [*B.B.S.W.* 2, p. 168].

It was pointed out by Maxwell, *Elect. and Mag.* ii. 693, that the self-inductance of a coil of rectangular section is too great if calculated on the assumption of uniform distribution of the current over the cross-section. There are three corrections, (1) for the space occupied by the insulation, which amounts to $4\pi a n(\log D - \log d)$, where D and d are the diameters of the covered and the bare wire respectively; (2) for reduction from a square to a circular section

$$[0{\cdot}1380606 = \tfrac{4}{3}\log 2 + \tfrac{1}{3}\pi - \tfrac{11}{6}];$$

(3) E for the differences in the mutual inductances of the assemblage of round wires on one another from the values they would have if they were of square wire, and fitted without loss of space occupied by insulation. Mr. Rosa has shown that the value of E is variable, and gives the following table of its values:

Turns.	Layers.	Value of E.
2	—	0·006528
3	1	·009045
4	2	·01691
4	1	·01035
8	2	·01335
10	1	·01276
20	1	·01357
16	4	·01512
100	10	·01713
400	20	·01764
1000	50 × 20	·01778
∞	—	·01806

NOTE (Dec. 6, 1920). Several of the results of 33, ..., 49 were given independently by Spielrein, *Archiv für Elektrotechnik*, Bd. 3, 1915. This journal, which began in 1912, was only received by the writer on the date of this note, after the foregoing chapter was in type, and further reference to Spielrein's paper was impossible. The results of Butterworth and Spielrein agree very closely.

CHAPTER XIV.

MEASUREMENT OF INDUCTANCES.

1. Coefficients of induction or "inductances." The experimental comparison of coefficients of induction, or, as they are now called, *inductances*, with one another, with known resistances, and with electrostatic capacities, received much attention during the last quarter of the nineteenth century. This was a consequence on the one hand of the efforts that were then made to obtain a more accurate realization of the ohm, and of the ratio of the electromagnetic to the electrostatic unit of quantity of electricity, and on the other of the vastly increased importance which induction has assumed in electrical theory and practice, through the enormous development during that period of the use of dynamos, and especially of alternate-current machines. In the last years of that century a very successful attempt was made by the late Professor Viriamu Jones, in a determination of the ohm, to apply the very great accuracy attained in the action of machine tools, by Sir Joseph Whitworth and others, to the design and construction of physical apparatus. This striving after extreme exactitude in physical measurement has continued and been increased mainly as a result of the establishment and activities of well equipped national laboratories of physics in Europe and America, such as the Bureau of Standards at Washington, the Bureau International des Poids et des Mésures at Paris, the Physikalische Reichsanstalt at Berlin, and the National Physical Laboratory in our own country. Exact electrical standards have been defined and constructed for international use, and the movement has been carried into other departments of physics, so that correct standards of all kinds are now available for the comparison of experimental results obtained all over the world. The effect of all this in promoting accurate physical research can hardly be overestimated.

In the present chapter an attempt is made to describe the chief methods of comparison and measurement of inductances which have been devised, with, as far as possible, illustrations of the processes used and results obtained, in accounts of actual experiments. We shall use Mr. Oliver Heaviside's term "inductance" to signify what is generally denoted by "coefficient of induction" distinguishing where necessary between *mutual inductance* and *self-inductance*; but as self-induction is, on the whole, relatively more important, and is much more frequently referred to than mutual induction, we shall, where no ambiguity is likely to arise, use the single word "inductance" in the sense of coefficient of self-induction.

2. General theory of network of conductors carrying varying currents. It is convenient to consider in the first place some points of general theory which are of importance in this connection. The equations of varying currents in any conductor, or circuit of conductors, are obtainable from the electrokinetic energy and the dissipation function, when these are known, if only electrokinetic phenomena are in question, or from these two expressions, together with that of the electrostatic energy, if, as will be the case in some of the problems in the present chapter, electrostatic phenomena have also to be taken into account.

Equations of currents have been obtained in VIII. 5 above by considering an assemblage of complete circuits as a dynamical system; but similar equations are obtainable in precisely the same way for the currents in the individual conductors of a network, provided that instead of resistances, inductances and electromotive forces in circuits, the resistances, inductances, self and mutual, of the conductors, and the impressed differences of potentials between their terminals are used. The only difficulty is as to the meaning of the self-inductance of a conductor joining two points in a circuit, or the mututal inductance of two such conductors in the same or different circuits. But all such questions are resolved by adopting some proper mode of calculating inductance [for example Neumann's formula, V. 23 (18)] which enables the inductance of a conductor to be found as that of a part of a circuit, in the sense that when the inductances of the parts are calculated in this way they give the proper value of the electrokinetic energy of the circuit or circuits for any possible arrangement of currents. A case in point is that of two or more *coils* joined in parallel between two points AB. The inductances for these conductors are very approximately those obtained by regarding the coils as made up of so many complete circuits given in dimensions and position by the turns of wire. In such cases the flux of magnetic induction through the part of the circuit considered is definite and calculable, and different methods lead to the same result. But there are other cases, for example that of a Hertzian vibrator, in which different processes lead to distinctly different values of the self-inductance of the apparatus.

3. Maxwell's cycle-method of a network. The difficulty here indicated is avoided by a device adopted by Maxwell. A network is made up of a series of meshes or "cells," in which each individual conductor, except those forming the outer edge of the network, is common to two meshes. Maxwell supposed a current to circulate round each mesh in the same direction, so that the actual current in each conductor was the difference of the currents round two adjoining meshes. Thus each mesh is a closed circuit with its own current in it, and the self- and mutual inductances of the system are perfectly definite, being those due to the various closed circuits each supposed to carry unit current.

Taking the former method first let L_1, L_2, ... denote the self-inductances of the different homogeneous conductors of the system supposed linear, $y_1, y_2, \ldots$, the quantities of electricity which these conductors have conveyed in the interval from some chosen epoch of time to the instant considered, so that $\dot{y}_1$, $\dot{y}_2$, ... are the currents in the conductors at the instant, M_{12}, M_{23}, ... , the mutual inductances of the conductors indicated by the suffixes, then the electrokinetic energy is given by

$$T = \tfrac{1}{2}(L_1\dot{y}_1^2 + 2M_{12}\dot{y}_1\dot{y}_2 + \ldots + L_2\dot{y}_2^2 + 2M_{23}\dot{y}_2\dot{y}_3 + \ldots). \quad \ldots\ldots(1)$$

Here L_1, L_2, ... , M_{12}, ... are constants. In many electric circuits, with which we are not in these chapters concerned, but which contain coils with iron cores, the inductances are functions of the currents.

The dissipation function is

$$F = \tfrac{1}{2}\Sigma R_k \dot{y}_k^2, \quad \ldots\ldots(2)$$

where R_k denotes the resistance of the conductor in which the current is $\dot{y}_k$. If E be the electrostatic energy due to the charge of condensers

$$\mathsf{E} = \tfrac{1}{2}\Sigma C_m V_m, \quad \ldots\ldots(3)$$

where C_m is the capacity of a typical condenser of the system changed to a difference of potential V_m between its coating.

The effect of the electrostatic capacity of the conductors concerned is something quite sensible, and may in certain cases be allowed for. When it can be expressed, the part of the electrostatic energy which depends on the capacity of the conductors, enables the terms in (6) below to be calculated.

By this expression, also, when it can be calculated for the different parts of the conductors, the electrostatic capacities of the connecting wires can be taken into account. In such cases, however, the capacity can in general only be roughly estimated.

4. Equation of current in a single conductor. Bringing then into the account the electrostatic energy regarded as potential energy, we have to add to the electrokinetic applied force corresponding to the current $\dot{y}_k$ the electrostatic force $-\partial\mathsf{E}/\partial y_k$. Thus if V_k denote the difference of potential between the terminals of the conductor in which the current is $\dot{y}_k$, we find by the dynamical method of Lagrange the typical equation of current

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}_k} + \frac{\partial F}{\partial \dot{y}_k} = V_k - \frac{\partial \mathsf{E}}{\partial y_k}. \quad \ldots\ldots(4)$$

Writing down the equation of this type for the successive homogeneous conductors taken in order round a circuit of a network, and adding both sides of the equations, we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_j}+\frac{\partial T}{\partial \dot{y}_{j+1}}+\ldots\right)+\frac{\partial F}{\partial \dot{y}_j}+\frac{\partial F}{\partial \dot{y}_{j+1}}+\ldots=E-\left(\frac{\partial \mathsf{E}}{\partial y_j}+\frac{\partial \mathsf{E}}{\partial y_{j+1}}+\ldots\right), \quad \ldots.(5)$$

where E is the total internal applied electromotive force in the circuit, since we know that the latter is the sum of the differences of potential between the terminals of the successive homogeneous conductors forming it. This equation may be written

$$\Sigma(L_j\ddot{y}_j+M_{jk}\ddot{y}_k+R_j\dot{y}_j)=E-\sum\frac{\partial \mathsf{E}}{\partial y_j}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

in which the summations are taken for all the conductors of the circuit considered.

This equation may be taken as the most general form of the so-called "second law" which Kirchhoff explicitly stated for steady currents in a system of linear conductors. It will be of constant use in what follows.

5. Principle of continuity for varying currents derived from law of magnetic force. The principle of continuity, commonly called Kirchhoff's first law, is generally assumed for variable currents, and it is also usual to assume, as has been done above, that at any instant the magnetic force at any point due to a varying current in a circuit is the same as would be produced by a steady current equal in intensity to that which exists in the circuit at that instant. The latter assumption is justified, for points which are *near* the circuit, by the theory, confirmed now by experiment, of propagation of electromagnetic action.

It does not seem to have been noticed that the principle of continuity for a linear circuit can be deduced from this fact regarding magnetic force as follows. Let three wires meet at a point O, then according to the principle of continuity the rate of flow from the point must be exactly equal at any instant to the rate of flow to the point at the same instant. Let O be taken as the centre of a small sphere, and let the wires pass through the surface of the sphere at A, B, C (Fig. 170). Let a path be drawn round the wire A on the sphere, then carried to B, then to C nearly round it, and finally back to the point of starting from A, so that a closed path is traced on the sphere, consisting of three nearly closed curves described in the same direction round A, B, C, and an infinitely nearly closed path A', B', C', not embracing any of the conductors. A magnetic pole carried round the complete path will have no work done on it on the whole, since the path does not really surround any conductor; in other words, it could be shrunk to a point, without cutting through the conductor, and the

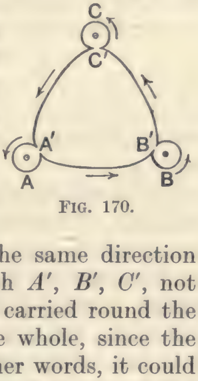

FIG. 170.

work done in carrying a pole round the infinitely nearly closed path A', B', C' also vanishes. [To see this it is only necessary to conceive the path opened out into an open loop, slipped back beyond the centre of the sphere and then shrunk up.] But if γ_1, γ_2, γ_3, be the currents in the wires at A, B, C, all reckoned as inflowing, or all as outflowing, the work done on the pole in the three paths closely surrounding the wires is $4\pi(\gamma_1+\gamma_2+\gamma_3)$, and thus must be zero, since the work done round A' B' C' is zero. Hence we have

$$4\pi(\gamma_1+\gamma_2+\gamma_3)=0$$

or

$$\gamma_1+\gamma_2+\gamma_3=0, \quad\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

that is the total current arriving at or flowing away from the point at any instant is zero. The same thing can obviously be proved in the same way for any number of conductors meeting at a point.

6. Theory of Maxwell's cycle-method. Method usual in practice. Returning to the dynamical equations of currents, the equations for Maxwell's method of meshes, each carrying its own current, are easily written down, as in (4) and (5) above. The quantities of electricity which have flowed round the different meshes from any era of reckoning up to the instant under consideration become the generalized conductors, and their time-rates of variation, or the currents at that instant, the corresponding velocities. If then L_1, L_2, ... denote the self-inductances of the different meshes, each regarded as a separate circuit, in which currents $\dot{y}_1$, $\dot{y}_2$, ... flow, M_{12}, M_{23}, ... the mutual inductances of the pairs of meshes indicated by the suffixes, we have

$$T=\tfrac{1}{2}(L_1\dot{y}_1^2+2M_{12}\dot{y}_1\dot{y}_2+\ldots). \quad\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Again, if R_{jk} denote the resistance of a conductor which adjoins two meshes distinguished by the suffixes j and k,

$$F=\tfrac{1}{2}\Sigma R_{jk}(\dot{y}_j-\dot{y}_k)^2. \quad\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

These two equations with (3) above enable the equations of currents for the different meshes to be written down. They are thus of the type

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{y}_j}+\frac{\partial F}{\partial\dot{y}_j}=E_j-\frac{\partial \mathsf{E}}{\partial y_j}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

where E_j is the electromotive force in the circuit indicated by the suffix j.

This method avoids the necessity for an explicit reference to the principle of continuity, inasmuch as this principle is assumed in the statement of the method, and it is convenient for the systematic working out of a complicated system; but with the electrokinetic energy expressed in the above form, which is the strictly accurate one when the generalized velocities are the mesh- or cycle-currents, it is not convenient for the derivation of equations from which the inductances of given conductors are to be obtained. In these applications, however, we modify the form of the electrokinetic energy by writing it

$$T=\tfrac{1}{2}\Sigma\{L_{jk}(\dot{y}_j-\dot{y}_k)^2+2M_{(jk)(lm)}(\dot{y}_j-\dot{y}_k)(\dot{y}_l-\dot{y}_m)\}, \quad \text{.........}(11)$$

where L_{jk} is the self-inductance of the conductor common to the two cycles indicated by the suffixes, and $M_{(jk)(lm)}$ the mutual inductance between that conductor and the conductor common to the two meshes indicated by the suffixes *lm*. But this merely amounts to using the first method after all. In general it is quite easy to write down the equations for the different conductors from (4) for the first method, applying the principle of continuity mentally; and as only one symbol is required for the current in each conductor, the first method has the advantage of greater brevity of expression.

7. Comparisons of inductances: problems. Ratio of inductances obtained as ratio of two resistances. The comparison of inductances comprises five problems with which we shall deal in succession: the comparison (1) of two mutual inductances, (2) of two self-inductances, (3) of a mutual inductance with a self-inductance, (4) of a mutual or self-inductance with a resistance, (5) of a mutual or self-inductance with an electrostatic capacity.

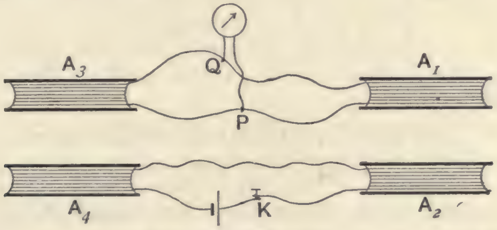

FIG. 171.

Of the first problem the following solution has been given by Clerk Maxwell. Let A_3, A_4 (Fig. 171), be the two coils, the mutual inductance M_{34} between which is to be compared with that, M_{12}, between two other coils A_1, A_2. A_1 and A_2, A_3 and A_4 are placed opposite one another at the required distance in the case of each pair. A circuit is made up of A_2, A_4, a battery and a make-and-break key K; while A_1, A_3 are joined up as a secondary circuit to which the former is the primary, and a branch containing a galvanometer is made to join two points P, Q, on this latter circuit.

The resistances R_1, R_3 of the coils A_1, A_3, respectively, with any additional resistance included with the coil in each case up to PQ, are adjusted by adding resistance coils from boxes, until there is no current through the galvanometer when the battery circuit is made or broken, and are then compared by means of a Wheatstone's bridge or other convenient method. We have then (see below)

$$\frac{M_{34}}{M_{12}}=\frac{R_3}{R_1}. \quad \text{..................................}(12)$$

To increase the sensibility of this and similar methods, some arrangement such as Ayrton and Perry's secohmmeter, described below, for successively making and breaking the battery circuit, and sending the successive integral flows through the galvanometer in the same direction, must be adopted.

8. Theory of Maxwell's method. To prove the condition (12) let L_1, L_3 be the self-inductances of the coils A_1, A_3, L the self-inductance of the battery circuit, and Γ that of the galvanometer. Then if $\dot{u}$ be the battery current at any instant, $\dot{x}$, $\dot{y}$, the currents in the same direction round A_1, A_3, respectively, the current through the galvanometer is $\dot{x}-\dot{y}$, and the electrokinetic energy of the system is given by the equation

$$T=\tfrac{1}{2}\{L\dot{u}^2+L_1\dot{x}^2+L_3\dot{y}^2+\Gamma(\dot{x}-\dot{y})^2+2M_{12}\dot{u}\dot{x}+2M_{34}\dot{u}\dot{y}\}. \quad \ldots(13)$$

If R be the resistance of the battery circuit, G the resistance of the galvanometer, we get for the dissipation function

$$F=\tfrac{1}{2}\{R\dot{u}^2+R_1\dot{x}^2+R_3\dot{y}^2+G(\dot{x}-\dot{y})^2\}. \quad \ldots\ldots\ldots\ldots\ldots(14)$$

Since the impressed electromotive forces corresponding to $\dot{x}$, $\dot{y}$, are zero, we have by (4),

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}}+\frac{\partial F}{\partial \dot{x}}=0, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{y}}+\frac{\partial F}{\partial \dot{y}}=0.$$

Hence, by (13) and (14),

$$L_1\ddot{x}+\Gamma(\ddot{x}-\ddot{y})+M_{12}\ddot{u}+R_1\dot{x}+G(\dot{x}-\dot{y})=0,$$
$$L_2\ddot{y}-\Gamma(\ddot{x}-\ddot{y})+M_{34}\ddot{u}+R_3\dot{y}-G(\dot{x}-\dot{y})=0,$$

or, integrating and writing $\dot{z}$ for $\dot{x}-\dot{y}$, and taking account of the fact that x, y, $\dot{x}$, $\dot{y}$, $\dot{u}$, are initially zero, we obtain two equations which may be written

$$\left.\begin{aligned}\left\{(L_1+\Gamma)\frac{d}{dt}+R_1+G\right\}z+\left(L_1\frac{d}{dt}+R_1\right)y+M_{12}\dot{u}=0,\\ -\left(\Gamma\frac{d}{dt}+G\right)z+\left(L_2\frac{d}{dt}+R_3\right)y+M_{34}\dot{u}=0.\end{aligned}\right\} \quad \ldots\ldots.(15)$$

Eliminating y between these we obtain an equation of the form

$$A\ddot{z}+B\dot{z}+Cz=D\ddot{u}+E\dot{u},$$

where A, B, C, D, E are constants.

Soon after completion of the primary circuit the current in the secondary will have died out. Then the last equation becomes

$$Cz=E\gamma, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

where γ is the steady current in the primary. By inspection of (15) it is easy to see that

$$C=R_3(R_1+G)+R_1G, \quad E=M_{34}R_1-M_{12}R_3.$$

Thus (16) becomes, since $z = x - y$,

$$x - y = \frac{M_{34}R_1 - M_{12}R_3}{R_3(R_1+G)+R_1G}\gamma. \quad \text{.................(16')}$$

Hence if $z (= x - y) = 0$,

$$\frac{M_{34}}{M_{12}} = \frac{R_3}{R_1}, \quad \text{..............................(17)}$$

the relation stated above.

9. Condition that Maxwell's method should be absolutely "null." It is to be noticed that if $z = 0$ at each instant, and the relation (17) be fulfilled, $D = 0$, that is by (15),

$$\frac{M_{34}}{M_{12}} = \frac{L_2}{L_1}. \quad \text{..............................(18)}$$

Thus the ratio R_3/R_1 is also the ratio of the self-inductances, if the arrangements be such that no current whatever passes in either direction through the galvanometer.

It is sometimes important, as Lord Rayleigh pointed out,* that this last condition, and in other cases a similar one if it exist, should be fulfilled in order that the method may be an absolutely null one. Very frequently, unless the galvanometer-needle is of very long period, it shows considerable uneasiness even if the condition for zero integral current is fulfilled. The fulfilment of (18) or a corresponding condition may be brought about by the insertion of self-inductance in addition to that associated with the conductors employed as resistances, and it is always desirable, if possible, to do so. The test will be the absence of uneasiness of the needle, and may be made a sensitive one by the use of a vibration galvanometer (see XIV. 60 below).

The magnetic moment of the needle may be, and no doubt often is, affected by the current in the coil, and this may interfere seriously with the ballistic action of the galvanometer. This is discussed in Chapter XII. 40 above.

10. Modification of Maxwell's method. The experiment may be arranged with a derived branch on both the primary and the secondary circuit, as shown in Fig. 172, and the galvanometer in the circuit of one of the coils as A_1. Let the resistance of the coil A_2 and connections

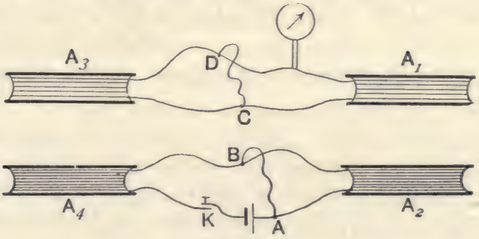

FIG. 172.

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* *British Association Report*, 1883, *Collected Papers*, 2, p. 228.

to the right of AB be R_2, the resistance to the left of AB, R_4, the resistances similarly to right and left of CD, R_1, R_3 (R_1 including the resistance of the galvanometer), the resistance of the derived branch AB on the primary R, of the derived branch CD on the secondary S.

With this arrangement, when no current through the galvanometer is produced on depression or raising the battery key, the relation

$$-\frac{M_{34}}{M_{12}}=\frac{R(R_3+S)}{S(R+R_2)} \quad \text{.............................(19)}$$

holds. Thus the mutual inductances are opposite in sign, that is the coils must be joined up so that the induced electromotive forces in the secondary circuit are opposed. In the test therefore the coils are joined up in this way, and the resistances are adjusted until no deflection of the galvanometer needle is produced by making or breaking the battery circuit.

If $R=\infty$, that is if there is no derived branch on the primary, the relation (19) becomes

$$-\frac{M_{34}}{M_{12}}=\frac{R_3+S}{S}. \quad \text{.............................(20)}$$

In this case, since numerically $M_{34}>M_{12}$, the galvanometer must be placed on that side of CD on which the induction is the weaker.

If $S=\infty$, that is if there is no derived branch on the secondary,

$$-\frac{M_{34}}{M_{12}}=\frac{R}{R+R_2}. \quad \text{..............................(21)}$$

If, for the coils in the positions of Fig. 172, $M_{12}>M_{34}$ numerically, (21) becomes $-M_{12}/M_{34}=R/(R+R_4)$.

11. Theory of the modified method. Let $\dot{u}$ be the current at any instant through the battery, and therefore through A_4, $\dot{u}'$ the current at the same instant through A_2, then the current in AB is $\dot{u}-\dot{u}'$. Denoting now by L_2, L_4, L, the inductances of the three parts into which the primary circuit is divided, namely A_2, A_4, and the derived branch AB, by L' the inductance of the derived branch CD, and by L_1, L_2, as before, the inductances of A_1, A_2, we have

$$T=\tfrac{1}{2}\{L\dot{x}^2+L_3\dot{y}^2+L'(\dot{x}-\dot{y})^2+L_2\dot{u}'^2+L_4\dot{u}^2+L(\dot{u}-\dot{u}')^2$$
$$+2M_{12}\dot{u}'\dot{x}+2M_{34}\dot{u}\dot{y}\}, \quad \text{....(22)}$$

$$F=\tfrac{1}{2}\{R_1\dot{x}^2+R_3\dot{y}^2+S(\dot{x}-\dot{y})^2+R_4\dot{u}^2+R_2\dot{u}'^2+R(\dot{u}-\dot{u}')^2\}, \quad \text{....(23)}$$

where $\dot{x}$, $\dot{y}$ denote as before the currents in A_1, A_3.

The equations of currents obtained from these and integrated over any interval from an instant just before the contact was made or broken, with attention to the fact that the initial values of the variable quantities are all zero, give equations which can be written in the form

$$\left.\begin{aligned}\left\{(L+L')\frac{d}{dt}+R_1+S\right\}x-\left(L'\frac{d}{dt}+S\right)y+M_{12}u'=0,\\ -\left(L'\frac{d}{dt}+S\right)x+\left\{(L'+L_3)\frac{d}{dt}+R_3+S\right\}y+M_{34}u=0.\end{aligned}\right\} \quad \text{.......(24)}$$

Elimination of y from these gives an equation of the form

$$A\ddot{x} + B\dot{x} + Cx = D\dot{u} + D'\dot{u}' + Eu + E'u'.$$

If the currents have become steady this reduces to

$$Cx = E\gamma + E'\gamma',$$

where x is the time-integral of the current which has passed through the galvanometer, and γ, γ' are the steady currents in the battery and the coil A_2. Hence $\gamma' = \gamma R/(R+R_2)$, and

$$Cx = \left(E + E'\frac{R}{R+R_2}\right)\gamma. \qquad \text{(25)}$$

Now by (24) $\quad C = (R_3+S)(R_1+S) - S^2,$

$$E = -M_{34}S, \quad E' = -M_{12}(R_3+S).$$

Hence (25) becomes

$$x = -\frac{M_{34}S(R+R_2) + M_{12}R(R_3+S)}{(R_1+R_2)\{(R_3+S)(R_1+S)-S^2\}}\gamma. \qquad \text{(25')}$$

If $x=0$, this gives at once

$$-\frac{M_{34}}{M_{12}} = \frac{R(R_3+S)}{S(R+R_2)}, \qquad \text{(26)}$$

the condition (19) above for no integral current through the galvanometer.

12. Ayrton and Perry's secohmmeter. As stated above, the sensibility of these methods may be greatly increased by using successive reversals of the battery current, with a proper arrangement for commutating the inductive flows through the galvanometer. An excellent contrivance for this purpose was provided by Professors Ayrton and Perry in the Secohmmeter. This is an arrangement of two rotary commutators, worked by the same spindle, one for periodically interchanging the points to which the galvanometer terminals are attached, the other for reversing the battery circuit. Each of these commutators, as will be seen from the diagrammatic sketches in Fig. 174 below, which show the mode of using the instrument, consists of four brushes pressing on a cylindrical surface made up of two nearly semi-cylindrical metal pieces separated by insulating material. The relative times of reversal by the two commutators can be adjusted to suit the purpose for which it is to be used.

The spindle can be driven by a handle or by any convenient small motor. For a given speed of driving, two speeds of the commutators can be arranged for. With one there are rather more than eight, and with the other twenty-four, reversals effected by each for one turn of the handle or driving pulley. The speed of the driving handle or pulley is governed by a fly-wheel.

For example, the instrument can be applied to the comparison of two mutual inductances by the methods just described. The battery commutator is arranged to reverse the battery circuit at an instant when the galvanometer circuit on the secondary is complete. An induction-flow takes place through the instrument unless the proper adjustment of resistances has already been made. After the battery current has reached its steady value, the galvanometer terminals are reversed by the commutator preparatory to a second reversal of the battery. The flow due to induction in this second case thus takes place through the instrument in the same direction as before, and so on as the commutator revolves. If the period of rotation is small in comparison with that of oscillation of the needle, the result is to give a steady deflection equal to that which would be produced by a current equal to nq, where n is the number of reversals of the battery per second, and q the quantity of electricity which passes at each of them.

The sensibility therefore increases with the speed of rotation; but in the present application, as in all others in which only the *integral flow* through the galvanometer, taken over the interval of variation of battery current, vanishes for certain experimental arrangements, the speed must not be so great as to prevent the battery current from reaching its steady value between each pair of reversals. In cases in which the method is really "null" the speed may be made as high as is thought desirable.

13. M. Brillouin's experiments. M. Brillouin* carried out some careful comparisons of mutual inductances by these methods. He used (1) a derived branch on the primary, (2) a derived branch on the secondary (with in each of these cases the galvanometer in series with one of the coils in the secondary), (3) the galvanometer in the derived branch on the secondary. We give here a short account of experiments (1) and (3).

In (1) the derived branch was made up of a resistance box reading to fractions of an ohm. As its coils were not wound double it was placed at a distance from the rest of the apparatus.

The galvanometer used had a resistance of 900 ohms and was an astatic needle mirror instrument. It was provided with a damping vane of wire gauze, and was enclosed in a case to shield off air currents. The observations were made in the ordinary way by means of a telescope and attached scale placed at a distance of 1 metre from the mirror.

The connecting wires were carried along side by side to reduce their external action as nearly as possible to zero.

As the galvanometer was not sensitive enough to enable measurements to be made satisfactorily with a single make or break, a rotating commutator driven by a Gramme motor was arranged, so that in each turn it (1) connected the galvanometer with the secondary circuit, (2) closed the primary circuit, (3) short circuited the galvanometer, (4) opened the primary.

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* *Thèses Présentées à la Faculté des Sciences de Paris*, 1882.

The secondary circuit was kept closed permanently and the galvanometer received only the transient current at each closing of the primary. About 10 impulses were given to the needle per second, and a permanent deflection was thus produced.

The coils used were first a pair consisting of an exterior coil made of a cable of twenty insulated wires lightly twisted together, surrounding an internal bobbin of somewhat thick wire. The mutual inductance between the internal bobbin and each of the twenty strands of the other was the same, M say. A commutator enabled any number p of the strands to be opposed to the rest, so that the coefficient of induction between the two bobbins was reduced to $(20-2p)M$. The wires however being kept in series the resistance did not vary.

The maximum mutual inductance of these coils will be denoted by M_{12}.

In experiments (1) of which results are quoted below a pair of coils was used of mutual inductance intermediate (for the positions adopted) between the maximum and minimum inductances of the apparatus just described. We shall denote the mutual inductance of these coils by M_{34}.

A pair of coils used in experiments (3) consisted of a very carefully wound bobbin of thick wire 19 cm long, and 10 cm in internal, 12 cm in external diameter, placed concentrically with a small coil of length 4·7 cm and internal and external diameters 1 cm, 5 cm respectively. The latter bobbin could be turned round through any required angle by means of an index and divided circle. The external coil being long, the two coils had a coefficient of mutual induction proportional to the cosine of the inclination of the axes.

The coefficient of induction between these coils in any given relative positions will be denoted by M'_{34}.

The following are the results of five experiments made with different fractions h of M_{12}, and no derived branch on the secondary. The ratio of the coefficients comes out, as shown in (21), in terms of the resistance R of the shunt on the primary, and R_2 the resistance of the coil A_2 in Fig. 172; R_2 was corrected to agreement at the temperature of experiment with the box from which R was taken.

h	Temp.	R	$\frac{R+R_2}{R_2}$	$-\frac{M_{12}}{M_{34}}$
1	15° C.	68·5 ± 0·1	1·671	1·671
0·9	14·2 ,,	91·6 ± 0·1	1·500	1·666
0·8′	14·7 ,,	138·9 ± 0·1	1·338	1·672
0·8	14·8 ,,	139·2 ± 0·1	1·330	1·662
0·8	14·2 ,,	137·6 ± 0·1	1·333	1·666
				Mean 1·667

A set of experiments was also made with the same arrangement, and at one temperature, 12°·6 C., with $hM_{12} < M_{34}$. The ratio in this case comes out in terms of the resistance R_4 of the coil A_4 and any non-inductive resistance in series with it, and the resistance R of the derived branch. R_4 was that of the bobbin A_4, together with a resistance seven times as great, making $R_4 = 18·49$ ohms in all.

The results are given in the table.

h	R	$\frac{R}{R+R_4}$	$-\frac{M_{12}}{M_{34}}$
0·1	3·69	·1663	1·663
0·2	9·35	·336	1·680
0·2′	9·33	·335	1·675
0·1 + 1·2	18·62	·5017	1·672
0·1 + 0·2′	18·70	·5028	1·676
0·2 + 0·2′	37·5	·669	1·672
0·1 + 0·2 + 0·2′	92·7	·8336	1·667
0·5	92·5	·8334	1·667

These results give by addition for the values 1, ·9, ·8, of h used in the former set of experiments,

$$M_{12}/M_{34} = 1·670, \quad 9M_{12}/M_{34} = 1·504, \quad ·8M_{12}/M_{34} = 1·334,$$

which closely agree with the values of $(R + R_2)/R_2$ then found.

14. Experiments by Maxwell's method. A set of experiments was also made with the galvanometer included in a derived branch on the secondary according to the arrangement of which the theory is given in 11 above.

The galvanometer was a very sensitive astatic instrument of the Thomson pattern with a coil of 7000 ohms resistance. The coils, which were the two pairs already described, were at distances of only about $2\frac{1}{2}$ metres from the galvanometer, but were placed in such positions that the direct action of each on the needle was zero. They could be turned through 10° from these positions without producing any sensible action. The induced current in the small bobbin of the second pair of coils, was found to produce no direct effect upon the needle in any position in which the bobbin was used.

All the joining wires had their outgoing and return parts together and were carefully insulated.

The primary circuit contained a battery of 10 Daniell's cells; and the rotating commutator was not employed, as the galvanometer was sufficiently sensitive to show a single impulse when the integral current through it was not zero. For the final adjustment the deflections were amplified by closing and opening the circuit when the needle was passing through zero alternately in opposite directions. Any want of perfect adjustment manifested itself by the aggregate effect of the successive exceedingly small impulses thus given, since these all tended to increase the kinetic energy of the needle.

But for balance in these circumstances it is necessary that the effects on the needle-system of completing the circuit and of breaking the circuit should both be zero. It was found at first that, while making the circuit produced no effect, breaking it always produced a slight impulse. This M. Brillouin traced to inductive action between the coils and the metallic vane attached to the needles for the purpose of damping. This induction depended on the law of variation of the induced current in the coils and took place notwithstanding the fact that the integral current at break was zero as well as that at make. By placing a condenser across the primary circuit and the make and break key, the law of variation of the current could be altered; and it was found that a corresponding change took place in the deflection. The electromagnetic action between the induced currents in the vane and the inducing current in the coils clearly ought to cause such effects as those observed.

It was found that this action had a maximum for any position of the needles when the capacity of the condenser was ·25 microfarad, and that when the vane was quite symmetrically placed relatively to the coils the effect always vanished. A condenser of this capacity was therefore applied, and the position of the needles adjusted by the directing magnet until the effect was zero. The experiment was then made, and the method of multiplication used for the deflections, with certainty that the effect of make was exactly equal and opposite to that of break.

By (12) above we have $$\frac{hM_{12}}{M'_{34}}=\frac{R_1}{R_3}.$$

In the experiments made R_3 was constant and $=974{\cdot}2$ ohms, while R_1 was made up of a constant part $R=1264{\cdot}1$ ohms, and a variable part r. The results of one set of experiments are given in the table. The fourth column is calculated by taking the fraction h of the sum of the results in column 3, and indicates the closeness of agreement of the results.

h	r	$\frac{hM_{12}}{M'_{34}}$	$\frac{hM_{12}}{M'_{34}}$ (Mean value from last col.).
0·1	191 ± 0·5	1·493 ± ·001	1·491
0·2	1639 ± 3	2·980 ± ·003	2·981
0·2′	1640 ± 3	2·981 ± ·003	2·981
0·5	5994 ± 3	7·450 ± ·003	7·452

15. Comparison of two self-inductances. The following method of comparing two self-inductances is due to Clerk Maxwell.* The two coils, the inductances, L_1, L_2, of which are to be compared, are placed in adjacent branches, AC, AD, of a Wheatstone bridge (Fig. 173), and balance is obtained for steady currents by properly adjusting the (non-inductive) resistances R, S of the branches CB, DB. If the resistances of the branches AC, AD be P, Q respectively, the relation fulfilled when balance is attained is, as we know, $PS=QR$ for steady currents. The test for this balance is carried out by depressing the battery key to establish steady flow, before the galvanometer key is put down. If besides this the relation

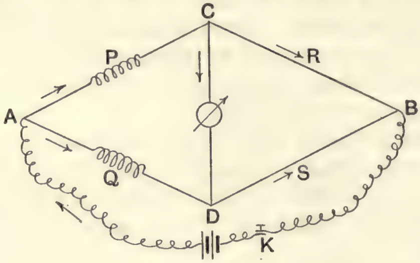

FIG. 173.

$$\frac{L_1}{L_2}=\frac{R}{S} \qquad \text{...(27)}$$

be fulfilled, there will be also balance for transient currents, and no deflection of the needle will be produced when, the galvanometer branch CD being complete, the battery circuit is made or broken. Or the coils may be placed in AC, CB so that L_1 is associated with P and L_2 with R; then balance is obtained when

$$\frac{L_1}{L_2}=\frac{P}{R}. \qquad \text{...(27')}$$

A secohmmeter may be used, as shown in Fig. 174, to increase the sensibility. Balance for induction currents is tested for by rotating the commutators. The arrangement of the apparatus will be obvious from the diagram.

16. Theory of method. To prove (27) and (27′) we write down by (6) the equations of currents of the circuits $ACDA$, $CBDC$, putting Γ, G, for the self-inductance and resistance of the galvanometer, L_1, L_2 for the inductances in the branches AC, AD, L_3, L_4, for those in the branches CB, DB, $\dot{x}$ for the current in AC, $\dot{y}$ for that from C to D, and $\dot{u}$ for that in the battery at any instant. The equations are by (6)

$$\left.\begin{array}{l} L_1\ddot{x}+P\dot{x}+\Gamma\ddot{y}+G\dot{y}-L_2(\ddot{u}-\ddot{x})-Q(\dot{u}-\dot{x})=0, \\ L_3(\ddot{x}-\ddot{y})+R(\dot{x}-\dot{y})-L_4(\ddot{u}-\ddot{x}+\ddot{y})-S(\dot{u}-\dot{x}+\dot{y})-\Gamma\ddot{y}-G\dot{y}=0. \end{array}\right\} \text{...(28)}$$

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* *El. and Mag.* vol. ii. p. 398. (Third edition.)

Integrated over the whole period of variation of currents these equations become, since there is finally zero current in CD,

$$\left.\begin{aligned}(P+Q)x \qquad\qquad\quad + Gy &= Qu + \frac{PL_2 - QL_1}{P+Q}\gamma,\\ (R+S)x-(G+R+S)y &= Su + \frac{PL_4 - QL_3}{P+Q}\gamma,\end{aligned}\right\} \quad \text{.................(29)}$$

where x, y, u denote the quantities of electricity which have flowed through AC, CD, and the battery, respectively, in the interval of integration, γ denotes the steady current through the battery, and for the steady current in the branch AC has been put its value $\gamma Q/(P+Q)$.

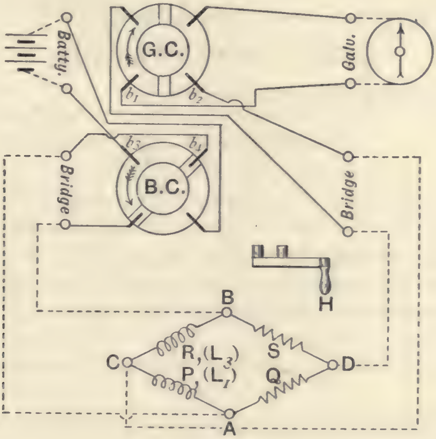

FIG. 174.

The continuous lines represent permanent connections inside the instrument, the dotted lines temporary connections, bridge, etc. The upper part shows the arrangement of the secohmmeter.

Elimination of x from (29) gives

$$y = -\gamma \frac{L_1 - P\left(\dfrac{L_2}{Q} + \dfrac{L_3}{R} - \dfrac{L_4}{S}\right)}{\dfrac{1}{S}\{G(P+Q+R+S)+(P+Q)(R+S)\}}. \quad \text{...........(30)}$$

Hence, in order that y may be zero, we must have

$$L_1 - P\left(\frac{L_2}{Q} + \frac{L_3}{R} - \frac{L_4}{S}\right) = 0, \quad \text{...........................(31)}$$

or if L_3, L_4 be negligible, $$\frac{L_1}{L_2} = \frac{P}{Q}, \quad \text{...(31')}$$

the relation stated above.

It is to be noticed that if L_1 be negligible in comparison with the other inductances, and P be finite, balance will be obtained if the resistances Q, R, S are such that

$$\frac{L_2}{Q}+\frac{L_3}{R}-\frac{L_4}{S}=0. \qquad \text{(31'')}$$

This result will be of use in connection with the comparison of a mutual inductance with a self-inductance.

17. Sensibility of the arrangement. We may now shortly investigate the sensibility of the arrangement. If r be the resistance of the battery branch AB, the resistance of the whole circuit for steady currents is evidently $r+PQ/(P+Q)+RS/(R+S)$, or since $PS=QR$,

$$r+S(P+R)/(R+S).$$

If E be the electromotive force of the battery,

$$\gamma=E/\{r+S(P+R)/(R+S)\}.$$

Thus (30) becomes with a little reduction

$$y=-E\frac{L_1-P\left(\frac{L_2}{Q}+\frac{L_3}{R}-\frac{L_4}{S}\right)}{\left\{P+R+r\left(1+\frac{R}{S}\right)\right\}\left\{G\left(1+\frac{P}{R}\right)+P\left(1+\frac{S}{R}\right)\right\}}. \qquad \text{(32)}$$

If the ratio $R/S(=P/Q)$ be taken as fixed, and P and G as given, R is to be taken so that the denominator, D say, of this expression for y may be a minimum. Denoting R/S by ρ, we have

$$D=\left\{G\left(1+\frac{P}{R}\right)+P\left(1+\frac{1}{\rho}\right)\right\}\{r(\rho+1)+P+R\}.$$

Calculating dD/dR from this and equating it to zero we find

$$R^2=\frac{GP\rho\{P+R+r(\rho+1)\}}{G\rho\left(1+\frac{P}{R}\right)+P(\rho+1)}. \qquad \text{(33)}$$

18. Conditions that the galvanometer current may be always zero. If the condition (31), and the relation $L_2L_3-L_1L_4=0$, are fulfilled, the difference of potential between C and D is always zero and therefore not only is there no integral flow from C to D, but the current at each instant is zero. This may be seen as follows. Assuming that the difference of potential at any instant is zero, there will be no current through the galvanometer. Hence

$$L_1\ddot{x}+P\dot{x}=L_2(\ddot{u}-\ddot{x})+Q(\dot{u}-\dot{x}),$$

and

$$L_3\ddot{x}+R\dot{x}=L_4(\ddot{u}-\ddot{x})+S(\dot{u}-\dot{x}).$$

Eliminating $\dot{u}$ and $\ddot{u}$ from these equations we get the relation

$$(L_3L_2-L_1L_4)\frac{d^2\dot{x}}{dt^2}+(L_3Q+L_2R-L_1S-L_1P)\frac{d\dot{x}}{dt}+(QR-PS)\dot{x}=0,$$

which must hold for all values of $\dot{x}$, $d\dot{x}/dt$, $d^2\dot{x}/dt^2$. Hence we must have, in the first place,

$$QR - PS = 0,$$

the condition for balance in the case of steady currents.

Equating the coefficient of $d\dot{x}/dt$ to zero, and using the relation $QR = PS$, we get

$$\frac{L_1}{P} - \frac{L_2}{Q} - \frac{L_3}{R} + \frac{L_4}{S} = 0,$$

which is the condition [(31)] that there should be no integral flow through the galvanometer at make (or break) of the battery circuit.

19. Use of a telephone in Wheatstone's bridge. Lastly, equating the coefficient of $d^2\dot{x}/dt^2$ to zero, we find

$$L_2L_3 - L_1L_4 = 0, \qquad \text{.....................................(34)}$$

which shows that if C and D are kept at one potential always, the inductances of the branches of the bridge must fulfil a relation precisely similar to that fulfilled by the resistances when there is balance for steady currents. The relations (31) and (34) must be fulfilled by the inductances in order that a telephone may be used in a Wheatstone's bridge. When the telephone was first introduced it was thought by many experimenters that by using a telephone and intermittent currents the Wheatstone's bridge method of testing could be made much more sensitive. As a matter of fact there can be silence in a telephone, substituted for a galvanometer in a Wheatstone's bridge, only if the inductances are balanced as well as the resistances by being made to fulfil the relation (34).

If L_3, L_4, are negligibly small each term of (34) vanishes, and the only condition to be fulfilled by the inductances is then (31), which takes the form

$$\frac{L_1}{L_2} = \frac{P}{Q}.$$

The converse proposition however, that if this condition, or in the more general case (31) and (34), be fulfilled, the current through the galvanometer is always zero is not proved. But if the points CD are not joined by a wire, and the conditions be fulfilled, CD will, it has just been shown, be at the same potential during the whole interval of variation of the currents. Hence, if at any instant during that interval a conductor, of any resistance and inductance, be supposed applied between C and D, no current would start in it, since there would be no difference of potential between its extremities. Thus, with fulfilment of the condition, varying flow in the network, with zero current in CD, is physically possible, and is the solution of the problem, otherwise there would be more than one solution, and this we know to be impossible if the currents can be regarded as a dynamical system.

20. Practice of the method. In the practice of the method the battery key is depressed first, then the galvanometer key, and balance is obtained in the ordinary way for steady currents. Then a test of balance is made for variable currents by putting down the galvanometer key first and observing whether there is any sudden deflection to one side or the other when the battery key is depressed.

If there is, the resistances *R*, *S* are altered, and balance for steady currents restored by adding non-inductive resistance to the coils in *AC*, *AD*. Then a test is made for an induction deflection as before, and if necessary a further change in *R*, *S* is made, and so on. Balance for steady currents is, at each step of the adjustment, obtained before a test for the variable currents is made, and thus confusion between a transient and a steady deflection is avoided.

21. Niven's modification of Maxwell's method. The repeated adjustments necessary in this method render it troublesome in the above form. The following modification of it, due to Prof. C. Niven,* overcomes this difficulty. One of the coils, say that of inductance *L* and resistance *P*, is made one arm of a Wheatstone bridge (Fig. 175), and balance is obtained with resistances *Q′*, *R*, *S* which form the other three branches. The other coil of inductance *L′* and resistance *Q* is then inserted at *FD*, and balance is restored by inserting a non-inductive resistance *P′* at *EC*. Non-inductive resistance, *K*, is then inserted between *E* and *F* until there is no induction current in the galvanometer, when putting down the battery key produces no current through the previously completed circuit of the galvanometer. When this is the case

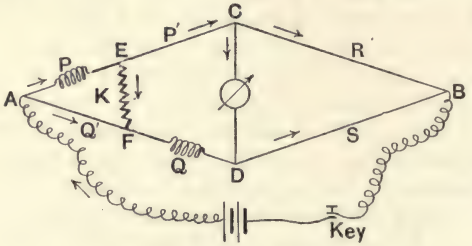

FIG. 175.

$$\frac{L}{L'}=\frac{(K+P+Q')R}{KS}. \qquad \text{.................................}(35)$$

22. Theory of Niven's modification. Let at any instant $\dot{u}$ be the current through the battery, $\dot{x}$ the current from *A* to *E*, $\dot{y}$ that from *C* to *D*, $\dot{z}$ that from *E* to *F*, then the other currents are, in *EC* $\dot{x}-\dot{z}$, in *FD* $\dot{u}-\dot{x}+\dot{z}$, in *CB* $\dot{x}-\dot{y}-\dot{z}$, in *DB* $\dot{u}-\dot{x}+\dot{y}+\dot{z}$. We get then by (6) from the three circuits *AEFA*, *ECDFE*, *CBDC*, the following equations of currents in which Γ, G, denote respectively the inductance and resistance of the branch CD.

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* *Phil. Mag.* Sept. 1887.

$$L\ddot{x} + P\dot{x} + K\dot{z} - Q'(\dot{u} - \dot{x}) = 0,$$
$$P'(\dot{x} - \dot{z}) + \Gamma\ddot{y} + G\dot{y} - L'(\ddot{u} - \ddot{x} + \ddot{z}) - Q(\dot{u} - \dot{x} + \dot{z}) - K\dot{z} = 0,$$
$$R(\dot{x} - \dot{y} - \dot{z}) - S(\dot{u} - \dot{x} + \dot{y} + \dot{z}) - \Gamma\ddot{y} - G\dot{y} = 0.$$

Integrating these from an instant just before closing the circuit of the battery to any instant after the steady state has been attained, denoting the steady currents in AE and the battery by $\dot{x}_s$ and γ respectively, and remembering that the adjustments have been supposed so made that the steady currents in EF, CD, are zero, we get

$$\left.\begin{aligned} &(P + Q')x + Kz = Q'u - L\dot{x}_s, \\ &Gy + (P' + Q)x - (Q + P' + K)z = Qu + L'(\gamma - \dot{x}_s), \\ &-(G + R + S)y + (R + S)x - (R + S)z = Su. \end{aligned}\right\} \quad (36)$$

Hence, if Δ denote the determinant of this system of equations, we get, by elimination of x and z,

$$\Delta \,.\, y = \begin{vmatrix} Q'u - L\dot{x}_s, & P + Q', & K, \\ Qu + L'(\gamma - \dot{x}_s), & P' + Q, & -(P' + Q + K), \\ Su, & R + S, & -(R + S). \end{vmatrix}$$

Expanding this determinant (first simplifying it by adding the second column to the third), remembering that since $P/Q' = P'/Q = R/S$, the relations $(R + S)Q' = (P + Q')S$, $(P' + Q)S = (R + S)Q$, hold, and putting $(\gamma - \dot{x}_s)/\dot{x}_s = R/S$, $\dot{x}_s = \gamma S/(R + S)$ we find

$$\Delta \,.\, y = \gamma\{(K + P + Q')RL' - KSL\}. \quad (37)$$

If the right-hand side be zero, and, as will generally be the case, the determinant Δ does not vanish, y must be zero. Hence, in order that there may be no integral current through the galvanometer, it is necessary and sufficient that, as stated in (35),

$$\frac{L}{L'} = \frac{(K + P + Q')R}{KS}.$$

23. Arrangement of bridge for sensibility. If r denote the resistance of the battery branch AB, we easily see, taking account of the relations $P/Q' = P'/Q = R/S$, that the resistance of the whole circuit for steady currents is

$$r + R(Q + Q' + S)/(R + S),$$

and that

$$\Delta = \frac{R + S}{S}(K + P + Q')\left\{GS + \left(Q + \frac{KQ'}{K + P + Q'}\right)(G + R + S)\right\}.$$

Hence putting E for the electromotive force of the battery we have $\gamma = E/\{r + R(K+Q+Q')/(R+S)\}$, and instead of (37)

$$y = E\frac{RL' - \dfrac{KS}{K+P+Q'}L}{\dfrac{1}{S}\{r(R+S)+R(Q+Q'+S)\}\{GS+W(G+R+S)\}}, \quad \text{...(37')}$$

in which W is written for $Q + KQ'/(K+P+Q')$.

If D denote the denominator in the expression for y, then in order that the arrangement may be as sensitive as possible D/R must be made a minimum. For simplicity let $P = Q'$, $P' = Q$, $R = S$. Then S is to be so chosen that D/S shall be a minimum. This by the ordinary method is found to be the case when

$$S^2 = \frac{(2r+Q+Q')GW}{G+2W}. \quad \text{...(38)}$$

24. Comparison of two inductances by differential galvanometer. This comparison may also be effected by means of a differential galvanometer. The two coils of inductances L_1, L_2, and resistances R_1, R_2, are joined as shown in the diagram with non-inductive adjustable resistances, and balance is obtained for steady currents without the cross-conductor of resistance S. It is plain that if, as we suppose, the resistance of each coil of the galvanometer is the same (G), and their effects on the needle are equal for equal currents, the additional resistances R'_1, R'_2 (including connections) must be equal to R_2, R_1 respectively. If E be the electromotive force and r the resistance of the battery the steady current in each coil is

$$\gamma = \frac{E}{R_1 + R'_1 + G + 2r} = \frac{E}{R_2 + R'_2 + G + 2r} = \frac{E}{R'_1 + R_2 + G + 2r}. \quad \text{...(39)}$$

The cross conductor is then applied at the points of junction P, Q, and the balance for steady currents is again tested, and if found to be disturbed is restored by slightly shifting one or both of the contacts P, Q. The resistance S is then adjusted until there is no deflection of the needle on depression of the battery key. When this adjustment has been made the relation is fulfilled

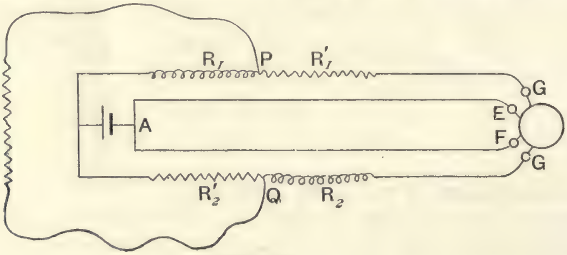

FIG. 176.

$$\frac{L_1}{L_2} = \frac{2R_1 + S}{S}. \quad \text{...(40)}$$

25. Theory of method. If $\dot{x}$, $\dot{y}$, be the currents from P, Q, respectively, to the galvanometer, $\dot{z}$ that from P to Q through the cross-connection, the current arriving from the battery is $\dot{x}+\dot{z}$ at P, and $\dot{y}-\dot{z}$ at Q. Hence if Γ be the inductance of each galvanometer coil, M their mutual inductance, and the inductances of other parts of the circuits be negligible, the equations of currents for the circuits $APGEA$, $AQGFA$, $APQA$ are by (6)

$$\Gamma\ddot{x}+M\ddot{y}+L_1(\ddot{x}+\ddot{z})+(R_1+R'_1+G)\dot{x}+r(\dot{x}+\dot{y})+R_1\dot{z}=E,$$
$$\Gamma\ddot{y}+M\ddot{x}+L_2\ddot{y}+(R_2+R'_2+G)\dot{y}+r(\dot{x}+\dot{y})-R'_2\dot{z}=E,$$
$$L_1(\ddot{x}+\ddot{z})+S\dot{z}+R_1(\dot{x}+\dot{z})-R'_2(\dot{y}-\dot{z})=0.$$

Integrating the first two of these equations over the rise of the current in each circuit from zero to the steady value γ, and subtracting the second integral from the first, we get, since $R'_1=R_2$, $R'_2=R_1$,

$$(L_1-L_2)\gamma+(R_1+R_2+G)(x-y)+(R_1+R'_2)z=0. \quad \text{.........(41)}$$

Also the third equation integrated gives

$$L_1\gamma+R_1(x-y)+(2R_1+S)z=0. \quad \text{.....................(42)}$$

Substituting in (41) the value of z given by (42) and solving for $x-y$, we obtain

$$x-y=\frac{(2R_1+S)L_2-SL_1}{2R_1(R_2+G)+S(R_1+R_2+G)}\gamma. \quad \text{................(43)}$$

In order that this may be zero we must have

$$\frac{L_1}{L_2}=\frac{2R_1+S}{S}. \quad \text{....................................(44)}$$

The value $E/(R_1+R_2+G+2r)$ substituted for γ in (43) gives

$$x-y=E\frac{(2R_1+S)L_2-SL_1}{\{2R_1(R_2+G)+S(R_1+R_2+G)\}(R_1+R_2+G+2r)}. \quad \text{...(45)}$$

The resistances R_1, R_2 are fixed, and in practice G also is given. If the galvanometer is too sensitive the magnetic field at the needles may be increased in intensity, or the coils may be shunted provided the shunt is precisely the same in inductance (if any) and resistance in both cases. The flow through each coil will, if S' be the resistance of the shunt, be simply $(x-y)S'/(G+S')$, as it would be if the galvanometer coils and shunt had no inductance.

26. Comparison of mutual inductance of two coils with self-inductance of one. Maxwell has also given the following method of comparing the mutual inductance M of two coils with the self-inductance of one of them. One of these coils, C_1, of inductance L_1 ($>M$) is included in the branch AC (Fig. 177) of a Wheatstone bridge, and the other coil, C_2, of the pair is joined up with the battery in the branch AB. The galvanometer is in the branch CD. Let P, Q, R, S be the resistances of the branches AC, AD, CB, DB, and let balance be obtained for steady currents so that $PS = QR$. Then if the coils be properly placed the ratio $P/Q = R/S$ can be so adjusted that there is no varying current through the galvanometer, and the relation

$$L_1 = -M\left(1 + \frac{P}{Q}\right) = -M\left(1 + \frac{R}{S}\right) \quad \text{..................} (46)$$

is fulfilled if the inductances of the other branches are negligible, or are balanced in the manner described below. The whole theory is given in (30) below.

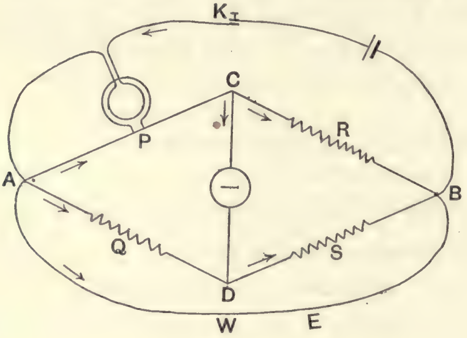

FIG. 177.

27. Avoidance of successive adjustments by shunting coil. In order that the bridge may be balanced for both steady and varying currents, the coils must be so placed that the inductive actions in the branch AC are opposed, and the resistances adjusted until no deflection is produced on depressing or raising the battery key. After each alteration of the ratio P/Q or R/S balance for steady currents must be restored before testing for varying currents. To avoid the repeated adjustments necessary in this process, a non-inductive coil is joined between A and B, and varied in resistance until no deflection is obtained on depressing or raising the battery key after the galvanometer circuit has been completed. The presence of this coil does not affect the balance for steady currents, so that when PS has once been made equal to QR, this adjustment is not disturbed. Now if W be the resistance supplied by this coil and E the point in it at the potential of C, D, it is divided into two parts AE, EB by the point E the resistances of which are

$$QW/(Q+S), \quad SW/(Q+S).$$

Since, if we please, E may be taken as in contact with D the former of these may be regarded as a shunt on AD, bringing it down to the resistance $QW/(Q+S+W)$, which gives by (46) the relation

$$L = -M\left(1 + \frac{P}{Q} + \frac{P+R}{W}\right). \quad \text{.......................} (47)$$

28. Brillouin's modification of method. It will be noticed that there is want of generality of application in this method, inasmuch as both (46) and (47) require that $L_1 > M$. It has been pointed out by M. Brillouin that the method is made perfectly general, and the relation between L_1 and M simplified by putting the coil C_2 in the shunt branch W between A and B. Balance for steady currents is first obtained, and then the total resistance W of the shunt branch is altered until balance is also obtained on making or breaking the battery circuit. The relation between L_1 and M is then

$$L_1 = -\frac{P+R}{W}M. \quad \text{.....................................(48)}$$

It is of great importance in this method that the inductances of the other branches of the bridge should be as nearly as possible zero, as sensible inductance of unknown amount not allowed for may very seriously affect the accuracy of the result obtained. The coils used for balance should therefore be as nearly as possible non-inductive.

It is shown below that if the branches AD, CB, DB have inductances L_2, L_3, L_4, the complete condition for balance when the battery key is depressed or raised,

$$kM + L_1 - P\left(\frac{L_2}{Q} + \frac{L_3}{R} - \frac{L_4}{S}\right) = 0, \quad \text{........................(49)}$$

where k denotes the factor $1 + P/Q + (P+R)/W$, or simply $(P+R)/W$, according as the coil C_2 is placed in the battery circuit or in its shunt AEB. Now we may begin by arranging so that L_2, L_3, L_4 shall be large in comparison with L_1. This may be done by first arranging a finite and as nearly as possible non-inductive resistance P in AC greater than that of the coil C_1, while inductive coils are included in the other three branches. Balance for steady as well as for varying currents is then obtained for this arrangement, and we know that then by (31″)

$$\frac{L_2}{Q} + \frac{L_3}{R} - \frac{L_4}{S} = 0. \quad \text{...................................(50)}$$

Without some special appliance this operation will involve successive adjustments to balance for steady currents at every alteration of the resistances, but this may be avoided by using for one of the coils, say that in DB, a coil of variable inductance such as two coils joined in series, one of which is within the other and capable of being turned round to any angle of inclination of the axes. The self-inductance of such a pair of coils is made up of two parts, the sum of the self-inductances of the component parts, and twice the mutual inductance between them. The latter part can be varied by varying the positions of the coils; and by this means when once balance for steady currents has been obtained, that for varying currents may be obtained also without altering the resistances of the branches.

This done, C_1 may be included in AC (thus making L_1 finite) and balance for steady currents restored by adjusting P to its former value. Balance for transient currents is then made by varying W, and we have accurately $L=kM$, since L_2, L_3, L_4, Q, R, S have not been altered.

29. Method of correcting for unknown inductances in the bridge. A different method of correction was employed by M. Brillouin. If the coils of a resistance box made of wire doubled on itself before being wound have identical dimensions and be made of wire of the same specific conductivity, but differ only in length and diameter of wire, and moreover be, as of course they generally are, without mutual inductance of sensible amount, the ratio of the small residual inductance of any coils which may be used from the box to their resistance will be approximately the same. This was found to be the case for a resistance box used by Brillouin in his investigations, and accordingly this box was used to give L_3/R. Balance both for steady and varying currents having first been obtained with certain values of P, Q, R, S, W, and L_1, L_2, L_3, L_4, a resistance r of inappreciable inductance was added to P, and the balances restored by varying R and W to new values R' and W'. The equations were then

$$\frac{kM+L_1}{P}-\frac{L_2}{Q}-\frac{L_3}{R}+\frac{L_4}{S}=0,$$

$$\frac{k'M+L_1}{P+r}-\frac{L_2}{Q}-\frac{L'_3}{R'}+\frac{L_4}{S}=0,$$

which, since $L_3/R=L'_3/R'$, gave

$$L_1=\left\{(k'-k)\frac{P}{r}-k\right\}M. \qquad\text{(51)}$$

A general investigation given by M. Brillouin shows that in order that this comparison may be carried out with all the exactness of which the method is capable, the galvanometer ought, if used without a commutator giving a steady deflection, to be from 100 times to 1000 times as sensitive for transient as for steady currents. Thus to obtain a sufficiently great galvanometer deflection, a rapidly rotating commutating arrangement, such as Ayrton and Perry's secohmmeter, Fig. 174, must be employed, if very high accuracy is aimed at.

30. Theory of the method. Referring to Fig. 177 let the inductances of AC, AD, CB, DB, AEB, and the galvanometer branch CD, be denoted by L_1, L_2, L_3, L_4, L_5, Γ, respectively, and let $\dot{u}$, $\dot{x}$, $\dot{y}$, $\dot{z}$, be the currents in the battery, AC, CD, and the shunt branch AEB at any instant, then integrating over the whole interval of variation of currents at "make" of the battery circuit, and putting γ, $\dot{x}_s$, $\dot{y}_s$, $\dot{z}_s$ for the corresponding values of the steady currents, we get for the finite equations of currents for the circuits $ACDA$, $CBDC$, $ACBEA$

$$\left.\begin{aligned}(P+Q)x \qquad\qquad\qquad + Gy + Qz &= -M\gamma+L_2(\gamma-\dot{z}_s)-(L_1+L_2)\dot{x}_s+Qu,\\ (R+S)x-(G+R+S)y+Sz &= L_4(\gamma-\dot{z}_s)-(L_3+L_4)\dot{x}_s+Su,\\ (P+R)x \qquad\qquad\qquad - Ry - Wz &= L_5\dot{z}_s-M\gamma-(L_1+L_3)\dot{x}_s.\end{aligned}\right\}\text{(52)}$$

But since the resistance of the bridge network is $S(P+R)/(R+S)$,

$$(\gamma - \dot{z}_s)/\dot{z}_s = W(R+S)/S(P+R),$$

and therefore
$$\dot{z}_s = \frac{S(P+R)}{S(P+R)+W(R+S)}\gamma.$$

Again $(\gamma - \dot{x}_s - \dot{z}_s)/\dot{x}_s = P/Q$, which gives

$$\dot{x}_s = \frac{Q}{P+Q}\,\frac{W(R+S)}{S(P+R)+W(R+S)}\gamma.$$

Substituting these values of $\dot{x}_s$, $\dot{z}_s$, in (52) and eliminating x and z, we see that since $PS = QR$, the coefficient of u identically vanishes, and we find after easy reductions

$$y = \frac{QW(R+S)}{P+Q}\,\frac{L_1 + M\left(1+\frac{P}{Q}+\frac{P+R}{W}\right) - P\left(\frac{L_2}{Q}+\frac{L_3}{R}-\frac{L_4}{S}\right)}{\Delta}\gamma, \quad (53)$$

where Δ denotes the determinant

$$\begin{vmatrix} G, & P+Q, & Q \\ -(G+R+S), & R+S, & S \\ -R, & P+R, & -W \end{vmatrix}.$$

31. Modification of formula for Brillouin's arrangement. If the coil C_2 is included in the shunt branch AEB, the term involving M in the first and third equation of (52) is $-M\dot{z}_s$ instead of $M\gamma$. Hence in the value of y given by (53) we have only to multiply M by $\dot{z}_s/\gamma$ to find the proper relation for this case. But

$$\dot{z}_s/\gamma = S(P+R)/\{S(P+R)+W(R+S)\}.$$

The multiplier of M in the numerator of the second fraction on the right of (53) therefore becomes

$$\left(1+\frac{P}{Q}+\frac{P+R}{W}\right)\frac{S(P+R)}{S(P+R)+W(R+S)}$$
$$= \frac{W(P+Q)+Q(P+R)}{W(R+S)+S(P+R)}\,\frac{S}{Q}\,\frac{P+R}{W} = \frac{Q}{S}\,\frac{S}{Q}\,\frac{P+R}{W} = \frac{P+R}{W},$$

since $P/R = Q/S$.

In order that y may vanish the necessary and sufficient condition is thus

$$L_1 + M\left(1+\frac{P}{Q}+\frac{P+R}{W}\right) - P\left(\frac{L_2}{Q}+\frac{L_3}{R}-\frac{L_4}{S}\right) = 0, \quad \text{.............}(54)$$

in the case of Maxwell's arrangement; or

$$L_1 + M\frac{P+R}{W} - P\left(\frac{L_2}{Q}+\frac{L_3}{R}-\frac{L_4}{S}\right) = 0, \quad \text{....................}(54')$$

in Brillouin's modification.

It is therefore necessary in order that no error of serious magnitude may enter into the results that L_2, L_3, L_4 may be either negligible or capable of approximate estimation. If the latter is the case the correcting term can be at once found from (54) or (54′).

32. Most sensitive arrangements of bridge. We may investigate the most sensitive arrangement of the bridge for this comparison. This we shall do by (*a*) finding the best value of R for a given value, ρ, of the ratio P/Q, and a given galvanometer, (*b*) finding the proper resistance of a galvanometer bobbin of given shape and dimensions for use with the bridge. Let r be the resistance of the battery (and the coil if included between A and B), then, if W be supposed infinite for the present, the resistance of the circuit is $r+S(P+R)/(R+S)$. Hence if E be the electromotive force of the battery we have

$$\gamma = E(R+S)/\{r(R+S)+S(P+R)\}.$$

Hence (53) becomes, with the value of k indicated in (49),

$$y = -\frac{L_1 + kM - P\left(\frac{L_2}{Q}+\frac{L_3}{R}-\frac{L_4}{S}\right)}{\{r(\rho+1)+P+R\}\left\{G\left(\frac{P}{R}+1\right)+P\frac{\rho+1}{\rho}\right\}}E. \quad \text{..........(55)}$$

The condition that the denominator of this expression may be a minimum is easily found in the ordinary way, and is

$$R^2 = \frac{GP\{r(\rho+1)+P\}\rho}{G\rho+P(\rho+1)}. \quad \text{..............................(56)}$$

This gives the best value of R for use with a given galvanometer. If however there is a choice of galvanometer-bobbins of the same volume and arrangement of wire but of different resistance, G, then for a given current the galvanometer effect produced by each bobbin varies as $\sqrt{G}$, provided the thickness of the insulating coating be in a constant ratio to the diameter of the wire, or be so small as to be negligible. Thus in order to find the condition for a maximum we have to substitute for the denominator (D say) of the expression on the right of (55) a new denominator $D' = D/\sqrt{G}$. Thus, calculating dD'/dG and equating to zero, we find in addition to (56) the condition

$$G = \frac{\rho+1}{\rho}\,\frac{PR}{P+R}. \quad \text{....................................(57)}$$

These give for R the quadratic

$$2R^2 + PR - P\{r(\rho+1)+P\} = 0. \quad \text{.........................(58)}$$

This has two real roots, one positive, the other negative. The former is therefore the required value of R and substituted in (57) gives the value of G.

A good practical example is that in which one of the two coils has a comparatively small resistance, as for example the primary of a Ruhmkorff induction coil. If this be put in the battery circuit, and the cells have a low internal resistance, r may be put equal to zero, and we have then

$$\left.\begin{aligned} R&=\frac{1}{2}P,\\ G&=\frac{1}{3}\,\frac{\rho+1}{\rho}P. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (59)$$

These results are not affected by the introduction of the wire of resistance W, since we should then have instead of Q, S, simply $QW/(Q+S+W)$, $SW/(Q+S+W)$ respectively, and ρ would have the value
$$R(Q+S+W)/SW=P(Q+S+W)/QW,$$
so that (56) and (57) would not be altered in form.

33. Practical example of method. The following are samples of results obtained by M. Brillouin in experiments made with two coaxial and concentric coils of the following dimensions:

	Mean Diameter.	Length.	No. of Turns.	
Large bobbin, - -	10·9 cm	48·5	3263	in four layers in each case.
Small ,, - -	4·98 ,,	48·5	3272	

Value of M calculated (without allowing for thickness of layers) $4{\cdot}79\times 10$ c.g.s.

In all the experiments here quoted the coil C_2 was placed in the battery circuit as shown in Fig. 177.

Q	R	S	W	k
117·72	100	100	81·886	4·659 ± ·002
117·72	1000	1000	420 ± ·3	4·661 ± ·002
117·73	10000	10000	3806 ± 10	4·658 ± ·005
235	100	200	68·85 ± ·05	4·661 ± ·002
587·4	200	1000	91·79 ± ·04	4·659 ± ·002

A series of eight experiments from which these results are selected gave a mean value of $k=4{\cdot}6595$.

Four other experiments made with R and S, 1000 ohms and 10000 ohms respectively, and with values of Q, 1176·3, 1176·3, 1165, 1164·8 ohms, gave results agreeing very well with one another, but furnishing a somewhat different mean value of k, namely 4·6397.

Two experiments in which these mean values of k were respectively used to find L/M, gave

Q	R	S	W'	k'	$-L_1/M$
379·6	1000	1000	518·2 ± ·5	4·662 ± ·004	4·661 ± ·003
3785	1000	10000	394·5 ± ·5	4·594 ± ·004	4·660 ± ·005

A rapidly rotating commutator was used as described above to make and break the battery circuit so as to increase the sensibility by giving a steady deflection of the galvanometer when the condition for balance was not fulfilled.

34. Mutual inductance of two coils compared with self-inductance of third. The mutual inductance M of two coils may be compared with the self-inductance L of a third coil by the following method, which is also due to Prof. C. Niven.* One of the mutually acting coils is included in the battery branch AB, Fig. 178, of a Wheatstone bridge, the other is placed as a shunt across the galvanometer branch CD.

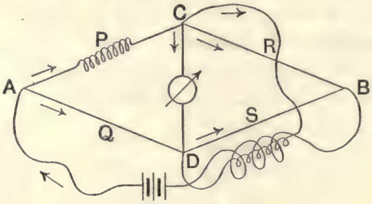

FIG. 178.

Balance is first obtained for steady currents, then the resistance, S', of the shunt is altered until there is no deflection of the needle at make and break of the battery circuit. Then

$$\frac{L}{M}=\frac{(P+Q)^2}{QS'}. \quad \text{......................................(60)}$$

35. Theory of method. We shall denote by P, Q, R, S, G, as before, the resistances of the four branches of the bridge and the galvanometer, by S', L', the resistance and inductance of the coil shunting the galvanometer branch, by Γ the inductance of the galvanometer, by $\dot{u}$, $\dot{x}$, $\dot{y}$, $\dot{z}$ the currents at any instant through the battery, the branch AC, the galvanometer, and S', and by γ the steady current through the battery. From the galvanometer circuit $ACDA$, $CBDC$, we get the integral equations of currents

$$\left.\begin{aligned}&(P+Q)x+Gy=-L\dot{x}_s+Qu,\\&(R+S)x-(R+S+G)y-(R+S)z=Su,\end{aligned}\right\} \quad \text{.................(61)}$$

in which the inductances of the galvanometer and the coil which shunts the branch CD do not appear, since there is no steady current from C to D and the inductances of the other branches are supposed negligible.

The difference of potential between C and D is

$$\Gamma\ddot{y}+G\dot{y}=M\ddot{u}+L'\ddot{z}+S'\dot{z}.$$

This integrated yields $\qquad Gy=M\gamma+S'z,$

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* *Phil. Mag.* Sept. 1887.

which converts the second equation of currents just found into

$$(R+S)x-\left(R+S+G+\frac{R+S}{S'}G\right)y=-M\frac{R+S}{S'}\gamma+Su. \quad \text{......(62)}$$

Eliminating x between (62) and the first of (61) with $Q\gamma/(P+Q)$ put for $\dot{x}$ and using the relation $PS=QR$, we find

$$y=\gamma\frac{M\dfrac{(P+Q)^2}{S'}-LQ}{(P+Q)\left\{P+Q+G\left(1+\dfrac{P}{R}+\dfrac{P+Q}{S'}\right)\right\}},$$

or since $\gamma=E/\{r+Q(P+R)/(P+Q)\}$, where r is the resistance of the battery branch AB, including coil and connections,

$$y=E\frac{M\dfrac{(P+Q)^2}{S'}-LQ}{\left\{P+Q+G\left(1+\dfrac{P}{R}+\dfrac{P+Q}{S'}\right)\right\}\{r(P+Q)+Q(P+R)\}}. \quad \text{......(63)}$$

The necessary and sufficient condition that y may be zero is thus

$$\frac{L}{M}=\frac{(P+Q)^2}{QS'},$$

which is (60). Hence when the resistances are so adjusted that there is no integral transient current in the galvanometer branch the inductances have this ratio.

It is clear that since P is fixed the value of S' depends on that chosen for Q. To a certain extent S' is fixed and therefore also Q, since S' cannot be less than the resistance of the coil and connections used across CD. If P and Q be supposed both given, the best value of R to choose would be given by the equation

$$R^2=\frac{\{QP+r(P+Q)\}GPS'}{GQ(S'+P+Q)+QS'(P+Q)}. \quad \text{......................(64)}$$

The following example is given by Prof. Niven. The field magnets of an old dynamo of the Ladd pattern were joined up in AC, and their self-inductance was compared with the mutual inductance of a pair of experimental coils. The resistance of that one of these coils which was placed in CD was 10·5 ohms, the resistance P of AC was 1·79 ohms, R was made equal to P, and Q was chosen 1000 ohms, so that S was also 1000 ohms. It was found that for balance an additional resistance of 167 ohms was required, making S' 177·5 ohms. Thus

$$\frac{L}{M}=\frac{(1001{\cdot}79)^2}{1000\times177{\cdot}5}=5{\cdot}65.$$

36. Comparison of an inductance with a resistance. The following method of determining a self-inductance in absolute measure, by comparing it with a resistance, was used by the late Lord Rayleigh in his determination of the absolute value of the B.A. unit of resistance.* The method is originally due to Clerk Maxwell, and is described in his paper on "A Dynamical Theory of the Electromagnetic Field." † Four resistances, P, Q, R, S, are joined as four branches of a Wheatstone bridge, as shown in Fig. 179. The branch AC has self-inductance L, but none of the others inductance of any kind. A battery is placed in the branch AB, and a ballistic galvanometer in the branch CD.

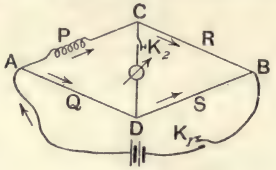

FIG. 179.

Balance for steady currents is first obtained by depressing the battery key K_1, and a second or so afterwards the key K_2. Then K_2 is depressed first, and the angular deflection θ_1, produced by putting down K_1, and caused by the inductance L in AC, is observed.

The balance for steady currents is now disturbed by altering the resistance P to $P+\delta P$, or Q to $Q+\delta Q$. We shall suppose that the latter change is made. The deflection θ_2, produced by the steady current which now flows through the galvanometer when both keys are put down, is read off and noted.

If $\dot{x}_s$, $\dot{z}_s$, be the steady currents which flow through the branches AC, AD respectively, after Q is changed to $Q+\delta Q$, and T be the period of oscillation of the needle, then it is shown below that, subject to correction for damping,

$$L = \delta Q \frac{\dot{z}_s}{\dot{x}_s} \frac{T}{\pi} \frac{\sin \frac{1}{2}\theta_1}{\tan \theta_2}. \qquad \text{.................................}(65)$$

The ratio $\dot{z}_s/\dot{x}_s$ can be found as described below, and thus L can be calculated.

37. Use of secohmmeter. The secohmmeter can be applied to increase the sensibility of this method, and the arrangement of the apparatus is shown in Fig. 179*a*. *BC* denotes the battery commutator, *GC* the galvanometer commutator. The arrows show the direction of rotation of each as seen from its side of the instrument. After the bridge has been balanced for steady currents, the instrument is rotated at a speed determined by a speed-measurer, and makes say n reversals per second. Let the steady deflection of the galvanometer needle be θ_1, then the uniform current equivalent to that producing the deflection is

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* *Phil. Trans. R.S.* Part II. 1882.

† *Phil. Trans. R.S.* vol. clv. 1865; or Clerk Maxwell's *Collected Papers*, vol. i. p. 547.

$H \tan\theta_1/G$, where H is the field intensity acting on the needle, and G is the constant of the galvanometer, supposed to be a tangent instrument.

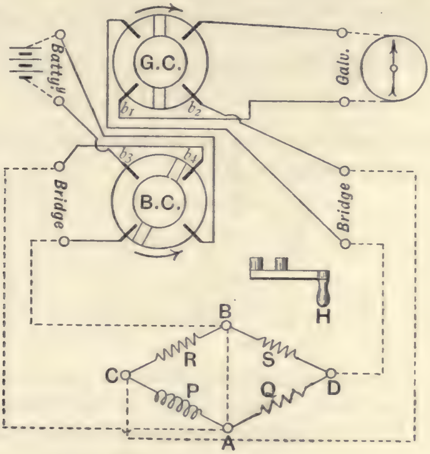

FIG. 179*a*.

The continuous lines here represent permanent connections inside the instrument, the dotted lines temporary connections to bridge, etc.

The secohmmeter is now stopped, and a steady current through the galvanometer is produced by altering Q to $Q+\delta Q$. Then it will be seen that

$$\left.\begin{aligned} L &= \frac{\delta Q}{n}\frac{P}{Q}\frac{\tan\theta_1}{\tan\theta_2} \\ \text{or}\qquad L &= \frac{\delta Q}{n}\frac{P}{Q}\frac{\theta_1}{\theta_2}, \end{aligned}\right\} \qquad \text{.................................(65')}$$

if the deflections are small.

By first balancing for steady currents, then altering Q by a convenient amount δQ, and rotating the commutators at a proper speed, the induction current may be made to balance that due to the disturbance of the ratio P/Q, so that no deflection is produced. When this is the case

$$L = k\frac{\delta Q}{n}\frac{P}{Q}. \qquad \text{.................................(65'')}$$

Here k is a coefficient depending on the relative positions of the galvanometer and battery commutators, and may be determined once for all by determining the other quantities for a known self-inductance L. The galvanometer must not be reversed exactly or very nearly midway between two reversals of the battery, as the more nearly this arrangement is made, the smaller must be the value of k and the greater $\delta Q\,.\,P/Q$ for the necessary balance.

38. Theory of method. The integral transient current through the galvanometer is easily found as follows. Let $\dot{x}$, $\dot{y}$, $\dot{u}$ be the currents in AC, through the galvanometer, and through the battery, at any instant, Γ, G, the self-inductance and resistance of the galvanometer, then from the circuits $ACDA$, $CBDC$ (Fig. 179) we get the equations of currents

$$L\ddot{x} + P\dot{x} + \Gamma\ddot{y} + G\dot{y} - Q(\dot{u} - \dot{x}) = 0,$$
$$R(\dot{x} - \dot{y}) - S(\dot{u} - \dot{x} + \dot{y}) - \Gamma\ddot{y} - G\dot{y} = 0.$$

These integrated over the whole interval of variation give, with $\dot{x}_s$ put for the steady current in AC,

$$\left.\begin{aligned} (P+Q)x + Gy &= Qu - L\dot{x}_s, \\ (R+S)x - (R+S+G)y &= Su. \end{aligned}\right\} \qquad \text{.......................(66)}$$

Hence
$$y = \frac{-L\dot{x}_s}{G\left(1+\dfrac{P}{R}\right) + P\left(1+\dfrac{S}{R}\right)}. \qquad \text{..............................(67)}$$

Thus the flow through the galvanometer is the same as that due to an electromotive impulse, $-L\dot{x}_s$ in AC, acting independently of the battery branch AB. For, any electromotive force e, thus acting, would give a current through the galvanometer of amount

$$\frac{e}{P+Q+\dfrac{G(R+S)}{G+R+S}}\,\frac{R+S}{G+R+S} = \frac{e}{G\left(1+\dfrac{P}{R}\right) + P\left(1+\dfrac{S}{R}\right)}.$$

The inductance of the galvanometer would not affect this result, and is therefore not introduced. Thus, if we put for the integral of e the value $-L\dot{x}_s$, we get the result stated above.

Now if $\dot{z}_s$ denote the steady current through the branch AD, the steady currents through the galvanometer and the other branches of the bridge satisfy the equations (obtained from the circuits $ACDA$, $CBDA$, and the circuit $ACBA$, through the battery),

$$\left.\begin{aligned} P\dot{x}_s + G\dot{y}_s - Q'\dot{z}_s &= 0, \\ R\dot{x}_s - (G+R+S)\dot{y}_s - S\dot{z}_s &= 0, \\ (P+R+r)\dot{x}_s - R\dot{y}_s + r\dot{z}_s &= E, \end{aligned}\right\} \qquad \text{.....................(68)}$$

where Q' denotes any value of the resistance of the branch AD, r the resistance and E the electromotive force of the battery. Putting $Q' = Q + \delta Q$, and using the relation $SP = QR$, we get from these equations

$$\dot{y}_s = E\frac{R\,\delta Q}{\Delta}, \qquad \text{.......................................(69)}$$

where Δ is the determinant of the system of equations (68).

But eliminating $\dot{x}_s$, $\dot{y}_s$, we find for the steady current $\dot{z}_s$ through the branch AD

$$\dot{z}_s = E\frac{GR + P(G+R+S)}{\Delta}.$$

Hence, from (69),
$$\dot{y}_s=\frac{\dot{z}_s\delta Q}{G+\dfrac{P}{R}(G+R+S)}.$$

It may be noticed that an electromotive force $\dot{z}_s\delta Q$ in AD, acting as if the battery branch did not exist, would produce through the galvanometer a steady current of amount
$$\frac{\dot{z}_s\delta Q}{P+Q'+\dfrac{G(R+S)}{G+R+S}}\,\frac{R+S}{G+R+S}=\frac{\dot{z}_s\delta Q}{G+\dfrac{P}{R}(G+R+S)+\delta Q\dfrac{G+R+S}{R+S}},$$
which is nearly the same thing as $\dot{y}_s$ if δQ be small. It is to be carefully noticed here that $\dot{z}_s$ is the current in the branch AD *after* the resistance Q has been altered to $Q+\delta Q$.

By the theory of the ballistic galvanometer (XII. 36, *et seq.* above) y is given by the equation
$$y=\frac{HT}{\pi G}\sin\tfrac{1}{2}\theta_1,$$
subject to a correction for damping. Also
$$\dot{y}_s=\frac{H}{G}\tan\theta_2,$$
so that $y/\dot{y}_s=L\dot{x}_s/\dot{z}_s\delta Q=T\sin\frac{1}{2}\theta_1/\pi\tan\theta_2$, or
$$L=\delta Q\frac{\dot{z}_s}{\dot{x}_s}\,\frac{T}{\pi}\cdot\frac{\sin\frac{1}{2}\theta_1}{\tan\theta_2},$$
which is equation (65).

39. Lord Rayleigh's experiments. In the late Lord Rayleigh's experiments the battery current was reversed to produce the induction-flow through the galvanometer; so that taking the deflection produced by reversal in each case we must use the ratio $\dot{z}_s/2\dot{x}_s$ in the above formula for L.

Lord Rayleigh used for R and S two coils of ten units each taken from a resistance box, while P was a copper coil of resistance rather less than 24 ohms, and inductance L to be determined. A coil of 24 units taken from the same resistance box with a coil of 753 units (which was taken from an auxiliary box) placed in parallel with it, balanced P. The resistance P was thus $24\times 753/777=23{\cdot}25869$, in units of the box.

Q was altered by substituting 853 units from the auxiliary box for the 753 units used in parallel with the coil of 24 units. Thus Q was made 23·34322 units, and therefore δQ was 0·08453 unit.

The battery current was reversed by a key placed in AB while the galvanometer branch was kept closed. Observations of θ_1, θ_2 were taken by means of telescope and scale in the ordinary manner; and were made as rapidly as possible, by properly manipulating the key, and opening and closing the galvanometer branch so as to stop the inductive deflections after the throw had been observed. The observer himself damped the vibrations of the needle by exciting temporarily at proper times a current in a coil for the purpose.

The induction throw was taken without waiting for the needle to come perfectly to rest, or arranging for perfect balance for steady currents. The amplitude of free swing was obtained by observing two successive elongations with the needle fairly quiet. Then the battery current was reversed as the needle passed through the position of equilibrium, and it was noted whether the induction throw was with or against the direction of free motion, and the four elongations after reversal were observed.

After reversal the zero for steady flow had of course shifted owing to imperfect balance, but the change gave a means of correcting the induction throw. Let a be double the true arc of deflection due to induction, a_0 the range of vibration from side to side just before reversal, and b the arc through which the zero had shifted, then at the moment after reversal the velocity which the needle had in consequence of free swing was numerically $\pi a_0/T$, in consequence of induction $\pi a/T$, and the displacement from the new zero was b. The velocity was thus $\pi(a \pm a_0)/T$.

If now s represent the displacement from the new zero at any subsequent time we have

$$s = A \sin\left(\frac{2\pi}{T}t - e\right),$$

where A and e are constants. Then

$$\frac{ds}{dt} = \frac{2\pi A}{T}\cos\left(\frac{2\pi}{T}t - e\right)$$

$$= \frac{2\pi A}{T}\cos e = \frac{\pi(a \pm a_0)}{T}$$

when $t=0$. Thus

$$A\cos e = \tfrac{1}{2}(a \pm a_0).$$

Again when $t=0$, $s=b$, and therefore

$$A\sin e = -b.$$

Hence we have

$$s = \tfrac{1}{2}(a \pm a_0)\sin\frac{2\pi}{T}t + b\cos\frac{2\pi}{T}t.$$

This represents a vibration of which the amplitude

$$A = \sqrt{\tfrac{1}{4}(a \pm a_0)^2 + b^2},$$

or, if b be small,

$$2A = a \pm a_0 + \frac{2b^2}{a},$$

so that

$$a = 2A \mp a_0 - \frac{2b^2}{a},$$

where A was the observed arc of deflection. The correction given by the last term was very small. $2A$ was the arc between the two turning points immediately following the reversal. As a check, readings of the two following turning points were also taken. The new zero was obtained from two successive elongations of the needle which were observed after the needle had nearly come to rest in its new position.

The next time the needle passed through the equilibrium position an induction throw in the opposite direction to the last was taken, and the four immediately following elongations observed.

40. Observations of steady current deflection. Readings were then taken as quickly as possible of the steady current deflection produced by changing the coil of 753 units to 853 units. Readings of three or four successive elongations were taken as soon as the amplitudes had become moderate. Then the galvanometer branch CD was opened, and the battery current was reversed while the needle was passing over to the other side of zero. When the needle had swung over, the galvanometer contact was restored, then four elongations were again observed. The arc between the two positions of equilibrium was thus twice the deflection due to the steady current produced by changing Q from 23·25869 to 23·34322 units.

A correction of course had to be made for the effect which would have been produced by reversing without changing Q. This was obtained from the observations of the effect of imperfect balance made before each induction throw; and any progressive change due to alteration of temperature was got rid of by using the mean of such observations made before and after a change from 753 to 853 units.

The following is a specimen set of observations. In the table E. P. stands for "equilibrium position," and I. T. for "induction throw."

Time of Observation.	Position of Battery Key.	Readings on Scale and Deflections in Scale Divisions.	
3 h. 36 m.	Left.	E. P. 264·4	Res. 753 units.
		I. T. 246·6	
3 h. 38 m.	Right.	E. P. 262·5	
		I. T. 245·6	
3 h. 40 m.	Right.	E. P. 182·3	Res. 853 units.
3 h. 41 m.	Left.	E. P. 344·7	
3 h. 44 m.	Left.	E. P. 264·4	Res. 753 units.
		I. T. 245·7	
3 h. 45 m.	Right.	E. P. 263·1	
		I. T. 245·6	

41. Reduction of results of observations. In the first set of these results the difference 1·9 between 264·4 and 262·5 was due to imperfection of balance, in the second set the difference was 1·3. The mean of these, 1·6, subtracted from 162·4 gave 160·8 as the deflection produced by replacing 753 units by 853 and reversing, corrected for imperfection of balance.

Thus the ratio of the two deflections obtained from this specimen set of observations was $245{\cdot}9/160{\cdot}8 = 1{\cdot}529$. Two sets, each of four similar observations, the second set made with the galvanometer reversed, gave each the mean value 1·5310 for this ratio, so that reversing the galvanometer produced no effect.

Calling D the distance of the mirror from the scale, $2A$ the induction deflection, $2B$ the deflection produced by reversing the battery current when balance is disturbed by the addition of 100 units to the 753, all three quantities being expressed in terms of the same unit of length, we have

$$\tan 2\theta_1 = \frac{A}{D}, \quad \tan 2\theta_2 = \frac{B}{D},$$

which give by successive approximation

$$\frac{2\sin\frac{1}{2}\theta_1}{\tan\theta_2} = \frac{A}{B}\,\frac{1-\frac{11}{32}\frac{A^2}{D^2}}{1-\frac{1}{4}\frac{B^2}{D^2}},$$

or, since $A = 122{\cdot}5$, $B = 80$, and $D = 2180$,

$$\frac{2\sin\frac{1}{2}\theta_1}{\tan\theta_2} = {\cdot}99925\,\frac{A}{B} = {\cdot}99925 \times 1{\cdot}5310.$$

Separate determinations of the logarithmic decrement gave $\lambda = {\cdot}0142$, and the period T was found to be 23·386 seconds. Since the effect of damping was to diminish the distances from zero at the first and second elongations by the fractions $\frac{1}{2}\lambda$, $\frac{3}{2}\lambda$ of their proper amount, the difference between these distances can be corrected by multiplying by the factor $1+\lambda$. It is sufficient to apply this factor to the value of $2\sin\frac{1}{2}\theta_1/\tan\theta_2$.

Thus the equation for L becomes

$$L = \delta Q\frac{\dot{z}_s}{\dot{x}_s}\,\frac{T}{4\pi}\,{\cdot}99925\,\frac{A}{B}\,(1+\lambda). \qquad \text{......................}(69')$$

The resistance of the galvanometer was 80 units, and calculation showed that the current through it might be neglected in estimating the ratio $\dot{z}_s/\dot{x}_s$. The resistance of the battery being low, the difference of potential between A and B was taken as given. Calling it V we have

$$\dot{x}_s = V/(10+23{\cdot}25869), \quad \dot{z}_s = V/(10+23{\cdot}34322),$$

so that

$$\frac{\dot{z}_s}{\dot{x}_s} = \frac{10+23{\cdot}25869}{10+23{\cdot}34322}.$$

Using then these data with the value

$${\cdot}08453 \times {\cdot}987 \text{ ohm} \quad \text{or} \quad {\cdot}08453 \times {\cdot}987 \times 10^9 \text{ c.g.s.}$$

for δQ obtained by regarding 1 B.A. unit as ·987 ohm, we get

$$L = 2{\cdot}4028 \times 10^8$$

in ordinary electromagnetic c.g.s units, that is, in centimetres.*

At the temperature of the room the resistances given by the boxes were not exactly multiples of the B.A. unit, and the resistance of 853 units had to be increased by fully one part in a thousand to give the necessary correction. Thus δQ was greater than the value given above by this fraction. Thus, finally,

$$L = 2{\cdot}4052 \times 10^8, \text{ in cm.}$$

Calculation from the specification of the coil gave

$$L = 2{\cdot}400 \times 10^8, \text{ in cm,}$$

about 1 in 500 less. In Lord Rayleigh's judgment the former value was just as likely to be correct.

42. Joubert's method of measuring self-inductance. A self-inductance may also be compared with a resistance by the following method due to M. Joubert. A circuit is made up of the coil the inductance of which is to be determined, and a non-inductive resistance. An alternating machine giving a suitable electromotive force as nearly as possible following the simple harmonic law is included, and the mean square of the difference of potential between the terminals is compared by means of an electrometer with that existing between the terminals of the non-inductive resistance. Denoting the mean squares of these differences, respectively, by $\overline{V_1^2}$, $\overline{V_2^2}$, and the resistances of the corresponding coils by R_1, R_2, we have

$$\frac{\overline{V_1^2}}{\overline{V_2^2}} = \frac{R_1^2 + n^2 L^2}{R_2^2}, \quad \text{.....................................(70)}$$

where $n = 4\pi/T$, T being the complete period of the alternating current. This equation gives

$$L = \frac{R_1}{n}\left(\frac{R_2^2}{R_1^2}\frac{\overline{V_1^2}}{\overline{V_2^2}} - 1\right)^{\frac{1}{2}}. \quad \text{.............................(71)}$$

The value of n can be found of course from the speed of the machine, and the number of alternations in each turn.

To find the ratio $\overline{V_1^2}/\overline{V_2^2}$ the electrometer must be used idiostatically as explained below,† that is, one terminal is connected to one pair of quadrants if the instrument is a quadrant electrometer; or to the stationary electrified system which acts on the movable system or indicator, while the other terminal is attached to the needle or indicator. Then the mean square of the difference of potential between the terminals will be proportional to the deflection if small, or if the needle is brought back to a sighted zero position, will be proportional to the couple required to keep it in that position. Any sensitive electrostatic voltmeter is well adapted for this measurement.

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* See Chapter XVI. † See the chapter on Electrostatic Measurements.

43. Theory of Joubert's method. To prove the formulas stated above let r be the part of the resistance which does not depend on the coils used for the comparison, $E \sin nt$ the electromotive force in the circuit at any instant, and $\dot{x}$ the current at that instant. Then if $L+L'$ is the total inductance in the circuit

$$(L+L')\ddot{x}+(R_1+R_2+r)\dot{x}=E\sin nt.$$

The part $L\ddot{x}+R_1\dot{x}$ is the difference of potential then existing between the terminals of the coil that is being tested, $R_2\dot{x}$ is that between the terminals of the non-inductive coil. We may write, therefore, if v_1, v_2, be constants,

$$\left.\begin{array}{r} L\ddot{x}+R_1\dot{x}=v_1\sin nt, \\ R_2\dot{x}=v_2\sin nt. \end{array}\right\} \quad\text{..........................(72)}$$

The complete solution of the first of these equations is

$$\left.\begin{array}{rl} & \dot{x}=A\epsilon^{-\frac{R_1}{L}t}+\dfrac{v_1}{\sqrt{R_1{}^2+n^2L^2}}\cos(nt-e), \\ \text{where} & \tan e=\dfrac{R_1}{nL}. \end{array}\right\} \quad\text{..................(73)}$$

The first term on the right dies out in a short time and has no further influence, if the machine works regularly, and so

$$\dot{x}=\frac{v_1}{\sqrt{R_1{}^2+n^2L^2}}\cos(nt-e).$$

By this result and the second of (72)

$$R_2\dot{x}=\frac{v_1R_2}{\sqrt{R_1{}^2+n^2L^2}}\cos(nt-e)=v_2\sin nt,$$

and therefore $$v_1{}^2\cos^2(nt-e)=\frac{R_1{}^2+n^2L^2}{R_2{}^2}\,v_2{}^2\sin^2 nt.$$

Hence, integrating over a complete period, we find

$$\frac{\overline{V}_1{}^2}{\overline{V}_2{}^2}=\frac{R_1{}^2+n^2L^2}{R^2},$$

which is (70), and the rest follows as above.

The quantity in the numerator is the square of what is called the "impedance" of the conductor; R_1 is often referred to as the "ohmic resistance."

44. Conditions for accuracy of Joubert's method. It will be observed that the accuracy of this method depends to some extent on obtaining an alternating current which is sufficiently nearly expressed by a simple harmonic function of the time, that is which varies as $\sin pt$, where

$2\pi/p$ is the period of alternation. Almost all alternating machines give currents which are more complex, and a difficulty arises unless a special machine is available. There are sources of error also; for example, the capacity of the electrometer if of sensible amount gives a current which is derived from the main alternating current in the circuit, and the coils have each a certain electrostatic capacity. Moreover, if the rate of alternation is high and the wire be thick, the ohmic resistances of the coils cease to be constants, and depend on the frequency of alternation in a manner which has been explained in Chapter IX. This effect, however, is not of sensible amount if the frequency is low, and the wire is of the thickness usually employed in induction coils and standards. Also the tendency to restriction of the current to the outer layers which rapid alternation produces takes place in both coils, and this affects the ohmic resistances of both, and the inductance of that which is inductive.

45. Rosa and Grover's investigation: correction for wave form. A determination of a self-inductance by a modification of this method was carried out with great care by Mr. E. B. Rosa and Mr. Frederick Grover in the laboratory of the Bureau of Standards at Washington.* The arrangement was that shown in Fig. 180. The alternating machine used was one which gave an electromotive force represented approximately by a simple sine curve. This curve was drawn by a curve tracer devised by Mr. Rosa, and was subjected to harmonic analysis in order to discover its various components. To accentuate the higher harmonics present in the curve a current was passed in a derived branch on *AC* (Fig. 180) which contained a condenser and was examined by the curve tracer. In such a branch the harmonics are magnified by the action of the condenser in proportion to their order.

A correcting factor f due to the presence of the higher harmonics was then found in the following manner. Denoting the current produced by the alternator at any instant by I, we have for I an expression of the form

$$I = I_1 \sin(pt - \phi_1) + I_3 \sin(3pt - \phi_3) + I_5 \sin(5pt - \phi_5) + \dots$$

as only the odd harmonics were present. The mean square of the current was then (see VIII. 24 above)

$$I^2 = I_1{}^2 + I_3{}^2 + I_5{}^2 + \dots.$$

For two coils in series one non-inductive and of resistance R, the other having self-inductance L and resistance r, through which the current I passes, the mean squares of the electromotive forces between the pairs of terminals are respectively

$$E_a{}^2 = R^2(I_1{}^2 + I_3{}^2 + I_5{}^2 + \dots) = R^2 I^2,$$
$$E_b{}^2 = r^2 I^2 + p^2 L^2 (I_1{}^2 + 9 I_3{}^2 + 25 I_5{}^2 + \dots).$$

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* See a paper presented at the Intern. Elect. Congress at St. Louis, 1904 (*B.B.S.W.* i. p. 125).

Now in the experiments carried out E_a^2 and E_b^2 were made identical in value, and so was obtained the equation

$$L=\frac{I}{p}\left\{\frac{R^2-r^2}{I_1^2+9I_3^2+25I_5^2+\ldots}\right\}^{\frac{1}{2}}.$$

The correction factor f is therefore given by the equation

$$f=\frac{I^2}{I_1^2+9I_3^2+25I_5^2+\ldots}.$$

The analysis of the curves of the machine is given with full particulars in the paper referred to above. We state here only the most probable value found for f, which was

$$f=0{\cdot}99858,$$

a number differing from unity by 142 parts in 100,000, a quite sensible difference.

46. Rosa and Grover's determination continued. In Fig. 180 the non-inductive coil is marked R, the other L, r. These were in series, and the terminals of the alternator were applied at the points A, C. An electrodynamometer joined between A and C acted as a sensitive voltmeter for the alternator. One terminal of an electrometer was connected to the point B, between the two coils, and the other was carried to a switch S by which connection was made between A, B or B, C as required.

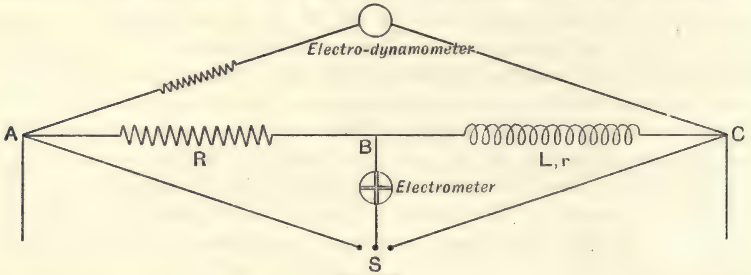

FIG. 180.

The value of R was varied until the same deflection was obtained for AB as for BC. The frequency was found at the same time by means of a chronograph and chronometer. The beats of the chronometer were registered by the chronograph, which made also an electric contact for every 50 revolutions of the alternator. Thus when the speed was kept constant the frequency $p/2\pi$ was at once obtained.

To control the speed a rotating commutator directly connected to the alternator so as to be driven by it, was used, with a Wheatstone bridge and condenser arranged with it. When this bridge was balanced at the required speed an observer kept the speed constant, by varying the resistance R_3 in the field coils of the driving motor by means of a rheostat, so as to keep the galvanometer needle stationary.

The electromotive force on *AB* could not easily be made exactly equal to that on *BC*, but the value of the resistance of *AB* was adjusted to the nearest ohm so as to give this equality, and electrometer readings were taken for *AB* and for *BC*. Three pairs of such readings were taken for the resistance chosen. Then if the resistance of *AB* was too low, it was increased by one ohm, and three pairs of electrometer readings again taken. The exact value of the resistance for equal readings on *AB* and *BC* was then deduced by interpolation.

47. Results of Rosa and Grover's experiments. Two coils from different makers, each having inductance of the nominal value of 1 henry were experimented with ; one is denoted in the following table of results by L_C, the other by L_F. The table will give an idea of the concordance of the results obtained in different sets of experiments ;

	1	2	3	4	5	6
	R	r	n	L_1 inductance uncorrected for wave form	Mean of L_1	L
L_C						
Run 1 -	1149·98	97·6	179·51	1·01567	1·06572	1·01428
Run 4 -	1141·90	97·75	178·284	1·01564		
Run 5 -	1133·93	97·9	177·009	1·01575		
Run 8 -	1132·69	98·05	176·793	1·01586		
Run 9 -	1159·76	98·2	181·083	1·01556		
L_F						
Run 2 -	1134·01	97·4	179·550	1·00145	1·00144	1·00002
Run 3	1125·93	97·6	178·272	1·00140		
Run 6 -	1118·19	97·7	177·009	1·00156		
Run 7 -	1116·68	97·8	176·804	1·00135		
Run 10 -	1143·62	97·9	181·081	1·00145		

In another set of experiments the corrected values obtained were respectively 1·01416 and 1·00018.

It was found that the inductances varied somewhat with the temperature of the laboratory, and that the temperature coefficients had opposite signs for the two coils. The coil L_F was found to be wound on a spool of serpentine imbedded in paraffin, while the other was made of dry silk-covered wire wound on a spool of mahogany. No doubt the expansion of the paraffin increased the geometric mean distance of the wires, and so diminished L more than it was increased by the increase of the radius caused by expansion of the copper.

48. Comparison of self-inductance with capacity of a condenser. Maxwell also showed how to compare the inductance of a coil with the capacity of a condenser, and his method has since been modified by various experimenters so as to obviate the necessity for successive adjustments which it involves. As originally given the method consisted in placing the coil in one branch of a Wheatstone bridge, as DB, Fig. 181, while the plates of the condenser were attached directly at AC. Balance for steady currents is first obtained and is not affected by the condenser; then the resistances are altered until no inductive flow through the galvanometer is produced by making or breaking the battery circuit. If C be the capacity of the condenser, P, S the resistances of the branches AC, DB, the relation fulfilled when balance is thus obtained is

$$L = PSC. \quad \text{.................................(74)}$$

49. Theory of the method. Let, as before, P, Q, R, S denote the resistances of AC, AD, CB, DB, L the inductance in the branch DB,

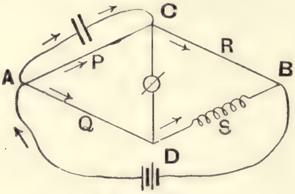

FIG. 181.

and put C for the capacity of the condenser. Let further for any instant $\dot{z}$ denote the current along AC, $\dot{x} - \dot{z}$ the current charging the condenser, $\dot{y}$ the current from A to D, and ξ, η the potentials at C and D. Suppose that balance for steady currents is first obtained so that $PS = QR$, then in order that at the instant in question ξ may be equal to η the conditions

$$P\dot{z} = Q\dot{y}, \quad \text{.................................(75)}$$

$$L\ddot{y} + S\dot{y} = R\dot{x} \quad \text{.................................(76)}$$

must hold. But $P\dot{z}$ is the difference of potential between A and C, and may be taken as that between the plates of the condenser. Hence the charge of the condenser is $CP\dot{z}$, and since $\dot{x} - \dot{z}$ is the rate of increase of this charge, we have

$$\dot{x} - \dot{z} = CP\ddot{z} = CQ\ddot{y}.$$

This with (75) converts (76) into

$$L\ddot{y} + S\dot{y} = \frac{RQ}{P}\dot{y} + RCQ\ddot{y}, \quad \text{..........................(77)}$$

which if ξ is always to be equal to μ must hold for all values of $\ddot{y}$ and $\dot{y}$. But $PS - RQ = 0$; hence we must have also

$$L = PSC, \quad \text{.................................(78)}$$

and S and P must be chosen so as to fulfil this condition if the current through the galvanometer is always to be zero.

50. Rimington's modification of Maxwell's method. A series of successive adjustments is thus necessary before the proper values of S and P and balance for steady currents are obtained. Mr. E. C. Rimington * has shown how these adjustments may be avoided by a very simple modification of the method. The balance for steady currents having been obtained as before, the condenser is applied at two points E, F, in AC (Fig. 182), including between them a resistance p ($<P$) such that with the inductance L in DB no deflection of the galvanometer needle takes place when the battery key is depressed or raised. The resistance p may be taken from a resistance slide, the whole (or the variable part) of which is included in AC, or preferably two slides in series may be used so as to give two adjustable sliding contact pieces to which to attach the plates of the condenser. The galvanometer needle should have sufficient moment of inertia to enable the inductive action to begin and end before the needle has sensibly moved, for the effect of the condenser AC, which is charged by the current from A to E, is to delay the rise of the potential at C to its final value after the battery key is put down, while the inductance L in DB produces a similar effect on the rise of the potential at C. Hence if the needle were not sufficiently ballistic it might show a deflection due to a difference in the rate of variation in the two cases, although the time-integral of the current through the galvanometer were really zero. The inductance is given by the equation

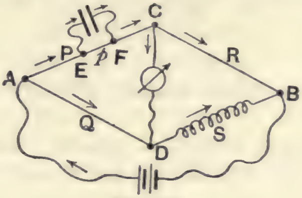

FIG. 182.

$$L = Cp^2 \frac{S}{P}. \qquad \text{(79)}$$

51. Theory of Rimington's modification. Writing down the equations of currents for the circuits $ACDA$, $CBDC$, putting $\dot{x}$ for the current in AE and FC, $\dot{z}$ for the current in EF, $\dot{y}$ for the galvanometer current, $\dot{u}$ for the current through the battery, using the same notation as before for the other quantities, and integrating over the time interval from the instant before completion of the battery circuit until the steady state has been attained, we find by (6)

$$\left.\begin{aligned} (P+Q)x \qquad\qquad\quad + Gy &= Qu + Cp^2\dot{x}_8, \\ (R+S)x - (G+R+S)y &= Su + L(\gamma - \dot{x}_8), \end{aligned}\right\} \qquad \text{(80)}$$

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* *Phil. Mag.* July, 1887.

where γ, $\dot{x}_s$, denote the steady currents in the battery and in the branch AC. Solving for y and putting $\gamma - \dot{x}_s = \dot{x}_s P/Q$, we find

$$y = \frac{(R+S)(Cp^2S - LP)\dot{x}_s}{S\{G(R+S) + (G+R+S)(P+Q)\}}. \quad \text{.................(81)}$$

Thus the necessary and sufficient condition that there should be no integral flow through the galvanometer is

$$L = Cp^2 \frac{S}{P},$$

as already stated.

If $p = P$ this gives the result already obtained for the case originally considered by Maxwell.

It ought to be noticed here that precisely the same equation may be obtained by integrating, in the same way, over the interval at break from the steady state to zero current in each conductor, so that the test may be repeated at breaking the circuit.

We may now investigate the most sensitive arrangement of the bridge. In general S is given in magnitude, and p, which must of course be less than P, will in most cases be some convenient resistance depending on the apparatus available, so that P may be regarded as given. Hence we have to choose the value of R (and that of Q will follow) so that y may for some chosen value of p be a maximum. By (81) and the equation

$$\dot{x}_s = \frac{SE}{r(R+S) + S(P+R)},$$

where E is the electromotive force of the battery, and r the resistance of the battery and the wires connecting it to A, B, we get easily

$$y = \frac{(Cp^2S - LP)E}{\left\{G\left(1+\frac{P}{R}\right) + P\left(1+\frac{S}{R}\right)\right\}\left\{r(R+S) + S(P+R)\right\}}. \quad \text{......(82)}$$

The numerator of this expression does not vary: hence calling the denominator D, calculating dD/dR, and equating to zero, we find after reduction

$$R^2 = \frac{SP(G+S)(r+P)}{(G+P)(r+S)}, \quad \text{..........................(83)}$$

which gives the best value of R if that of G is given.

If however there is a choice of similar galvanometer bobbins of different resistances, then as before (XII. 22) we must substitute for D a value $D' = D/\sqrt{G}$, calculate dD'/dG, and equate the result also to zero. This gives another equation for G and R, viz.

$$G = \frac{P(R+S)}{P+R}. \quad \text{...............................(84)}$$

From (83) and (84) as simultaneous equations the values of G and R are to be found.

If p is at the disposal of the experimenter and can be varied by small steps, the best arrangement is that for which, when y is almost zero, a given small change in p gives a maximum change in y. Hence if possible we have to arrange so that dy/dp may be a maximum when $y=0$. The conditions for this however are so complicated as to be unserviceable.

52. Anderson's ballistic method. In the same paper Professor Anderson gives the following simple ballistic method of comparing the capacity of a condenser with an inductance. A bridge is made up as before of four conductors, and a condenser and galvanometer are arranged as in Fig. 183, so that by means of mercury cups the galvanometer can be connected either to CD by the cups a, b, c, d, or in series with the condenser in the branch AB by the cups c, d, e, f. A rocking key is conveniently made to effect either of these connections at a single operation. A coil of inductance L is placed in AC, all the other branches with the exception of the galvanometer are destitute of inductance.

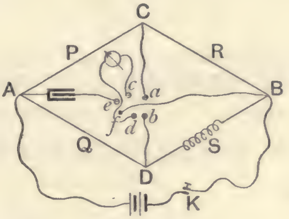

FIG. 183.

Balance for steady currents is first obtained with the galvanometer in CD. Then when the key K is depressed or raised an inductive flow of integral amount y passes through the galvanometer. If $\dot{x}_s$ is the steady current in DB, the value of y is given by

$$y=\frac{L\dot{x}_s}{G\left(1+\frac{S}{Q}\right)+S\left(1+\frac{P}{Q}\right)}. \qquad \text{..........................(85)}$$

The deflection θ_1 produced by this is noted.

By means of the rocking key the galvanometer is joined in series with the condenser between the points A and B, so that the plates of the condenser are charged to a difference of potential $\dot{x}_s(Q+S)$. If C be the capacity of the condenser a quantity of electricity $C\dot{x}_s(Q+S)$ passes through the galvanometer. The resulting deflection θ_2 is observed.

We have then by the theory of the ballistic galvanometer and (85)

$$L=C(Q+S)\left\{G\left(1+\frac{S}{Q}\right)+S\left(1+\frac{P}{Q}\right)\right\}\frac{\sin\frac{1}{2}\theta_1}{\sin\frac{1}{2}\theta_2}. \qquad \text{...............(86)}$$

53. Practical example of Anderson's ballistic method. The following are the details of an actual measurement made by the author of the method. A coil of mean radius 20·9 cm wound with 278 turns of wire in a groove of breadth 1·894 cm and depth 1·116 cm, was placed in *DB*. The galvanometer was an ordinary reflecting instrument of resistance 164·8 ohms, with its period made as long as possible by means of a controlling magnet. A non-inductive resistance of 100 ohms was added to the coil, and P and R were each 10 ohms. Balance was obtained by making S 150·51 ohms. The mean results of several readings agreeing well together were

Deflection due to induction - - - -	43·208 divisions.
Deflection due to charge of condenser of ·5 microfarad - - - - - - -	46·125 ,,
Deflection due to charge of condenser of ·45 microfarad - - - - - - -	41·875 ,,

By interpolation it was found from these results that a condenser of ·4657 microfarad capacity would just give a deflection of 43·208 divisions. Thus in c.g.s. units *

$$L = \cdot 4657 \times 10^{-15} \times 2 \times 150\cdot 51 \times (329\cdot 6 + 150\cdot 51 + 10) \times 10^{18}$$
$$= \cdot 0687 \times 10^{9}.$$

54. Comparison of mutual inductance and capacity by Anderson's ballistic method. To determine a mutual inductance the method is used thus: One coil, C_1, of the mutually influencing pair is joined in *DB* as before, the other, C_2, has its terminals joined to a pair of mercury cups g, h, which are arranged so that a rocking-key can put the galvanometer between A and B, or between the cups g, h, so as to connect the terminals of the coil C_2.

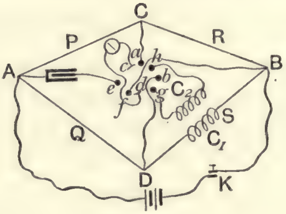

FIG. 184.

Balance for steady currents having been obtained as before, the terminals of the galvanometer are connected to g, h, and the battery circuit is completed or broken. Calling θ_3 the deflection produced and denoting by θ_1, θ_2, as before, the deflections obtained by operating with the coil C, as already described (p. 573), we have

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* See Chapter I. A microfarad is 10^{-15} c.g.s. units of capacity, and an ohm 10^{9} c.g.s. units of resistance.

$$M = C(Q+S)(r_2+G)\frac{\sin\frac{1}{2}\theta_3}{\sin\frac{1}{2}\theta_2} \quad \text{......................(87)}$$

and

$$\frac{M}{L} = \frac{r_2+G}{G\left(1+\frac{S}{Q}\right)+S\left(1+\frac{P}{Q}\right)}\frac{\sin\frac{1}{2}\theta_3}{\sin\frac{1}{2}\theta_1}. \quad \text{..................(88)}$$

The inductive electromotive force at any instant in the coil C_2 is $M\ddot{x}$, hence the integral electromotive force is $M\dot{x}_s$. The whole quantity of electricity which flows through the galvanometer is thus $M\dot{x}_s/(r_2+G)$, where r_2 is the resistance of the coil C_2. But the quantity of electricity which passes when the throw θ_2 is produced is $C\dot{x}_s(Q+S)$. Hence we get (87), and combining (87) with (86) we get (88).

As an example Professor Anderson gives the following:

$Q=S=1{\cdot}003$ ohm, the resistance of the coil C_1;

$r_2=157{\cdot}7$ ohms, $G=164{\cdot}8$ ohms;

$C=1$ microfarad, θ_3, $\theta_2=72$ and 5 scale divisions respectively.

Hence roughly, in c.g.s. units,

$$M = 10^3 \times 2{\cdot}006 \times 322{\cdot}5 \times 14{\cdot}4$$
$$= 9316 \times 10^3,$$

or about ·0093 henry.

55. Comparison of mutual inductance and capacity by a differential galvanometer. In the paper already referred to in (21) above, Professor Niven * has shown how to compare the inductance of a coil with the capacity of a condenser by means of a differential galvanometer. A circuit is made up as shown in Fig. 185, of one coil of the differential galvanometer, the coil (of inductance L and resistance R_1) to be compared, an additional resistance in the branch AE and the battery B. A corresponding circuit is arranged with the other coil of the galvanometer, a non-inductive resistance R_2, an additional resistance in the branch AF, and the battery as before, so that the battery serves both circuits as shown in the figure. After balance for steady currents has been obtained by adjusting the additional resistances, the condenser is joined across the two branches AE, AF, and the terminals shifted until no deflection is produced, when the battery-key is depressed, or

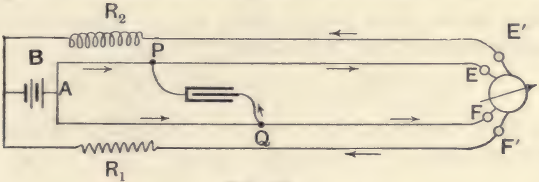

FIG. 185.

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* *Phil. Mag.* Sept. 1887.

raised, the circuits having been otherwise completed previously. When this is the case the following condition is fulfilled:

$$L = C(R'^2_1 - R'^2_2), \quad \text{.........} (89)$$

where R'_1, R'_2, are the resistances from A to P and Q respectively (see Fig. 185).

56. Theory of method by differential galvanometer. We shall suppose the coils of the galvanometer exactly equal for equal currents in magnetic effect on the needle, and that each has the same resistance G. Clearly, for balance with steady currents, the resistance of each circuit must be the same. Denoting therefore by R the resistance in each circuit, exclusive of the battery resistance r and the resistance G of the galvanometer coil, and putting E for the electromotive force of the battery, we have for the steady current γ through either of the galvanometer coils $\gamma(R+G)+2\gamma r = E$, or

$$\gamma = \frac{E}{R+G+2r}. \quad \text{.........} (90)$$

Let PQ be the points at which the terminals of the condenser are attached, R'_1 denote the resistance from A to P, R''_1, that from P to the nearest galvanometer terminal, R'_2, R''_2, the resistances from A to Q, and from Q to the galvanometer, Γ the inductance of each galvanometer coil, M their mutual inductance, $\dot{x}-\dot{z}$ the current from A to P, $\dot{y}+\dot{z}$ that from A to Q, and $\dot{z}$ the current from Q charging the condenser. The equations of currents obtained from the two circuits $AEGE'A$, $AFGF'A$, are (since $R_1 + R'_1 + R''_1 = R_2 + R'_2 + R''_2 = R$)

$$(L+\Gamma)\ddot{x} + M\ddot{y} + (R+G+r)\dot{x} + r\dot{y} - R'_1\dot{z} = E,$$
$$M\ddot{x} + \Gamma\ddot{y} + r\dot{x} + (R+G+r)\dot{y} + R'_2\dot{z} = E.$$

Integrating these from before make to the steady state, putting γ for the steady current, and subtracting, we find

$$(R+G)(x-y) + L\gamma - (R'_1 + R'_2)z = 0. \quad \text{.........} (91)$$

But the final charge of the condenser is $C(R'_1 - R'_2)\gamma$ if C denote its capacity, so that

$$z = C(R'_1 - R'_2)\gamma.$$

Substituting in the last equation, we get

$$x - y = \gamma\frac{C(R'^2_1 - R'^2_2 - L}{R+G}$$

or

$$x - y = E\frac{C(R'^2_1 - R'^2_2) - L}{(R+G)(R+G+2r)} \quad \text{.........} (92)$$

by (90).

If no deflection of the galvanometer needle takes place x must be equal to y, and for this the necessary and sufficient condition is

$$L = C(R'^2_1 - R'^2_2),$$

as already stated above in (89).

With regard to the sensibility of the arrangement it is to be observed that R_1 is given, being the resistance of the coil to be compared, and in general G also is given, so that all that can be done to make the arrangement sensitive is to keep down the value of the resistance additional to R_1.

If the resistance of the battery is negligible and the galvanometer bobbins be a matter of choice, the best arrangement is to make the additional resistance as small as possible, and make $G=R$.

If the galvanometer coils be each shunted by a wire of resistance S the resistance of each galvanometer bobbin will become $GS/(G+S)$, which we denote by G', and this, if the inductance of each shunt is the same, takes the place of G in (92). The integral flow through the coils is then $Sx/(G+S)$ for one, and $Sy/(G+S)$ for the other. Hence the total flow affecting the needle is $S(x-y)/(G+S)$, or $(x-y)G'/G$. But we now have

$$x-y=E\frac{C(R'^2_1-R'^2_2)-L}{(R+G')(R+G'+2r)}. \qquad (93)$$

Hence in order that $(x-y)G'/G$ may be a maximum, we must make $(R+G')(R+G'+2r)/G'$ a minimum. Differentiating with respect to G', we find that the condition for a minimum is

$$G'^2=R(R+2r). \qquad (94)$$

Thus if the galvanometer have a high resistance so that the deflections are small, an improvement can be effected by shunting down each coil of the instrument to an effective resistance given by this equation.

57. Anderson's null method for comparison of self-inductance and capacity. A modification of Maxwell's method which has the advantage of being a perfectly null method, and therefore of permitting a telephone to be used instead of a galvanometer has also been given by Prof. A. Anderson.* The arrangement of resistances is the same as before, but the condenser instead of being placed between A and C is placed between A and a point E on CD (Fig. 186). The galvanometer (or telephone) is supposed included in the part ED of CD, and the resistance, g say, of CE is varied until no deflection of the galvanometer needle is produced by making or breaking the battery circuit.

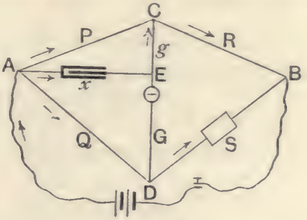

FIG. 186.

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* *Phil. Mag.* April, 1891.

Let the resistance of ED be denoted by G, the currents through the galvanometer (from E to D) and to the condenser by $\dot{y}$, $\dot{z}$, so that the current from E to C is $\dot{z}-\dot{y}$. Thus from the circuits $ACDA$, $CBDA$, by integrating over the interval of variation, and using the value $Q\gamma/(P+Q)$ for $\dot{x}_s$ the steady current in AC, and $CP\dot{x}_s$ for the final charge z of the condenser, we get, if the inductances of the other arms of the bridge are negligible,

$$\left.\begin{aligned}(P+Q)x+(G+g)y&=-\frac{Q-g}{P+Q}CPQ\gamma+Qu,\\(R+S)x-(R+S+G+g)y&=\frac{P\gamma}{P+Q}\{L-CQ(R+S+g)\}+Su.\end{aligned}\right\}\quad\ldots(95)$$

Eliminating x we find

$$y=\gamma\frac{P[C\{RQ+g(Q+S)\}-L]}{(R+S)(G+g)+(R+S+G+g)(P+Q)}.\quad\ldots\ldots\ldots\ldots(96)$$

The value of y is zero if the numerator vanish, that is if

$$L=C\{RQ+g(Q+S)\}.\quad\ldots\ldots\ldots\ldots(97)$$

If $g=0$ we fall back on Maxwell's solution, viz.

$$L=CRQ=CPS.\quad\ldots\ldots\ldots\ldots(98)$$

58. Condition that the method may be null. That this is the necessary condition that the method may be a null one may be seen in the following manner. Whatever be the conductor between A and E the difference of potential between A and E is $P\dot{x}+g(\dot{y}-\dot{z})$, while that between A and D is $Q(\dot{u}-\dot{x}-\dot{z})$. If there is no difference of potential between E and D, $\dot{y}=0$, and we have $P\dot{x}-g\dot{z}=Q(\dot{u}-\dot{x}-\dot{z})$. Integrating from just before the completion of the circuit to any instant during the interval of variation, we find

$$Px-gz=Q(u-x-z).\quad\ldots\ldots\ldots\ldots(99)$$

Also from the branches ECB, DB we get in like manner

$$R(x+z)+gz=S(u-x-z)+L(\dot{u}-\dot{x}-\dot{z}).$$

But by (99) the last equation may be written

$$Rx+(g+R)z-\frac{L}{Q}(P\dot{x}-g\dot{z})=S(u-x-z).\quad\ldots\ldots\ldots\ldots(100)$$

Equation (99) multiplied by S and subtracted from the last equation multiplied by Q gives, since $PS=QR$,

$$\{QR+g(Q+S)\}z-L(P\dot{x}-g\dot{z})=0,$$

and since $P\dot{x}=z/C+g\dot{z}$, this is

$$C\{QR+g(Q+S)\}z-Lz=0.$$

Hence

$$L=C\{QR+g(Q+S)\}.\quad\ldots\ldots\ldots\ldots(101)$$

That, conversely, the difference of potential between E and D is zero if this condition is fulfilled can be seen, as in 18 above, from the consideration that otherwise there would be more than one solution of the problem of flow of electricity in the given network between A and B.

If g be small the main part of L is CQR, if g is made great the main part will be $Cg(Q+S)$. Thus by merely changing the resistance between C and D and its distribution between g and D, a large range of inductances can be measured. Returning to (96), putting for γ its value

$$E/\{r+S(P+R)/(R+S)\},$$

we write the equation in the form

$$y=E\frac{P[C\{RQ+g(Q+S)\}-L]}{\left\{G+g+(R+S+G+g)\dfrac{P}{R}\right\}\{r(R+S)+S(P+R)\}}. \quad \text{...(102)}$$

For the greatest sensitiveness a given change in g, the adjustable resistance, must produce a maximum change in y when y is nearly zero, that is dy/dg must be a maximum when $y=0$. In all practical cases we may neglect r, the resistance of the battery, so that we have

$$\frac{dy}{dg_{y=0}}=\frac{CP(Q+S)E}{\left\{G+g+(R+S+G+g)\dfrac{P}{R}\right\}S(P+R)}.$$

But since
$$\frac{P(Q+S)}{S(P+R)}=\frac{P(Q+S)}{R(Q+S)}=\frac{P}{R},$$

this equation may be written

$$\frac{dy}{dg_{y=0}}=\frac{CE}{\dfrac{R}{P}(G+g)+R+S+G+g}. \quad \text{..................(103)}$$

Hence, in order that the denominator may be small, we must take R and g small and P large, and therefore Q also large.

59. Stroud and Oates' modification. Anderson's method has been modified by Messrs. Stroud and Oates* for the measurement of inductances and the comparison of capacities. The battery is replaced by an A.C. generator. The resistance g in the bridge scheme discussed above is in some cases of very considerable magnitude, and placing it in series with the galvanometer reduces the sensibility. Accordingly the condenser and the resistance g were arranged as would be shown by Fig. 186 if modified as follows: The condenser and the resistance g are connected in series across P between C and A, and one terminal of the generator is connected to the common terminal A' of the condenser and the resistance. Thus the condenser is between C and A', and the resistance between A' and A. The second terminal of the generator was connected to B, and the galvanometer was connected between C and D. The procedure is as before, the resistance g is varied until the

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depression of the battery key after the connection between C and D had been closed by the galvanometer key produced no deflection.

It may be observed that turning Fig. 186 round so that the condenser is placed between B and the point E on the connection between C and D, and the inductive resistance is in AD, in no way alters the expression for L. For we should have as in (101) above

$$L = C\{g(Q+S)+PS\},$$

which, since $PS=QR$, is the same equation as before. It is not difficult to prove that in the arrangement of Stroud and Oates the equation for L is

$$L = C\{g(P+Q)+PS\}. \quad \text{.............................(104)}$$

If this arrangement is represented by a drawing, it will be seen at once that it is in a sense conjugate to that of Anderson, which explains the similarity of the formulae for the two cases; in point of fact one formula can be deduced from the other.

Messrs. Stroud and Oates used an alternating current, with a moving coil galvanometer, the field magnet of which was of laminated soft iron strongly excited by an alternating current supplied by the alternating generator. The result was a large increase of sensibility, which enabled very small inductances to be tested.

60. Rosa and Grover's determination by Anderson's null method. Very accurate measurements have been made by Anderson's method

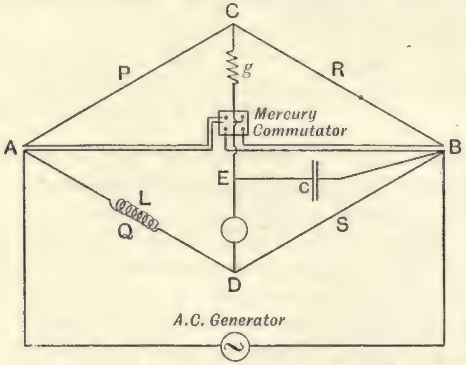

FIG. 187.—Showing commutator for interchanging two arms of the Anderson Bridge.

by Messrs. Rosa and Grover at the Bureau of Standards at Washington [*B.B.S.W.* 1, p. 291]. The arrangement of apparatus is shown in Fig. 187. An alternating current generator was connected to the points A, B, and a mercury commutator enabled the resistances P, R to be interchanged. This enabled errors from slight changes of P and R to be eliminated. These resistances are made equal, but of course there are nearly always differences in residual inductance and in capacity between coils which are equal in resistance, and these also have their effects practically annulled by the interchange of the arms of the bridge, as shown in Fig. 187.

The resistances were made of manganin and were submerged in oil to enable their temperatures to be measured and to prevent heating. Slide wires enabled the resistances g and Q to be adjusted to 0·001 ohm, and for all accurate determinations a new measurement of resistance was made every day.

A vibration galvanometer was employed, that is a galvanometer the needle system of which is capable of oscillating in time with the short period alternating forces applied by the alternating current. It is of course the case that when the conditions for balance of resistances and balance of inductances are satisfied, that no current can pass through the galvanometer at any instant. But if the steady current balance be slightly disturbed, as it is apt to be by heating of the coils, then it is impossible to adjust the resistance g so as to make the galvanometer current zero. All that can be done is first to adjust g so that the needle of the galvanometer has a minimum amplitude of oscillation as the circuit of the battery is made and broken, or the alternating current generator is run, and then to alter one of the resistances (say Q) so that a complete balance is obtained, and the needle remains stationary. This obviates the necessity when an alternating machine is used for returning to the use of a direct current to test whether the steady current balance still holds, and moreover it is possible, at the moment when the vibration range is a minimum, to make sure that the resistance balance is exactly restored. Also, as Messrs. Rosa and Grover point out, the vibration galvanometer takes account of the variation of the resistance of the inductive coil, or of the arms of the bridge, caused by the alternation of the current. This variation, as has already been pointed out, is small for fine wire and low frequencies of alternation.

A vibration galvanometer having a resistance of 200 ohms was used at a frequency of about 110. The curve of sensibility of the galvanometer had two peaks of sensibility, at 110·6 and 120 vibrations per second, and a minimum of sensibility at a frequency of 115. This rapid falling off from sensibility when the frequency varied from that natural to the needle was the chief inconvenience of the instrument.

The whole of the adjustments for balance were made by the vibration galvanometer, starting from a rough direct current balance. A graduated scale was viewed by means of a telescope which received rays reflected from the mirror of the galvanometer. The filament of an incandescent lamp which illuminated the scale was also seen in the telescope. When approximate adjustments of the resistances to balance for steady currents and of g were made the filament appeared somewhat broadened by the vibration of the galvanometer mirror when the alternator was run. Small changes of resistance were then made until the filament and the lines on the scale were quite sharp and clear.

61. Residual inductances and capacities in "non-inductive" coils. The difficulties in making exact determinations of inductances by this method, or by any other, arise from residual inductance in the non-inductive coils used, and the electrostatic capacity of the coil g and of the arms of the bridge. The inductive coils used must of course be placed at considerable distances (of the order of a metre) from one another, and from the arms of the bridge, and the necessary leads, if twisted together to avoid inductance, will have appreciable capacity. This capacity effect of the leads is small in small coils and is obviously relatively much more important in large coils. In the latter case the leads may be kept apart in definite positions for which their inductance can be calculated or determined.

The subject of errors due to residual inductance and to electrostatic capacity is well illustrated by the manner in which Messrs. Rosa and Grover's research on Anderson's method was carried out, and in some instructive remarks which the authors make. As regards residual inductance or capacity in the arms of the bridge and the coil g, they state first that the resistances P and R of the coils were always made equal and were interchanged by a commutator, so as to eliminate the effects of any difference in their resistances, their inductances, or their capacities. This left however the differences between Q and S (apart from the inductance L) to produce an effect. Residual inductance in the non-inductive part of Q would make L too large, and capacity, by counteracting inductance, would make L too small. Inductance or capacity in S would produce an opposite effect to that of the same thing in Q, and so the effects could be balanced if the coils used in Q and S were similar. In Rosa and Grover's experiments S was fixed and Q varied for balance; so that Q contained a number of small resistances which could not, as regards inductance and capacity, be balanced exactly by S.

As they point out, in measuring small inductance coils the capacity effect is small, and the leads should be made as short and placed as close together as is possible with safety; for large coils it is better, as has already been stated, to place the leads so that their inductance can be calculated and their capacity neglected.

The inductance l of the wires joining the condenser to the bridge reduces the capacity C to a slight extent, which is in the ratio of pl to $1/pC$, where $p = \text{frequency} \times 2\pi$; when the leads are close together their capacity forms an addition to C. When currents of high frequency are used with large condenser capacity C the error due to inductance of the leads may be of appreciable amount; with small values of C, the capacity of leads which are twisted together or laid close to one another, may be considerable, and this error should be guarded against.

62. Determination of the electrostatic capacity of an induction coil. The error due to the electrostatic capacity of the induction coil itself may be serious, if it is so constructed that spires in its winding which are in its use at considerable differences of potential are closely adjacent and the frequency is high. We shall prove that if c be the capacity of the coil and L the true inductance, the measured inductance L' is given approximately by the equation

$$L' = L(1 + p^2 cL). \quad \text{.............................}(105)$$

In the very high frequencies involved in the oscillatory discharge of a condenser this effect may lead to a considerable difference between the actual period and that calculated from the value of L.

To obtain an idea of the effect of capacity, consider the current produced by an alternating difference of potential $E_0 \sin pt$, applied to a coil, of resistance R and inductance L, which is joined in parallel with a condenser of capacity c. We suppose the resistance and inductance in the condenser branch to be zero. If the coil current be γ_1, and the condenser current γ, we have [see VIII.]

$$L\dot{\gamma}_1 + R\gamma_1 = E_0 \sin pt, \quad \frac{1}{c}\gamma = pE_0 \cos pt.$$

Now
$$\gamma_1 = \frac{E_0}{\sqrt{R^2 + p^2L^2}} \sin(pt - \theta), \quad [\theta = \tan^{-1}(pL/R)].$$

Hence

$$\gamma_1 + \gamma = E_0 \left\{ \frac{1}{(R^2 + p^2L^2)^{\frac{1}{2}}} \sin(pt - \theta) + cp \cos pt \right\} = E_0 A \sin(pt - \phi),$$

with
$$\phi = \tan^{-1}\left[\frac{p}{R} \{L - c(R^2 + p^2L^2)\} \right].$$

After reduction, as the reader may verify, we obtain

$$\gamma_1 + \gamma = \frac{E_0}{(R'^2 + p^2L'^2)^{\frac{1}{2}}} \sin(pt - \phi), \quad [\phi = \tan^{-1}(pL'/R')],$$

where
$$R' = \frac{R}{(1 - p^2cL)^2 + p^2c^2R^2}, \quad L' = \frac{L - c(R^2 + p^2L^2)}{(1 - p^2cL)^2 + p^2c^2R^2}.$$

Thus the effective inductance and resistance are L', R', and the impedance is $\sqrt{R'^2 + p^2L'^2}$. It will be noticed that if c be small and p be not extremely great, we have approximately

$$R' = R(1 + 2cp^2L), \quad L' = L(1 + cp^2L) - cR^2,$$

so that if R be small, $L' = L(1 + cp^2L)$, the value quoted in (105) above.

It is not difficult to find (see Rosa and Grover's paper, *loc. cit.*) the effect of placing an inductionless resistance r, (1) in series, (2) in parallel with the condenser [capacity C] of Fig. 187.

In (1) $L = CS\left(g\frac{P+R}{R} + P\right) - p^2 r_1^2 LC^2.$

In (2) the effect is zero.

It is to be observed that in an actual coil the capacity is distributed, and on that account the result here stated does not accurately represent the actual case.

For an actual induction coil carrying alternating currents any theoretical determination of the effect of distributed capacity is a practical impossibility, and this introduces a serious difficulty in the discussion of electrical oscillations in which that capacity can play a sensible part.

[It may be observed that discussions of impedance such as that above may be carried out very easily if we express the alternating difference of potential by E_0e^{-ipt}, and the impedance of the coil by $R+ipL$. Then we have two parallel impedances, $R+ipL$ in the coil and $1/ipc$ in the condenser branch. Thus if the combined impedance is I we have, adding the reciprocals of the impedances, just as we add the conductances of two parallel conductors,

$$\frac{1}{I}=\frac{1}{R+ipL}+ipc=\frac{1-p^2cL+ipcR}{R+ipL},$$

that is

$$I=\frac{R+ip\{L-c(R^2+p^2L^2)\}}{(1-p^2cL)^2+p^2c^2R^2}=R'+iL'.$$

Thus we get at once the results already found above,

$$R'=\frac{R}{(1-p^2cL)^2+p^2c^2R^2},\quad L'=\frac{L-c(R^2+p^2L^2)}{(1-p^2cL)^2+p^2c^2R^2}.\ ]$$

The value of c can be found by measuring L' at the different frequencies $p/2\pi$ and $p'/2\pi$. For one of the induction coils used by Rosa and Grover which had an inductance of 1 henry, the capacity was 1×10^{-4} microfarad. For this the correction term p^2cL was ·00005, not a large error, but one which with ten times the frequency would be ·005.

The capacity of a coil may be kept low by winding it in a deep channel so that there are many layers each of a small number of turns; and of course large induction coils have their secondaries wound in a number of such deep sections to avoid risk of breakdown of the coil by internal sparking.

This mode of winding, however, though it does not lessen the mutual inductance between primary and secondary, keeps down to some extent the inductance. When large self-inductance is required it is advisable to choose the form which gives maximum inductance and determine the capacity in the manner explained above.

There is also the effect of absorption in the condenser on the measured value of L. The current is made to lag in phase to a certain extent, so that the difference between its phase and that of the electromotive force is somewhat less than 90°. For good mica condensers the effect of absorption was found by Messrs. Rosa and Grover to be inappreciable.

The investigation of the effect of capacity given above is of importance in connection with the construction of coils for use as shunts for alternating current ampere-meters. It has been seen that a condenser of capacity c across the terminals of a coil of resistance R gives a reduction of inductance of amount cR^2, or for the same capacity distributed between two parallel wires of $\frac{1}{3}cR^2$. This is not affected appreciably by altering the frequency up to 3000 cycles per second, a frequency up to which the so-called "skin effect" is not of great amount. Keeping the effective inductance low gives a small phase angle. For example, for a 100-ohm coil with a condenser across its terminals the effective inductance would be $L - cR^2$. That this may be zero we should have $c = L/R^2$. Thus if it is found by calculation or experiment that L is 2 microhenrys the condenser which would neutralize L would have a capacity of $2 \times 10^3/100^2 \times 10^{18} = 2 \times 10^{-19}$, that is 2×10^{-4} microfarad.

The capacity alters with the frequency if that becomes very high, and the variation is as the square of the frequency and is more important for coils of large resistance than for small, as the formulae show. For a 100-ohm coil the change between the frequencies 100 and 3000 is not more than 10 per cent. Thus if the capacity for a 100-ohm coil were 3×10^{-4} microfarad, the change of effective inductance would not exceed, in absolute units,

$$3 \times 10^{-19} \times 10^4 \times 10^{18} \times 10^{-1} = 3 \times 10^2,$$

that is 0·3 microhenry. For coils of smaller resistance the effect is inappreciable.

If there is sensible absorption in the dielectric there is an apparent leakage between the turns of the coil, and the resistance suffers an apparent diminution, of amount proportional to the frequency of the alternation, to a phase angle which arises from the absorption and to the resistance itself. The apparent resistance R' may in fact be written

$$R' = R(1 - \tfrac{1}{3}pcR \tan \theta).$$

This formula has been tested by experiments carried out by Curtis and Grover at the U.S. Bureau of Standards on a manganin 1000-ohm coil, covered with shellacked silk, and found to give results fairly concordant with fact. The effects are not large. The phase angle was 2°·6, 1°·5, 1°·2 for frequencies 100, 1500, 2700.

Reference should be made for further information to Messrs. Curtis and Grover's paper, *B.B.S.W.* 8, p. 495.

63. Estimation of error due to residual inductance and capacity. The effects of small residual inductances and of capacities were fully investigated, and it was found that they could be summed up in two correction terms, α, β, as in the equation

$$L = L_0 + \alpha - \beta. \quad \text{..............................(106)}$$

The composition of these terms is as follows. Let l_1, l_2, l_3, l_4, l_5, be the residual inductances in the four arms AC, AD, CB, CB of the bridge, and the resistance g [see Fig. 187]. Then an investigation of the kind explained in Chapter XV. below gives

$$L = CS\left\{g\frac{P+R}{R}+P\right\}+\frac{1}{R}(l_1S-l_2R-l_3Q+l_4P)$$
$$+\frac{p^2C}{R}\{gl_4(l_1+l_3)+S(l_1l_3+l_3l_5+l_5l_1)+Rl_4(l_1+l_5)+Pl_4(l_3+l_5)\}. \quad (107)$$

When $P=Q=R=S$ this of course is considerably simplified.

If $L_0=CS\left\{g\frac{P+R}{R}+P\right\}$, the equation can be written in the form of (106) above. We can, unless the frequency is very high or the residual inductances great, neglect the term β. Further, multiplying and dividing by p, we have

$$\alpha=\frac{Q}{p}(\phi_1-\phi_2-\phi_3+\phi_4), \quad \dots\dots\dots\dots\dots\dots\dots(107')$$

where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are the phase angles of the currents in the arms of the bridge. For good mica condensers the effect of absorption was found by Rosa and Grover to be inappreciable.

It will be seen that the effect of small residual virtual inductances l_1, l_2, l_3, l_4, l_5, in the four arms of the bridge and the resistance g (where it is to be understood that the inductances proper and the small capacities are combined) can be summed up in the two correction terms α, β, in (106).

Here
$$\left.\begin{aligned} L_0&=CS\left\{g\frac{P+R}{R}+P\right\},\\ \alpha&=\frac{Q}{p}\left(\frac{pl_1}{P}-\frac{pl_2}{Q}-\frac{pl_3}{R}+\frac{pl_4}{S}\right)=\frac{Q}{p}(\phi_1-\phi_2-\phi_3+\phi_4),\\ \beta&=\frac{p^2C}{R}\{gl_4(l_1+l_3)+S(l_1l_3+l_3l_5+l_5l_1)+Rl_4(l_1+l_5)+Pl_4(l_3+l_5)\}.\end{aligned}\right\} \quad (108)$$

The expression $PS-RQ$ is not now zero, but has a value which will be given in the chapter referred to above.

Unless the frequency is very great or the inductances l_1, l_2, ... large, the value of β may be neglected. The quantities ϕ_1, ϕ_2, ϕ_3, ϕ_4 are the phase angles of the currents in the arms of the bridge due to combined inductance and capacity in the resistances P, Q, R, S (leaving L of course out of account). If they are all equal we have $\alpha=0$. The amounts of the correcting terms for some suggested cases will be computed. Here however we give the results of Rosa and Grover's determinations for four coils given as of 100 millihenrys each.

RESULTS OF MEASUREMENTS OF FOUR INDUCTANCE COILS OF 100 MILLIHENRYS EACH. COILS MEASURED SINGLY AND IN SERIES IN GROUPS OF TWO.

[$P = R = 249{\cdot}994$ ohms. $S = 280{\cdot}113$ ohms.]

1	2	3	4	5	6	7	8	9	10	11	12
No.	Coils.	r position 1.	r position 2.	r mean corrected.	$2r+P$	Temp. of condenser.	C	L_0 millihenrys.	$L=L_0+a$ millihenrys.	L_s sums of single values	$\Delta = L_s - L$ microhenrys.
1	A	52·638	52·598	52·628	355·250	18°·7	1.00518	100·025	100·024	....	....
2	B	52·578	52·543	52·570	355·135	18·75	1·00518	99·993	99·985	....	....
3	H	52·511	52·476	52·504	355·001	18·75	1·00518	99·955	99·954	....	....
4	C	57·021	56·987	57·014	364·022	18·8	1·00517	102·494	102·486	....	....
5	C + B	234·605	234·529	234·567	719·128	18·85	1·00516	202·477	202·472	202·471	− 1
6	C + A	234·673	234·604	234·638	719·271	18·85	1·00516	202·513	202·504	202·510	+ 6
7	H + A	230·151	230·082	230·116	710·227	18·9	1·00515	199·969	199·971	199·978	+ 7
8	H + B	230·085	230·015	230·050	710·094	18·9	1·00515	199·932	199·930	199·939	+ 9
											mean = + 5
9	A	52·651	52·613	52·642	355·278	18·9	1·00515	100·031	100·023	....	....
10	H	52·507	52·471	52·499	354·992	18·95	1·00514	99·950	99·949	....	....
11	B	52·573	52·534	52·564	355·121	18·95	1·00514	99·986	99·983	....	....
12	C	57·027	56·991	57·019	364·032	19·0	1·00514	102·494	102·485	....	....
13	C + H	234·526	234·465	234·500	718·995	19·0	1·00514	202·435	202·434	202·434	0
14	C + A	234·680	234·608	234·644	719·282	19·0	1·00514	202·516	202·508	202·508	0
15	B + A	230·219	230·146	230·182	710·359	19·0	1·00514	200·004	199·999	200·006	+ 7
16	B + H	230·083	230·015	230·049	710·092	19·0	1·00514	199·928	199·927	199·932	+ 5
											mean + 3

64. Comparison of mutual inductance and capacity : Carey Foster's method. A method of comparing a coefficient of mutual induction with the capacity of a condenser has been given by Prof. Carey Foster.* It is based on the following considerations. Let the two coils C_1, C_2, the mutual inductance for which is required, be given in position as in Fig. 188, and be joined, one, C_1, through a battery, a coil of resistance R_1, a make and break key K, and the other, C_2, as a secondary circuit through a galvanometer G. Then if R_2 be the resistance of the secondary circuit, M the mutual inductance of the two coils, the whole quantity of electricity which flows through the secondary, when a steady current of strength γ is produced or annulled in the primary, is $M\gamma/R_2$.

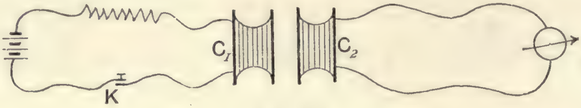

FIG. 188.

Again if the resistance coil in the circuit of C_1 have its terminals connected to a condenser of capacity C (Fig. 189) and the primary circuit be made or broken, the quantity of electricity which traverses the galvanometer G is $CR_1\gamma$. Thus if the same deflection as before is obtained we have

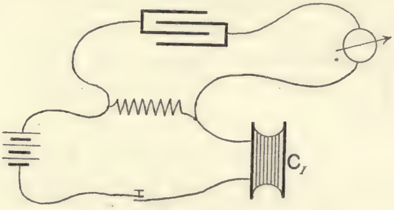

FIG. 189.

$$M = CR_1R_2. \quad \text{.............................(109)}$$

If however deflections are obtained indicating currents γ_1, γ_2, in the two cases, then

$$M = CR_1R_2\frac{\gamma_1}{\gamma_2}. \quad \text{...........................(110)}$$

Now let a combination of these two arrangements be made as shown in Fig. 190, including a resistance box in the secondary circuit to enable the resistance R_2 of that circuit between the points A and E to be varied at pleasure. Then let the resistances R_1 (in the primary between the terminals of the condenser), and R_2 be varied until on making or breaking the battery circuit no deflection is produced. When this is the case the integral flow through the galvanometer due to the charging of the condenser (that is the charge of the condenser) is exactly equal and opposite to that due to the induction current in the secondary circuit. Thus noticing that the inductance in C_2 cannot effect the integral flow through it we see that $CR_1\gamma = M\gamma/R_2$, or

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* *Phil. Mag.* Feb. 1887.

$$M = CR_1R_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(111)$$

65. Most sensitive arrangement for Carey Foster's method. We can easily find the most sensitive arrangement for the experiment. In the first place it is to be noticed that the resistance (R'_1 say) other than R_1 in the primary circuit depends on the primary coil and the battery and is to be taken as fixed. We shall regard the galvanometer bobbin (1) as given, (2) as a matter of choice from similar bobbins of different resistances.

Let us suppose that the potential at A is not equal to that at E. Then putting $\dot{u}$, $\dot{x}$ for the currents in the primary and secondary, $\dot{y}$ for the current through the galvanometer, Γ for the inductance and G for the resistance of the galvanometer bobbin, we get from the circuit AC_2EA (Fig. 190) the equation $L\ddot{x} + M\ddot{u} + R_2\dot{x} + \Gamma\ddot{y} + G\dot{y} = 0$. This gives the integral equation

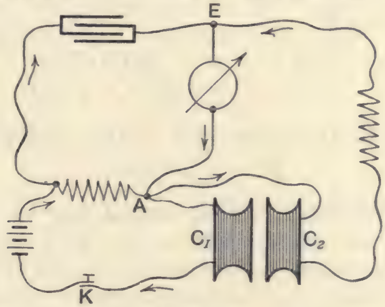

FIG. 190.

$$R_2x + Gy = M\gamma.$$

Further we have for the total charge of the condenser

$$x - y = CR_1\gamma.$$

Solving for y from these we find

$$y = \frac{(M - CR_1R_2)\gamma}{G + R_2},$$

or, since $\gamma = E/(R_1 + R'_1)$, where E is the electromotive force of the battery,

$$y = E\frac{M - CR_1R_2}{(G + R_2)(R_1 + R'_1)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(112)$$

which gives the same condition as before that y may be zero.

In order that y may be a maximum the value of the denominator must be a minimum. Calling it D, and noting that only R_1, R_2 vary, and are connected by the relation $R_1 R_2 = eM/C$, where e is a small quantity, we find

$$\frac{dD}{dR_1} = G + R_2 + (R_1 + R'_1)\frac{dR_2}{dR_1} = 0, \quad R_1\frac{dR_2}{dR_1} + R_2 = 0.$$

Eliminating dR_2/dR_1 we get, as the required condition of maximum sensitiveness with a given galvanometer,

$$\frac{R_2}{R_1} = \frac{G}{R'_1}. \qquad \text{..............................(113)}$$

If the galvanometer bobbin is also at our disposal we have, instead of the value of D found above, to use

$$D' = D/\sqrt{G} = (R_1 + R'_1)(\sqrt{G} + R_2/\sqrt{G}).$$

This gives in addition to (113)

$$\frac{dD'}{dG} = \tfrac{1}{2}(R_1 + R'_1)\left(\frac{1}{\sqrt{G}} - \frac{R_2}{G^{\frac{3}{2}}}\right) = 0,$$

that is,

$$G = R_2. \qquad \text{..(114)}$$

Thus we have in the latter case as the conditions for maximum sensibility,

$$R'_1 = R_1, \quad R_2 = G. \qquad \text{................................(115)}$$

66. Condition that Carey Foster's method may be "null." If it can be arranged to maintain the two points A, E *always* at the same potential, we may use a telephone instead of a galvanometer as observing instrument. To find the necessary condition consider the secondary circuit AC_2EA. Since there is no current between A and E, we have

$$L\ddot{x} + M\ddot{u} + R_2\dot{x} = 0.$$

But if $\dot{z}$ be the current passing the condenser, that is, through the resistance R_1, at this instant we must have (Fig. 190)

$$\dot{u} + \dot{x} = \dot{z},$$

and so $-\dot{x}$ is the current which charges the condenser. This gives

$$\ddot{x} = \ddot{z} - \ddot{u},$$

so that the former equation becomes

$$(M - L)\ddot{u} + L\ddot{z} + R_2\dot{x} = 0,$$

or

$$\dot{x} = -\frac{1}{R_2}\{(M - L)\ddot{u} + L\ddot{z}\}.$$

The charge of the condenser is then $CR_1\dot{z}$, so that

$$CR_1\dot{z} = -\int_0^t \dot{x}\,dt = \frac{1}{R_2}\{(M - L)\dot{u} + L\dot{z}\},$$

or

$$(M - L)\dot{u} = (CR_1R_2 - L)\dot{z}.$$

But in any case in which there has been no integral flow through the galvanometer during the rising of the current from zero to its steady value we have seen that $CR_1R_2 = M$. Thus the equation just found becomes

$$(M - L)(\dot{u} - \dot{z}) = 0,$$

which asserts that either $M = L$, or $\dot{u} = \dot{z}$. The latter is only true when the current $\dot{u}$ in the battery has attained its steady value γ. If however $M = L$ it will be possible to make the difference of potential between A and E always zero and to employ a telephone.

67. Practical example of Carey Foster's method. The following results obtained in Prof. Carey Foster's laboratory by Mr. F. Womack illustrate the method. A small induction coil was used with fixed primary and coaxial secondary capable of being moved in the direction of the axis so as to alter the mutual inductance of the coils. The dimensions, etc., of the coils were :—*Primary*, length 11·5 cm, mean radius 2 cm, wire 1·65 ohms of No. 20 B.W.G: *Secondary*, length 10·4 cm, inside radius 2·55 cm, outside radius 3·53 cm, wire 194 ohms of No. 30 B.W.G. Two Grove's cells were used and a condenser of 4·926 microfarads capacity, with a galvanometer of about 135 ohms resistance.

R_1	R_2 Res. of Secondary + Res. from Box.	$R_1R_2 = M/C$.
15 ohms.	441 ohms.	6165×10^{18}
14 ,,	441 ,,	6174
13 ,,	476 ,,	6188
12 ,,	516 ,,	6192
11 ,,	561 ,,	6171
10 ,,	617 ,,	6170
9 ,,	684 ,,	6156
8 ,,	770 ,,	6160
7 ,,	882 ,,	6174
6 ,,	1029 ,,	6174
		Mean $6172{\cdot}4 \times 10^{18}$

Thus in c.g.s. units

$$M = 4{\cdot}926 \times 10^{-15} \times 6172 \times 10^{18} = 3{\cdot}0403 \times 10^7.$$

The total resistance in the battery circuit was about $1{\cdot}65 + {\cdot}6 + R_1$, or $R'_1 = 2{\cdot}25$. Thus for greatest sensibility

$$R_2/R_1 = G/R'_1 = 135/2{\cdot}25 = 60.$$

Some very concordant results were also obtained with a 7-inch spark induction coil. The resistance of the primary was ·278 ohm ; of the secondary 7394 ohms. One Grove's cell was used with the same condenser as before and a galvanometer of resistance 135·6 ohms.

R_1	R_2	R_1R_2
27 ohms.	8944 ohms.	$2{\cdot}415 \times 10^{23}$ c.g.s.
28 ,,	8640 ,,	2·419
29 ,,	8334 ,,	2·417
30 ,,	8044 ,,	2·413
31 ,,	7784 ,,	2·413
32 ,,	7544 ,,	2·414
		Mean $2{\cdot}415 \times 10^{23}$ c.g.s.

Thus $M = 4{\cdot}926 \times 10^{-15} \times 2{\cdot}415 \times 10^{23} = 1{\cdot}1896 \times 10^{9}$,

in c.g.s. units, or 1·1896 henry.

CHAPTER XV.

ABSOLUTE MEASUREMENT OF RESISTANCE.

1. Importance of realized standards of resistance. In order that all the results of electrical experiments may be expressed in absolute units, realized absolute units of resistance must be available. An electric current can be measured at any time in absolute units, as we have seen, by means of a proper standard galvanometer or current balance. When the absolute value, R, of the resistance of a coil of wire is known, a difference of potential expressed by any chosen number of absolute units can be produced by causing a current of the proper strength, γ, to flow through the wire. If the wire is not the seat of any electromotive force, the difference of potential between two points in the wire, close to the ends, is γR. By this mode of realizing differences of potential the electromotive forces of voltaic cells have been determined; and such cells can be used in their turn as practical standards for the comparison of differences of potential. A realized standard of resistance is thus of fundamental importance in absolute electrical measurement.

2. Absolute measurement of resistance: methods. Various methods for the absolute measurement of resistances have been devised, and a few of these most suited to give exact results have been carried out with great care and experimental skill by several experimenters. We give here a general account of such investigations, going however into full detail regarding only one or two of the more recent, and, on account of the accumulation of experience, presumably the more exact of them.

The methods may be classed in three divisions: I. Those in which electromagnetic induction, of which the amount can be calculated, is employed to generate a current in the conductor the resistance of which is to be determined. The strength of the current depends on this resistance, and is measured directly or indirectly so that it enables the resistance to be found. II. Those based on Lorenz's method, in which a continuous difference of potential between the terminals of the given conductor is produced by electromagnetic induction, and is balanced by a difference of potential independently produced by a current γ flowing in the conductor. III. Joule's method, in which the rate, $\gamma^2 R$, of generation of heat produced by a measured current γ in the conductor is determined, and the resistance deduced by dividing by γ^2.

3. Kirchhoff's method. The first method of type I. which we describe is that due to Kirchhoff.* Two coils, C_1, C_2, between which there is a mutual inductance, M, are joined up, as shown diagrammatically in Fig. 191, with a battery and galvanometer, and the resistance R to be determined. The steady current deflection of the needle is first observed. C_1 is then removed from the position in which the mutual inductance is M, to one in which the mutual inductance is zero, and the first throw of the galvanometer is noted (together with the succeeding deflections to enable a correction for damping to be applied). If x be the total induction-flow through the galvanometer, $\dot{x}_s$ the steady current in G, and the resistances P, Q, of AC_1B, BC_2A, respectively, be each great in comparison with R, we have very approximately, as will be shown below,

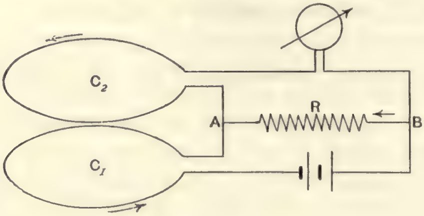

FIG. 191.

$$R = M\frac{\dot{x}_s}{x}. \qquad \text{.......................................(1)}$$

If the galvanometer deflections for steady currents follow the tangent law, and θ_1 be the deflection produced by the steady current, θ_2 the induction throw, corrected for damping and torsion of the fibre if it exists, and T the complete period of oscillation of the needle,

$$\frac{\dot{x}_s}{x} = \frac{\pi}{T}\frac{\tan\theta_1}{\sin\frac{1}{2}\theta_2},$$

so that

$$R = M\frac{\pi\tan\theta_1}{T\sin\frac{1}{2}\theta_2}. \qquad \text{.......................................(2)}$$

If two different galvanometers are used, one of constant G_1 to measure the steady current, and a ballistic galvanometer, of constant G_2, to measure the transient current, and H, H', be the values of the earth's horizontal components at their respective needles, then instead of (2) we have

$$R = M\frac{H}{H'}\frac{G_2}{G_1}\frac{\pi\tan\theta_1}{T\sin\frac{1}{2}\theta_2}. \qquad \text{...............................(2')}$$

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* *Pogg. Ann.* 76 (1849).

4. Theory of Kirchhoff's method. To prove equation (1) let $\dot{u}$, $\dot{x}$, be the current in the battery and in the galvanometer at any instant during the change of the inductance, and L_1, L_2, the self-inductances of the two circuits AC_1BA, ABC_2A, R being supposed devoid of self-inductance. Then these circuits give

$$\left.\begin{aligned} L_1\ddot{u} + M\ddot{x} + (P+R)\dot{u} - R\dot{x} = E, \\ L_2\ddot{x} + M\ddot{u} + (Q+R)\dot{x} - R\dot{u} = 0, \end{aligned}\right\} \quad \text{..........................(3)}$$

where E is the electromotive force of the battery.

Integrating these equations over the (very short) interval τ of change of the mutual inductance from M to 0, we get

$$\left.\begin{aligned} -M\dot{x}_s + (P+R)u - Rx = \int_0^\tau E\,dt = 0, \\ -M\dot{u}_s + (Q+R)x - Ru = 0. \end{aligned}\right\} \quad \text{....................(4)}$$

But when the currents are steady the second of (3) is

$$(Q+R)\dot{x}_s - R\dot{u}_s = 0.$$

Eliminating u between the two equations of (4), and putting $\dot{u}_s = \dot{x}_s(Q+R)/R$, as given by the last equation, we find after reduction

$$\begin{aligned} \frac{x}{\dot{x}_s} &= \frac{M}{R}\,\frac{(P+R)(Q+R)+R^2}{(P+R)(Q+R)-R^2} \\ &= \frac{M}{R}\left\{1 + \frac{2R^2}{(P+R)(Q+R)} + \text{etc.}\right\}. \quad \text{.................(5)} \end{aligned}$$

If P, Q, be each great in comparison with R, this gives (1). Equation (2) follows by the theory of the ballistic galvanometer.

This investigation is practically Maxwell's version of the process followed originally by Kirchhoff. The result may however be obtained somewhat more directly as follows. When M is annulled an integral electromotive force of amount $M\dot{u}_s$ acts in C_2, and another of amount $M\dot{x}_s$ in C_1. The induction-flow due to each through the galvanometer has the same direction, since, on account of the opposite signs of the inductions through C_1, C_2, the currents induced in them are in opposite directions round these coils. The flow through the galvanometer due to $M\dot{u}_s$ is

$$\frac{M\dot{u}_s}{Q + \dfrac{PR}{P+R}} = \frac{M\dfrac{Q+R}{R}\dot{x}_s}{Q + \dfrac{PR}{P+R}} = \frac{M}{R}\,\frac{(P+R)(Q+R)}{Q(P+R)+PR}\dot{x}_s.$$

That due to $M\dot{x}_s$ is

$$\frac{M\dot{x}_s}{P + \dfrac{QR}{Q+R}}\,\frac{R}{Q+R} = \frac{M}{R}\,\frac{R^2}{P(Q+R)+QR}\dot{x}_s.$$

Adding these we get for the total flow through the galvanometer

$$x=\frac{M}{R}\frac{(P+R)(Q+R)+R^2}{(P+R)(Q+R)-R^2}\dot{x}_s,$$

which agrees with (5).

5. Glazebrook's experiments. Determinations by this method have been made by Rowland at Baltimore and Glazebrook at Cambridge. In both sets of experiments the arrangement of coils was not disturbed; but the induction-flow was produced by the simple expedient of reversing the current in the coil C_1. Rowland used a special ballistic galvanometer to measure the transient current, and a comparison of its constant with that of the galvanometer used for the steady current gave the necessary data for calculating $\dot{x}_s/x$.

In Glazebrook's determination,* however, the same galvanometer was used for the measurement of both transient and steady currents, being shunted for the latter purpose so that only a fraction h of the current $\dot{x}_s$ produced the deflection θ_1 of the needle. Thus instead of (2) the formula of calculation

$$R=2M\frac{\pi\tan\theta_1}{T\sin\frac{1}{2}\theta_2}\frac{1}{h} \qquad\qquad (6)$$

was applied, the deflections of course being corrected for damping, etc. The factor 2 is introduced on the right-hand side as the current was reversed, and therefore the induction changed by $2M$.

The following are the particulars of the coils used by Glazebrook, which were wound with great care by Professor Chrystal for a similar investigation. The two coils are distinguished as A and B. They were wound with well-insulated copper wire.

	A	B	Mean.
Mean radius in cm (a)	25·753	25·766	25·760
Axial breadth of section ($2b$)	1·896	1·899	1·897
Radial depth of section ($2d$)	1·92	1·90	1·91
Number of turns of wire	797	791	794
Resistance (approx.) in B.A. units	84	83	167/2

The positions of the mean planes were estimated from the dimensions of the ring channels in which the wire was wound, and any doubt as to the exact positions in these channels was eliminated by reversing the bobbins relatively to the distance pieces between them.

The galvanometer used was an instrument also specially wound by Professor Chrystal. It consisted of two coils about 4 inches in diameter and 23/32 of an inch apart. These coils were movable about a vertical axis round a graduated circle, and could be fixed in the magnetic meridian.

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* *Phil. Trans. R.S.* 1883.

The needle was of hard steel, and 1·5 cm in length, and weighed ·708 gramme. It was suspended in a stirrup of brass on which was fixed the mirror, and a projecting stem of brass, on which brass weights were screwed to increase the period. The whole weighed 6·6 grammes, and was suspended by three fibres of silk 60 cm long.

The scale was of paper divided to millimetres, and compared with a standard scale.

Each experiment made included eight observations of throw, and two of steady current deflection, and each set of experiments consisted of four, one for each of the four positions in which the pair of coils could be placed by reversing them without changing the distance between their centres. Three such sets were made for the distance 15·019 cm of mean planes. These gave as a result

$$1 \text{ B.A. unit} = \cdot 98598 \times 10^9 \text{ cm per second.}$$

[By a "B.A. unit" is meant the value of the ohm as determined by the British Association Committee in 1863, using the method of the revolving coil. See 16 below.]

Three series of experiments were afterwards made in like manner for three different distances of mean planes 15·019 cm, 18·252 cm, 26·692 cm.

Different batteries were used so that the currents through the coil were varied. The mean result obtained was

$$R = 158 \times 10^9 \text{ c.g.s.}$$

As a precaution when the conductor, the resistance of which is to be determined, is a coil of copper wire, it is necessary lest the result should be affected by variation of temperature to make frequent comparisons of the resistance of the coil with that of a platinum, silver or German silver standard.

Expressed in B.A. units, R was found by such a comparison with B.A. standards to be 160·520 at 12° C., and the results reduced to this temperature for comparison gave for the B.A. unit the following values:

Series A. 4 sets.

$$\cdot 98633 \times 10^9 \text{ c.g.s.}$$

Series B. 2 sets.

$$\cdot 98558 \times 10^9 \text{ c.g.s.}$$

Series C. 3 sets.

$$\cdot 98676 \times 10^9 \text{ c.g.s.}$$

Including the preliminary results, with half-weights given to them, the whole investigation gave

$$1 \text{ B.A. unit} = \cdot 986271 \times 10^9 \text{ c.g.s.}$$

Glazebrook made a redetermination by this method of the value of the B.A. unit, and gave * as the mean of all his results

$$1 \text{ B.A. unit} = \cdot 98665 \times 10^9 \text{ c.g.s.}$$

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* *B.A. Report*, 1890.

6. Accuracy of method. In this method, apart from observations of galvanometer deflections, accuracy depends on the exact determination of M, which is a linear quantity. The coils used have had generally the same radius, and the effect of errors in the measurement of their radii and distances apart were estimated as follows by the late Lord Rayleigh.* If we denote the mean radius of the coils (supposed the same in both) by a, and the distance apart of their mean planes by b, and $(a/M)dM/da$, $(b/M)dM/db$, by λ, μ,

$$\lambda+\mu=1$$

since M is linear, and $$\frac{dM}{M}=\lambda\frac{da}{a}+\mu\frac{db}{b},$$

which enables the effects of the errors da/a, db/b, to be estimated.

The expression for M in terms of a and b is given by (32) at p. 192 above, and the known values (see Appendix) of $M/(4\pi\sqrt{aa'})$ for different values of $\gamma[=\sin^{-1}\{2\sqrt{aa'}/\sqrt{(a+a')^2+b^2}\}]$, enable those of λ and μ to be found. It is clear that since M increases as b diminishes, and *vice versâ*, μ must always be negative; λ must therefore be always greater than unity.

If b be great in comparison with a it is clear that M will vary as a^4/b^3, and therefore $\lambda=4$, $\mu=-3$. This is a very unfavourable case, as then errors in a and b are unduly multiplied in M.

Again, if b be small, it is clear that μ is nearly zero, and this may be verified by differentiating the approximate expression

$$4\pi a\log(8a/b-2).$$

Still any error db in b may, if b is small, be comparable with b itself, and thus, although μ may be small, $\mu\, db/b$ may be sensible. Further, the correction for cross-section is of greater relative importance in this case; and thus for two reasons it is preferable to keep b of moderate value. Lord Rayleigh gives the following table for intermediate values of b:

γ	$b/2a$	λ	μ	M
60°	·577	2·61	−1·61	·316
70°	·364	2·18	−1·18	·597
75°	·268	1·98	−0·98	·829
80°	·176	1·76	−0·76	1·186

This table shows that for equal values of da/a, and db/b, the numerical values of the errors in M are roughly as 2 to 1.

With regard to the current measurements, it is to be noticed that the method does not involve any determinations of distances of scales from mirrors, except as a means of correcting the approximate value of $\tan\theta_1/\sin\frac{1}{2}\theta_2$ given by the ratio of the deflections as read off in scale divisions (see 5 above).

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* *Phil. Mag.* Nov. 1882; *Collected Papers*, ii. p. 134.

The late Lord Rayleigh was (*loc. cit.*) of opinion that by using still larger coils than those employed by Glazebrook, with the same number of turns of wire, the accuracy of experiments by this method might probably be still further increased. The greater value of M, and the greater conductance of the wire, would give greater sensibility, and the linear measurements could be more exactly made. A relatively small value of the radial breadth of section, the chief element in the correction of cross-section, might then also be used.

7. Rowland's experiments. The induction coils used in Rowland's experiments * were made by winding 154 turns of fine silk-covered wire in each of three accurately turned brass bobbins (A, B, C). Their mean radii were respectively 13·710 cm, 13·690 cm, 13·720 cm, and each had a radial depth of ·90 cm and an axial width of ·84 cm.

These bobbins were used two at a time, and were made with carefully ground ends so that they could be fitted end to end with their axes in line. Each pair could of course be placed in four positions relative to one another without altering the distance between their mean planes, and as all four were used in each case, the slightest uncertainty as to the exact distance of the coils apart was eliminated by combination of the results. The distance of the bobbins was measured for each position by means of a cathetometer applied at three different points in the circumference.

The values of M were calculated by the elliptic integral formula already given, and a correction was made for the cross-section of each coil according to the formula at p. 434 above [see also XIII. 31]. The results were as follows :

	A and B.	A and C.	B and C.
Mean distance apart -	6·534 cm	9·574 cm	11·471 cm
Value of M - - -	3775500 cm	2561974 cm	2051320 cm

The ballistic galvanometer was composed of two coils containing between them 1790 turns of No. 22 silk-covered copper wire, wound on a brass cylinder 8·2 cm long, and 11·6 cm in diameter, in rectangular grooves 3 cm deep and 2·5 cm wide. A saw-cut along the cylinder prevented the circulation of induction currents round it. The coil was mounted so that it could be turned about a vertical axis to any required azimuth, and its position determined by a horizontal circle below. This circle was finely graduated, and was read to 30″ by a couple of verniers.

* Silliman's *American Journal*, 15 (1878).

Two different needles were used in each, consisting of two thin laminae of hard steel attached to the two sides of a square piece of wood so that the magnetic axis could not vary in position. One needle was 1·27 cm long, and had a period of 7·8 seconds ; the length of the other was 1·20 cm, and its period 11·5 seconds. The moment of inertia of each was augmented by brass weights carried by wires extending in the direction of the magnetic axis. Each needle was suspended by three single fibres 43 cm long. The torsion of these fibres was eliminated from the result, as will be seen below, except as regarded the period of vibration, and for this an allowance was made.

A brass bar, passing through the opening below the needle, carried a small telescope by which the mirror was observed when the constant of the coil was compared with that of another.

The constant, G_2, of the coil was determined first by calculation from its dimensions, and by comparison with that of the large double coil of an electrodynamometer constructed on Helmholtz's plan (p. 217 above). This coil had a constant of 78·37 by calculation. In the comparison the ballistic galvanometer was used with its graduated horizontal circle as a sine galvanometer.

After a comparison had been made the instruments were interchanged, and the comparison repeated to eliminate the ratio of the values of H at the two places.

Seven determinations gave as a mean result $G_2 = 1833{\cdot}67$, with a probable error of $\pm{\cdot}09$, and calculation gave $G_2 = 1832{\cdot}24$. The former result, being probably considerably the more accurate, was given double weight, and a mean then taken with the latter, which gave $G_2 = 1833{\cdot}19$.

8. Details and use of tangent galvanometer in Rowland's experiments. A tangent galvanometer was used to measure the steady current. This was a circle 50 cm in diameter, and had a needle 2·7 cm long, the deflection of which was read by a pointer moving round a graduated circle 20 cm in diameter. Parallactic error was avoided by placing the circle on a level with the needle which moved round inside it.

The constant of this galvanometer was compared with that of a single circle of wire 82·7 cm in diameter, wound on a ring made of pieces of wood laid together with the grain in the direction of the circumference, and carefully turned with a small groove near one side to receive the wire. The length of the wire was 259·58 cm, giving a mean radius of 41·31344 cm. This circle was made to surround the ballistic galvanometer coil, but at a distance of 1·1 cm on one side, to allow the tube carrying the suspension fibre to pass. Thus the constant of the circle was ·151925.

The same current being sent through the tangent galvanometer coil and the ring, and G_1, G', being their respective constants, we have, if a, a', be the angular deflections of the needles,

$$\frac{H}{G_1}\tan a = \frac{H'}{G'}\tan a',$$

so that
$$\frac{H}{H'}=\frac{G_1}{G'}\frac{\tan\alpha'}{\tan\alpha},$$
and this replaces H/H' in (2′), which becomes
$$R=M\frac{\pi}{T}\frac{G_2}{G'}\frac{\tan\alpha'}{\tan\alpha}\frac{\tan\theta_1}{\sin\frac{1}{2}\theta_2},\quad\text{.............................}(7)$$
where θ_2 is the ballistic deflection corrected for damping.

This method avoids the difficulty of accurately determining H/H' by vibration of a needle at the two places, and gives the further great advantage that the distance of the mirror from the scale of the ballistic galvanometer only enters as a correction on the ratio $\tan\alpha'/\sin\frac{1}{2}\theta_2$. The same factor of correction for torsion affected both $\tan\alpha'$ and $\sin\frac{1}{2}\theta_2$, so that, with the exception of a small correction on the period T of the needle of the ballistic galvanometer, all allowances for torsion were eliminated. Still further, since α and θ can be made nearly equal, the correction for length of needle in $\tan\theta/\tan\alpha$ is almost entirely obviated.

The apparatus was set up in a separate building in two rooms on the ground floor. The galvanometers were on brick piers, with marble tops, and were very carefully adjusted, and all connecting wires were twisted together to avoid magnetic effect. This adjustment, as well as the insulation everywhere, was carefully tested.

The experiments were mainly made by simply reversing the battery current and observing the throw; but the method of recoil was also used. Series of experiments were made with each pair of induction coils *A* and *B*, *B* and *C*, *C* and *A*.

The time of vibration was observed at the beginning and end of each series of observations. The needle was allowed to vibrate for 10 seconds, and ten observations were made before and after that interval. Time was taken on an accurate marine chronometer.

The mean result of a long series of experiments gave, after all corrections for temperature of coils, etc., $34{\cdot}719\times10^9$ cm per sec. as the value of *R*. Comparing with "10 ohm" standard coils in his possession, and with a resistance box by Elliott, Professor Rowland came to the conclusion that (in ordinary electromagnetic units)

$$1\ \text{B.A. unit}={\cdot}9911\times10^9\ \text{cm per sec.}$$

9. Weber's earth inductor method. Two methods of the first class are due to W. Weber. The first is very simple. A coil mounted with its axis of figure horizontal and in the magnetic meridian, and having its circuit completed through a ballistic galvanometer, is quickly turned through half a revolution round a vertical axis. If *A* be the effective area of the coil (the sum of the areas of its spires), and *H* the horizontal component of the earth's field-intensity, a change of induction of amount $2AH$ through the coil is produced. This measures the integral electromotive force in the coil, and hence if the circuit be completed, and include a total resistance R, the total quantity of electricity which flows through the circuit is $2AH/R$. This is not affected in the least by the inductance of the circuit.

The galvanometer deflection is observed, and also the elongations following, to allow damping to be corrected for. By the theory of the ballistic galvanometer, if T be the complete period of the needle, G the principal galvanometer constant, H' the horizontal component of the earth's magnetic field *at the needle*, and θ the observed deflection, the total flow through the instrument is $HT\sin\frac{1}{2}\theta/\pi G$. Thus

$$\frac{2AH'}{R}=\frac{HT}{\pi G}\sin\tfrac{1}{2}\theta,$$

or

$$R=2\pi GA\frac{H'}{H}\frac{1}{T\sin\frac{1}{2}\theta}. \qquad \text{..............................(8)}$$

In general H is very nearly equal to H', but it will not do to assume absolute equality; and the two quantities must be compared by observing the periods of vibration of a horizontally suspended needle at the two places.

10. Weber's mode of experimenting. Weber employed the method of recoil (XII. 42 above) in his observations. Turning the coil first through 180° from the initial position, he observed one deflection (positive, say) and the following elongation. Then when the needle was passing through zero the second time, he brought the coil back to its original position. This brought the needle to rest, and finally deflected it to the negative side of zero. This deflection was observed, and the following elongation, and then, at the second passage through zero, the same series of operations was begun afresh.

It was pointed out by the late Lord Rayleigh that if a, a', be the mean radius of the inductor and galvanometer coils respectively, the product

$$GA=2\pi^2\frac{a^2}{a'},$$

so that error of mean radius has double the importance in the inductor coil that it has in the galvanometer.

Great care is necessary in levelling the inductor as, on account of the largeness of the vertical component of the earth's field in high latitudes, any deviation in the plane of the meridian of the axis of rotation from verticality will lead to error of the same order in the result. Thus if the axis be inclined to the vertical at a small angle α in the plane of the meridian, we must use instead of A the value $A(1+\alpha\tan D)$, where D is the magnetic dip.

11. Weber and Zöllner's experiments. This method was used by Weber himself, and later by Weber and F. Zöllner. In the latter experiments very large inductor and galvanometer coils were used. Each consisted of 12 layers of copper wire 3 mm thick, 66 turns in a layer, wound on bobbins of well-seasoned, oil-soaked mahogany. The dimensions were:

	Int. Radius.	Ext. Radius.	Length.
Inductor - -	48·0414 cm	51·9461 cm	25·420 cm
Galvanometer -	48·032 cm	52·0797 cm	25·420 cm

For the galvanometer needle was used one or other of two magnets of lengths 10 cm and 20 cm respectively, and the deflections were read by means of a telescope and scale in the ordinary manner. The research was carried out in a room of the observatory at Leipzig, subject to varying magnetic disturbances and to variations of temperature, and was intended merely as a test of the apparatus.

The resistance of the circuit of the inductor given by the experiments came out slightly greater with the shorter needle than with the other. This was to be expected as the deflection, θ, with the shorter magnet must, on account of the greater distance on the whole of its magnetic distribution from the current, have been slightly smaller than the deflection in the other case. It is obvious that the needles were much too long.

12. Wiedemann's experiments. A careful determination of the ohm has been made with these coils by Professor G. Wiedemann.* The apparatus was set up in a room of very constant temperature in the University of Leipzig. A rhombus-shaped steel plate, with attached glass mirror, was hung with its plane vertical and its longest diameter horizontal, and being magnetized in the direction of this diagonal served as needle. The needle carried beneath it a horizontal metal bar on which weights could be slided to alter the moment of inertia of the suspended system.

The coils, having been levelled, were each adjusted until the same current sent in opposite directions produced equal deflections of a needle hung within the coil. Their axes were then at right angles to the magnetic meridian. The galvanometer coil was then fixed, and the inductor turned through an angle of 90°. This angle was measured by means of a right-angled glass prism, by observing a telescope scale by reflection in one of the rectangular faces (which were vertical), and turning the coil until the same division came to the cross-wires by reflection from the other face.

An arrangement of stops was then provided so that the coil could be turned from this position through exactly 180° and back again. The coil was turned a number of times in succession suddenly through this angle, always when the needle had returned to its zero position, so that the deflection was multiplied as far as the limits of the scale would allow.

The successive deflections θ_1, θ_2, etc., if the current was applied when the needle was accurately at zero in each case, were related to the quantity Q of electricity which flowed through the circuit at each half turn of the coil as follows:

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* *Abhandl. Berlin Akad. der Wissensch.* 1884, or Wiedemann's *Elektricität*, Band 4, p. 913.

$$\theta_1 = KQ, \quad \theta_2 = KQ\epsilon^{-\lambda}, \quad KQ(1-\epsilon^{-2\lambda}), \ldots,$$

where K has the value stated in XII. 42 (45) above. These were observed and the observations combined in a single formula for Q, which equated to $2AH/R$ enabled R to be calculated.

The periods T, T', of a needle vibrated at the galvanometer and inductor respectively were observed, and the ratio T^2/T'^2 gave the value of H'/H required as shown in (8). These were obtained by observing the oscillations with a telescope and scale, and registering the passages of different points of the scale across the wires by means of a chronograph.

The effect of torsion of the suspension fibre was found by turning a torsion head, to which the fibre was attached, through a measured angle, and observing the corresponding deflection of the needle. Thus when the torsion head was turned through an angle α, and the needle through an angle β, the return couple on the needle was $MH \sin\beta$, and the torsional couple $C(\alpha-\beta)$, where C is a constant. Thus

$$C = \frac{MH\sin\beta}{\alpha-\beta} = MH\tau, \text{ say.}$$

Hence, when the needle in the experiments was deflected through an angle θ, the return couple upon it was $MH(\sin\theta + \tau\theta)$, or nearly enough, as the deflections were small, $MH(1+\tau)\theta$. Thus instead of the value of H at the galvanometer needle $H(1+\tau)$ was used.

The dimensions of the coils were measured by determining their inner and outer circumferences with a steel tape, and as a check by measuring three diameters at intervals of 60° apart, by means of a cathetometer.

The distance of the scale from the mirror was first measured by means of a steel tape on which were sliding pieces furnished with points, which were brought against the mirror and scale respectively; then, by means of an auxiliary scale placed horizontally in the vertical plane through the centres of the telescope and mirror, on which the corresponding positions of the mirror and reading scale were observed by means of a cathetometer.

Experiments were made first with Weber and Zöllner's coils in the state in which they were left by these experimenters; then with the same coils rewound, and the number of turns increased from 792 to 804.

The experiments were then repeated with 10 mercury (Siemens) units included with the coils in the circuit.

Different series were made with the sliding weights on the needle at distances 2 cm, 1·5 cm, 1 cm, 0, from the end of the bar, so that the periods were altered through a considerable range.

The resistance of the Siemens' units was compared with a standard resistance of pure mercury, consisting of a mercury column contained in a carefully calibrated tube 106·398 cm long, the ends of which communicated with electrodes made of amalgamated copper-foil immersed in mercury in two vessels terminating the tube. It was found as a final mean result that 1 ohm or 10^9 c.g.s. units of resistance is equal to the resistance at 0° of a column of mercury 106·162 cm long and 1 sq. mm in cross-section.

13. Experiments of Mascart, de Nerville, and Benoit. This method has also been used by Mascart, de Nerville, and Benoit,* in a very elaborate series of experiments. Five coils were used, two of 27 cm internal and 30 cm external diameter, and 3 cm length, and three smaller coils each of 14 cm internal and 17 cm external diameter, and the same length as before. These were wound with silk-covered wire ·5 mm in diameter. One of the large coils and two of the small ones were wound with separate layers, so that, by joining these layers up differently, nine different arrangements could be obtained. The winding was performed with the wire under tension produced by passing it over loaded rollers when on its way from the reel to the bobbin. The length of the wire was measured as it was laid on, and the diameter of every turn was also observed by means of callipers.

Both the smaller and larger coils were mounted after completion on stands with suitable stops so as to admit of being turned when required through an angle of exactly 180°, and were set up with their axes horizontal and in the magnetic meridian.

At the centre of the larger coil when in position was placed a small magnetometer needle suspended by a single fibre of silk. By turning the coil round a vertical axis through 90° from its position when arranged for inductive use, and fixing it in its new position, it could be used as a galvanometer bobbin, and its galvanometer constant compared with that of the galvanometer bobbin itself. By this process, previously used by Rowland, the ratio of the horizontal magnetic forces, H'/H, at the inductor and the galvanometer was eliminated from the formula of calculation. For suppose the same current to be sent through the two coils, and α, α', to be the deflections for the galvanometer and the inductor respectively, G, G', the galvanometer constants of the two coils, we have, as at p. 600,

$$\frac{H'}{H}=\frac{G'\tan\alpha}{G\tan\alpha'}.$$

This substituted in (8) gives

$$R=2\pi G'A\,\frac{\tan\alpha}{\tan\alpha'}\,\frac{1}{T\sin\frac{1}{2}\theta}. \qquad\text{.........................(9)}$$

[Full details of the mode of comparing two galvanometer constants are given in XII. 50 above.]

This proceeding had the advantage (already pointed out in 8 above) that since the ratio of $\tan\alpha/\sin\frac{1}{2}\theta$ appears in the value of R the importance of an exact determination of the distance of the galvanometer scale from the mirror was greatly lessened. The value however of $\tan\alpha'$ had to be accurately known, and involved careful measurement of the corresponding distance for the other scale.

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* *Ann. de Ch. et de Phys.* 6, p. 5 (1885).

From the measured length of the wire the value of $G'A$ which appears in (9) could be approximated to. For a being the mean radius of the coil, and n the number of turns $A=n\pi a^2$, and $G'=2n\pi/a$, nearly, so that $G'A=2n^2\pi^2a=n\pi l$, where l is the length of the wire. The quantities therefore which required accurate determination were l, T, and the distance of the scale from the mirror of the magnetometer in the induction coil. The latter was found by means of a graduated measuring bar carrying sliding pieces, which were run up to the fibre and scale respectively. The positions of the contact faces of these pieces were read off from the scale and gave the distance required.

Observations were made by first reading off two successive elongations of the needle when it had nearly come to rest, and then turning the inductor when the needle was passing through zero, and reading the following deflections on the same side of zero.

If r, r', be the first two of these readings on the scale (supposed graduated from one end), in an induction throw, the (uncorrected) zero reading is $(r'+r)/2$. If the next two readings be r_1, r_2 the first deflection from zero is $r_1-(r'+r)/2$. The next reading being r_2 the diminution in one swing due to damping is $(r_1-r_2)/2$. The diminution of the first elongation must have been approximately $\frac{1}{2}$ of this or $(r_1-r_2)/4$. This correction applied to the first elongation gives for the deflection $r_1-(r'+r)/2+(r_1-r_2)/4$. There remains the correction for the initial motion, which is simply the correction of the zero for the decrement of that motion. If r' be taken as the greater reading the correction is $(r'-r)/2$, and must be added or subtracted according to the direction of the initial motion. Thus the deflection was

$$r_1-\frac{r'+r}{2}+\frac{r_1-r_2}{4}\pm\frac{r'-r}{2}.$$

The readings it was found did not vary more than $\frac{1}{3}$ per cent.

The torsion of the suspension fibre of the ballistic galvanometer was eliminated, as approximately it multiplied $\tan\alpha$ and $\sin\frac{1}{2}\theta$ in (9) by a common factor. That of the suspension fibre of the inductor was determined in the usual way by turning the upper end of the fibre round through 360°.

Experiments were made with the various coils arranged in different ways; and their effective areas were also compared by observing the effects which they produced on the galvanometer needle when turned in the earth's field.

The absolute resistance of the circuit in the various experiments having been obtained it was compared by Carey Foster's method of resistance comparison with four B.A. units, with four Siemens' mercury units, and with six specially constructed mercury units in spiral tubes. Careful comparisons of the temperature coefficients of the different coils were made, and all the resistances corrected to the temperature of experiment. The results were expressed finally as the absolute resistance of four mercury standards made of carefully calibrated tubes filled with mercury. These tubes were terminated by wide electrodes of mercury, and an allowance of a length of the tube equal to ·82 of its diameter was made to correct for the additional resistance due to the abrupt change of section of the tube at each end. The final result obtained was

$$1 \text{ ohm} = 1{\cdot}0142 \text{ B.A. unit},$$

or

1 ohm = resistance at 0° C. of a column of mercury 106·37 cm long and 1 sq. mm in section.

14. Weber's method by damping. Weber's second method consists in oscillating a magnet suspended within a coil, when the circuit is open, and again when the circuit is closed, and observing the period and logarithmic decrement in both cases. The induced currents assist the damping in the second case, and hence from a comparison of the results the resistance of the coil can be calculated.

When the circuit is open the equation of motion of the swinging needle is

$$\frac{d^2\phi}{dt^2}+2k\frac{d\phi}{dt}+\frac{MH}{\mu}\phi=0, \quad \text{..............................(10)}$$

where M is the magnetic moment, H the horizontal field intensity, and μ the moment of inertia of the magnet. Putting n^2 for MH/μ we get for the solution of the equation

$$\phi=A\epsilon^{-kt}\cos(\sqrt{n^2-k^2}\,t+e). \quad \text{............................(11)}$$

Here $k=2\lambda/T$ if λ be the logarithmic decrement of the oscillation and T the observed period $(=2\pi/(n^2-k^2)^{\frac{1}{2}}$.

If now the circuit be parallel to the meridian and be closed, the magnet will be acted on by the induced current produced by its motion. The magnetic induction through the coil due to the needle is $MG\sin\phi$ approximately, where G is the principal galvanometer constant of the coil. For let a current γ flow in the coil, then the mutual energy of the coil and magnet is equal to the product of the magnetic induction of the magnet through the coil and the current. But when $\phi=0$ this energy is obviously zero and the work done against the current in deflecting the magnet through the angle ϕ is $MG\gamma\sin\phi$, and so the magnetic induction through the circuit is $MG\sin\phi$. Supposing then the magnet swinging through a small range there will be a force exerted on the magnet by the current of amount $MG\gamma$. Hence the equation of motion of the magnet is

$$\frac{d^2\phi}{dt^2}+2k\frac{d\phi}{dt}+n^2\phi-\frac{MG}{\mu}\gamma=0. \quad \text{.........................(12)}$$

But we have also for the electromotive force in the circuit $-MG\,d\phi/dt$, and if L be the self-inductance of the coil

$$L\frac{d\gamma}{dt}+R\gamma+MG\frac{d\phi}{dt}=0. \quad \text{......................}(13)$$

Operating on equation (12) by $Ld/dt+R$, and on (13) by MG/μ, and adding, we eliminate γ, and find

$$\left(L\frac{d}{dt}+R\right)\left\{\frac{d^2}{dt^2}+2k\frac{d}{dt}+n^2\right\}\phi+\frac{M^2G^2}{\mu}\frac{d\phi}{dt}=0. \quad \text{............}(14)$$

If we suppose that the motion is simple harmonic with diminishing range, and put λ', T', for the logarithmic decrement and period we may write conveniently for our present purpose

$$\phi=\epsilon^{-(k'+ia)t},$$

where $i=\sqrt{-1}$, $k'=2\lambda'/T$, $a=2\pi/T'$. Thus we find

$$\frac{d}{dt}=-(k'+ia),$$

and (14) becomes

$$\{-(k'+ia)L+R\}\{k'^2-a^2+2ik'a-2k(k'+ia)+n^2\}$$
$$-\frac{M^2G^2}{\mu}(k'+ia)=0. \quad \text{................}(15)$$

The real and imaginary parts of this equation must vanish separately, and therefore picking out the imaginary terms, equating them to zero and solving for R, we obtain, since $n^2-k'^2=a^2$,

$$R=\frac{M^2G^2}{2\mu(k'-k)}+\tfrac{1}{2}L\left(3k'-k+\frac{n^2-k^2-a^2}{k'-k}\right). \quad \text{..............}(16)$$

A controlling equation is obtained in like manner from the real terms in (15).

This method has been used by W. Weber himself, and with modifications by H. F. Weber, Dorn, Wild, and F. Kohlrausch. It is against the method that M^2, G^2, enter to the second power, inasmuch as the very exact determination of either quantity is a matter of some difficulty. The value of μ also involves the square of the dimensions of the magnet.

15. Kohlrausch's modification of method by damping. The modification of this method used by Kohlrausch amounted to a combination of the first and second methods of Weber, in which he eliminated the constant of the galvanometer with which the earth-inductor was connected by determining the logarithmic decrement of the motion of the needle first when the circuit of the galvanometer was open, and again when it was closed. Calling these decrements λ_0, λ, and putting α, β, for the arcs of vibration in the method of recoil (which was used), T_0

the period of the needle when the circuit was open, we may write Kohlrausch's formula in the approximate form

$$R = \frac{16A^2H^2T_0(\lambda - \lambda_0)}{\pi^2\mu} \frac{\alpha\beta}{(\alpha^2+\beta^2)^2}.$$

This formula includes several quantities which are difficult to observe with accuracy, but its chief defect lies in the fact that it involves the fourth power of the radius of the inductor. Kohlrausch's final result, corrected for an error in the data used in his original calculations, is

$$1 \text{ B.A. unit} = \cdot 990 \times 10^9 \text{ c.g.s.}$$

16. Method of revolving coil. Another method of this class, suggested also by Sir William Thomson to the Committee of the British Association, seems to have been first proposed by Weber. It consists in spinning with uniform velocity about a vertical axis a circular coil, at the centre of which is suspended a small magnetic needle. A periodic current is thus made to flow in the coil in one direction (relative to the coil) in one half-turn from a position at right angles to the magnetic meridian, and in the opposite direction in the next half-turn. But the position of the coil being reversed in every half-turn as well as the current in it, the current flows on the whole in the same average direction relative to the needle and (apart from self-induction) has its maximum value always when the plane of the coil is in the magnetic meridian.

This method was used by the British Association Committee in their famous experiments, carried out principally by Clerk Maxwell, Balfour Stewart, and Fleeming Jenkin in 1863. Its theory was first fully given by Maxwell, and the following statement follows on the whole his notation and method.

17. Theory of the revolving coil method. If L be the self-inductance, γ the current at any time t, the electrokinetic energy of the circuit due to its own induction is $\frac{1}{2}L\gamma^2$. Again if M be the magnetic moment of the needle, and G the galvanometer constant of the coil, that is, the magnetic force at the centre which unit current in the coil would produce, the magnetic force at the needle due to the current γ is $G\gamma$. If ϕ be the angle which the axis of the needle makes with the magnetic meridian, and θ the angle which the coil makes with the same plane, the direction of the magnetic force due to the coil and the axis of the needle are inclined at an angle $\pi/2 - (\theta - \phi)$. Thus the mutual energy of the needle and current is numerically $MG\gamma \sin(\theta - \phi)$. This, if taken as potential energy, must be written with the positive sign, and if taken as kinetic energy with the negative sign prefixed to give the corresponding force. For the magnet is deflected in the direction of rotation, and hence, if $\theta > \phi$ say, the magnetic force on the needle due to the coil must be in the direction to increase ϕ, that is to diminish $\theta - \phi$. Hence $MG\gamma \sin(\theta - \phi)$ tends to diminution by the action of the mutual forces. We shall reckon it as kinetic energy of amount $-MG\gamma\sin(\theta-\phi)$.

Again if the effective area of the coil be A, there is mutual energy between it and the field of numerical amount $AH\gamma\sin\theta$. This may be taken as kinetic energy of amount $-AH\gamma\sin\theta$. Also the magnet is deflected in the field, and therefore between it and the field there is mutual energy $MH\cos\phi$ when reckoned as kinetic.

Lastly, if mk^2 be the moment of inertia of the needle about the axis of suspension it has kinetic energy $\frac{1}{2}mk^2\dot{\phi}^2$.

Collecting these terms we get for the total kinetic energy

$$T=\tfrac{1}{2}L\gamma^2-AH\gamma\sin\theta-MG\gamma\sin(\theta-\phi)+MH\cos\phi+\tfrac{1}{2}mk^2\dot{\phi}^2. \quad\text{...(17)}$$

Besides this there is potential energy V, due to the torsion of the fibre, depending on the angle through which the needle has been turned from the position of no torsion. If α be the angle which the needle makes with the meridian when the torsion is zero, the angle through which the fibre has been turned is $\phi-\alpha$. Denoting by $MH\tau$ the torsional couple which the wire gives when the lower end is turned through unit angle relatively to the upper, we have

$$V=\int_\alpha^\phi MH\tau(\phi-\alpha)\,d\phi=\tfrac{1}{2}MH\tau(\phi-\alpha)^2. \quad\text{.................(18)}$$

The equation of currents is

$$\frac{d}{dt}\frac{\partial T}{\partial\gamma}+\frac{\partial F}{\partial\gamma}=0,$$

where F is the dissipation function. This gives by (17)

$$L\frac{d\gamma}{dt}+R\gamma=AH\dot{\theta}\cos\theta+MG(\dot{\theta}-\dot{\phi})\cos(\theta-\phi). \quad\text{...........(19)}$$

There are two possible distinct motions for the magnet, one of oscillation in its own proper period (which we suppose great in comparison with the period of rotation of the coil), and the other of period equal to half that of rotation. So far as the former is concerned, we may take the magnet as at rest in computing the current, and for the latter we shall suppose at present the amplitude very small, so that the part of $\dot{\phi}$ depending upon it may also be neglected and ϕ may be taken as constant. Thus $\dot{\theta}$ being constant, $=\omega$, say, and $\theta=\omega t$, we have

$$L\frac{d\gamma}{dt}+R\gamma=AH\omega\cos\omega t+MG\omega\cos(\omega t-\phi). \quad\text{..............(20)}$$

Let a solution of this equation be

$$\gamma=C\cos\omega t+C'\sin\omega t.$$

Then $$L\frac{d\gamma}{dt}+R\gamma=(L\omega C'+RC)\cos\omega t-(L\omega C-RC')\sin\omega t. \quad\text{.....(21)}$$

This with (19) gives by equation of coefficients

$$\gamma = \frac{\omega}{R^2 + \omega^2 L^2} [AH(R\cos\theta + \omega L \sin\theta)$$
$$+ MG\{R\cos(\theta - \phi) + L\sin(\theta - \phi)\}]. \quad \text{....(22)}$$

A term, $C\,\exp.\,(-Rt/L)$, is required to complete the solution, but this dies out soon after the starting of the coil, and has no effect provided the rotation is uniform. The current therefore on the supposition made is given by (22).

18. Equation of motion of the needle and deduction of resistance of circuit. The expression for the kinetic and potential energies gives for the equation of motion of the magnet

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} + \frac{\partial V}{\partial \phi} = 0,$$

or $$mk^2\ddot{\phi} - MG\gamma\cos(\theta - \phi) + MH\sin\phi + MH\tau(\phi - a) = 0. \quad \text{.....(23)}$$

This equation may be obtained also by considering that the needle is acted on by three couples, one due to the current tending to produce further deflection, the second a return couple due to the earth's magnetic field, and the third also a return couple due to the torsion of the fibre. The numerical values of these are from the notation already explained, $MG\gamma\cos(\theta - \phi)$, $MH\sin\phi$, $MH\tau(\phi - a)$. Hence the total deflecting couple is

$$MG\gamma\cos(\theta - \phi) - MH\{\sin\phi + \tau(\phi - a)\},$$

and this is equal to the rate of increase $mk^2\ddot{\phi}$ of angular momentum.

The needle is found to take up a nearly constant position if the rotation of the coil is kept uniform, and in this case ϕ may be taken as very nearly zero. Thus we have, integrating over any finite interval of time, $\int \ddot{\phi}\,dt = 0$. The position must therefore be such that the mean resultant deflecting couple applied by the current must be equal to the return couple $MH\{\sin\phi + \tau(\phi - a)\}$ due to the combined action of the magnetic field and torsion. This average couple is obtained from $MG\gamma\cos(\theta - \phi)$ by inserting the value of γ given by (22) and integrating each term over a whole turn on the supposition that ϕ is a constant, and dividing the result by 2π. The following integrals enter into the expression

$$\frac{1}{2\pi}\int_0^{2\pi} \cos\theta\cos(\theta - \phi)\,d\theta = \tfrac{1}{2}\cos\phi.$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\theta\cos(\theta - \phi)\,d\theta = \tfrac{1}{2}\sin\phi.$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^2(\theta - \phi)\,d\theta = \tfrac{1}{2}.$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(\theta - \phi)\cos(\theta - \phi)\,d\theta = 0.$$

Therefore the average couple is

$$\frac{1}{2}\frac{\omega MG}{R^2+\omega^2L^2}\{AH(R\cos\phi+\omega L\sin\phi)+MGR\}=MH(\sin\phi+\tau\phi) \quad \text{...(24)}$$

for equilibrium, if $\alpha=0$. Since τ is very small we may write $\tau\sin\phi$ for $\tau\phi$, and we get

$$R^2-\tfrac{1}{2}R\frac{\omega AG}{1+\tau}\cot\phi\left(1+\frac{MG}{AH}\sec.\phi\right)+L^2\omega^2-\frac{1}{2}\frac{AGL\omega^2}{1+\tau}=0. \quad \text{...(24}'\text{)}$$

This may be written in the form

$$R^2-aR-b=0,$$

the solution of which is $$R=\frac{a\pm\sqrt{a^2+4b}}{2}.$$

The value of b is positive in the experiments made, and hence, since R cannot be negative, the + sign in the solution must be taken. Expanding the radical, having regard to the fact that MG/AH and τ are small, we get

$$R=\tfrac{1}{2}AG\omega\cot\phi\left\{1+\frac{MG}{AH}\sec\phi-\frac{2L}{AG}\left(\frac{2L}{AG}-1\right)\tan^2\phi\right.$$
$$\left.-\left(\frac{2L}{AG}\right)^2\left(\frac{2L}{AG}-1\right)^2\tan^4\phi-\ldots\right\}. \quad \text{...(25)}$$

[If τ is not negligible and α is not zero, instead of the coefficient $\frac{1}{2}$ on the right of (25) should be written $1/[2\{1+\tau(\phi-\alpha)/\sin\phi\}]$. It appears that the term in $\tan^6\phi$ is not quite insensible, so that on the whole it would probably have been more convenient to use the value $R=\{a+\sqrt{a^2+4b}\}/2$ directly. This was done in Rayleigh and Schuster's repetition of the experiments with the B.A. apparatus. See 21 below.]

This is the expression for R used by the B.A. Committee in the reduction of the results of their experiments.

19. Criticisms of method. Taking the first term only we may write

$$R=\tfrac{1}{2}AG\omega\cot\phi=\pi^2n^2a\omega\cot\phi, \quad \text{......................(26)}$$

where a denotes the mean radius and n the number of turns in the coil. This formula is convenient for the discussion of the advantages and disadvantages of the method. These were examined by the late Lord Rayleigh in papers on this method * and in his "Comparison of Methods for the Determination of Resistances in Absolute Measure." †

As regards the measurement of dimensions of the apparatus, it is to be noticed that the method involves only one fundamental linear quantity a, and that only to the first power. The observation of the deflection corresponding to ϕ and the evaluation of $\cot\phi$ involve no greater difficulty than those which attend ordinary angular measurement, and in this respect the method is on a par with Weber's method by earth inductor. The main difficulties lie in the determination of ω and the avoidance of mechanical disturbance, and of error due to currents in the ring and alterations in the magnetization of the needle.

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* Lord Rayleigh and Arthur Schuster, "On the Determination of the Ohm," *Proc. R.S.* No. 213, 1881. Lord Rayleigh, *Phil. Trans. R.S.* Part II. 1882.

† *Phil. Mag.* Nov. 1882.

It will be seen below that, by the employment of what may be called the stroboscopic method of observation, the late Lord Rayleigh, who repeated the determination with the same apparatus, was able to control and measure the speed with great exactness. A correction is easily made for the currents induced in the coil in consequence of its motion in the field of the needle, in fact a small term appears in the result above [$MG \sec \phi / AH$ in (24′)], by means of which this correction is made. This involves the determination of MG/AH, but, as will be seen below, about this there is no difficulty whatever.

The currents produced in the metal ring can be allowed for by rotating the coil (1) with the wire circuit open, (2) with that circuit closed. Further, these currents can be reduced by dividing the ring into two parts along a diameter and putting them together with ebonite separating pieces. The currents are then confined to circuits which are on the whole at right angles to the plane of the coil, and their effect can easily be eliminated by the method just stated. The existence of these currents in the ring has one advantage, pointed out by Lord Rayleigh, that by rotating the ring before winding, and again with the wire circuit open after winding, the insulation can be tested. For if any difference is found between the deflections of the needle it must be due to leakage.

20. Effect of self-induction. The method has been objected to on the ground of the influence of self-induction in the result, that is on account of the terms in (25) which involve L. Now the value of the coefficient (U, say) of $\tan^2\phi$ in (25), and therefore of $\tan^4\phi$, etc., may be calculated with considerable accuracy from the dimensions and arrangement of the rotating coil, and any want of exact knowledge of the value of U can be eliminated by using different speeds of rotation.

In comparing Weber's method by earth inductor with the present method, it is to be noticed that at half the lowest speed used by Lord Rayleigh the sensitiveness of the latter method would be considerably greater than that of the former, and the correction for self-induction, known with fair accuracy, would be only about $\frac{1}{4}$ per cent.

The effect of self-induction could be diminished, as pointed out by Lord Rayleigh, by duplicating the revolving coil by the addition of a second coil at right angles to the other, and giving an independent circuit. Thus the sensitiveness of the arrangement would be increased without entailing an increased correction for self-induction such as would be necessary if the increase of deflection were produced by running the coil at a higher speed. The two circuits in this arrangement also would be conjugate, that is the currents in one would be unaffected by those in the other, and would give a more nearly constant field of magnetic force.

21. Later experiments with the revolving coil method. We now give some account of later determinations by this method, beginning with the experiments made by Lord Rayleigh and Prof. Schuster in 1881.* The coil used by the B.A. Committee was employed, but its constants were carefully redetermined. The constant A of the coil was found by unwinding the wire, and carefully measuring the circumference of the successive layers. The thickness of the wire used was 1·37 mm, which ought to have produced a difference in the circumference of the successive layers of $2{\cdot}74\pi$ mm. The turns in each layer sinking a little into those below gave on the average 8·1 mm for this difference. The coil was made up of two parts, between which the needle was suspended. On each part there were 156·5 turns arranged in one case in 12 layers of 13 turns each, with half a turn outside, and in the other in 12 layers containing 155 turns with $1\frac{1}{2}$ turns outside. Allowing for the outside parts these measurements gave

Mean radius of double coil - - - -	15·789 cm.
Axial dimension of each groove - - - -	1·833 cm.
Distance of mean plane from axis of motion -	1·918 cm.

The value of A was calculated by the formula

$$A = \pi n a^2\left(1 + \frac{1}{3}\frac{d^2}{a^2}\right), \quad \text{.................................}(27)$$

where a denotes the mean radius, $2d$ the radial dimension of the section, and n the total number of turns. This formula may be proved thus. Since the number of layers in each coil was 12,

$$A = \frac{1}{12} n\pi \left\{\left(a - \frac{11d}{12}\right)^2 + \left(a - \frac{9d}{12}\right)^2 + \dots + \left(a + \frac{11d}{12}\right)^2\right\} = n\pi a^2\left(1 + \frac{d^2}{3a^2}\right)$$

nearly.

The value of the galvanometer constant G was calculated by an equation equivalent to that obtained from (9), p. 212 above, by taking the first term $2\pi\gamma a^2/r^3$, putting $\gamma = 1$, multiplying by n, and substituting for a^2/r^3, on account of the axial breadth $2b$ and radial depth $2d$, of the sections the value given in (20), p. 219, that is from

$$G = 2\pi n\left\{\frac{a^2}{r^3} + \frac{b^2}{6}\,\frac{3a^2}{r^7}\,(4x^2 - a^2) + \frac{d^2}{6}\,\frac{1}{r^7}(2x^4 - 11x^2a^2 + 2a^4)\right\}, \ \dots(28)$$

where x = distance of the mean plane of either coil from the suspension fibre, and n is the total number of turns in the double coil.

The value of GA obtained after applying all corrections, and including in it allowances for non-verticality of the axis and for torsion of the fibre, was 29887600. The axis of rotation was found to be inclined towards the north at an angle 0·0003 radian. This necessitated a correcting factor in GA of $1 + 0{\cdot}0003 \tan D$, where D is the magnetic dip, that is, a factor $1 + 0{\cdot}0008$.

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* *Proc. R.S.* No. 213, 1881.

22. Calculation of self-induction of coil. The value of L was found by calculating the inductance for a coil of mean radius a and rectangular cross-section of which the length of diagonal was r. This was found from the formula $[\theta = \tan^{-1}(b/a)]$

$$L = 4\pi n^2 a\{\log \frac{8a}{r} + \tfrac{1}{12} - \tfrac{4}{3}\theta \cot 2\theta - \tfrac{1}{3}\pi \tan\theta$$
$$- \tfrac{1}{6}\cot^2\theta \log\cos\theta - \tfrac{1}{6}\tan^2\theta \log\sin\theta\}, \quad \text{...(29)}$$

which is simply the formula $4\pi n^2 a\{\log(8a/R) - 2\}$ (see XIII. 13 above), with the value of the logarithm of the geometric mean distance of the cross-section from itself, given by XIII. 8 (35), put for $\log R$. The dimensions of the coil used were those given by the B.A. Committee, viz. $a = 15{\cdot}8194$ cm, axial breadth of each coil 1·841 cm, radial depth 1·608 cm, and distance of mean planes apart 3·851 cm.

The inductance was computed for the double coil by adding together the self-inductances of the coils taken separately, and twice the mutual inductance of the two coils. For if L_1, L_2, be the self-inductances, and M their mutual inductance, the whole electrokinetic energy of a current γ is $\frac{1}{2}\gamma^2(L_1 + L_2 + 2M) = \frac{1}{2}\gamma^2 L$ if L be the self-inductance of the whole system. To the approximation given by (29) Lord Rayleigh found for $L_1 + L_2$ 30192000 cm, and for $2M$ 14582000 cm. Corrections for the finite size of the cross-section, and (since the introduction of the geometric mean distance is made on the supposition that the coil may be regarded as straight) for curvature were made. The latter can be calculated by the series (54), p. 199, or by the elliptic integral formula by dividing the coil up into concentric circular filaments, and integrating over the cross-section. [See XIII. 29 above.] Lord Rayleigh found that for a single coil of circular cross-section of radius ρ the value of L is given by the equation

$$L = 4\pi n^2 a\left\{\log\frac{8a}{\rho} - \frac{7}{4} + \frac{\rho^2}{8a^2}\left(\log\frac{8a}{\rho} + \frac{1}{3}\right)\right\}, \quad \text{...............(30)}$$

so that the correction for curvature increases L. The correction term for curvature in the case of a coil of the same mean radius a and square cross-section of the same area is very nearly the same as in this formula. It is thus an addition to the approximate value given by the equation (29) above. The corrections in L_1 and L_2 were each 11950 cm, and the correction on $2M$ 346900 cm, so that finally

$$L = 45144800 \text{ cm.}$$

[See IX. 15 above for the derivation of (30).]

The value of $2M$ found by the formula of quadratures given on p. 434, from the value given by the elliptic integral formula for two circles, was 14939400 cm, agreeing very closely with the value 14928900 cm, (14582000 + 346900) cm, already obtained.

23. Experimental determination of inductance. An experimental determination of L was made by the method described above, and gave 45000000 cm on the supposition that the B.A. unit was 1 per cent. less than the ohm. The value given by Maxwell,* uncorrected for curvature, is 43744000, and the allowance for curvature, 734500 cm, was apparently subtracted from instead of added to this value, giving finally, with a correction for the finite diameters of the wires and variation of the current over the cross-section, $L=43016500$ cm. It was suggested by Lord Rayleigh that the discrepance might be due mainly to an interchange of the breadth and depth of the coils, together with the mistake just noticed as to the correction for curvature.

The observations included (1) the resistance of the experimental coil as compared with a standard coil of German silver of nearly the same resistance, viz. 4·6 ohms, (2) the deflections produced by the spinning of the coil, (3) the speed of rotation.

The comparison of resistances was made by a balance arranged by Mr. J. A. Fleming, in which Prof. Carey Foster's method (see XI.) of interchanging the resistance to be compared with the standard was used to give the difference between the two resistances in terms of a certain length of the bridge wire. Error due to thermo-electric currents was eliminated by making the comparison with the battery current first in one direction, then in the other. A comparison was made at the beginning and end of each set of spinnings.

The needle consisted of four magnetized needles, each 0·5 cm long, mounted on four parallel edges of a small cube of cork, to which the mirror was also fixed. This arrangement was adopted because four equal, thin, uniformly magnetized magnets placed along the parallel edges of a cube of length of edge $1/\sqrt{3}$ of the length of the magnets gave a lighter arrangement than a magnetized sphere of steel which was used by the B.A. Committee, and formed a needle the action of which was to a high degree of approximation the same as that of an infinitely small needle at the centre of the cube. The magnets were made about 2·3 times the edge of the cube in length to allow for non-uniformity of magnetization.

The needle was adjusted in position by raising or lowering the cube until it was midway between the highest and lowest points of the circular frame, and then adjusting it in the two other directions, by attaching a pointer to the frame reaching in nearly to the centre, then turning the plane round, and observing whether the centre of the cube coincided with the centre of the small circle described by the point.

The needle was in the usual manner caused to deflect another horizontally suspended needle in order to determine the ratio M/H of the magnetic moment to the horizontal component of the earth's magnetic field. At a distance of one foot the suspended needle was deflected through $\tan^{-1} 0{\cdot}000298$, and hence at a distance equal to the mean radius of the coil, 15·85 cm, the deflection of the needle would have been ·0021 approximately.

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* "On a Dynamical Theory of the Electromagnetic Field," *Phil. Trans. R.S.* vol. clv. (1864), and Reprint of Papers, vol. i. p. 596.

Now if r denote the mean radius of the coil, and μ the deflection of the needle, we have by (21), p. 96 above, since the length of the magnet was small compared with r,

$$\tan \mu = \frac{2M}{Hr^3};$$

and approximately $G = 2\pi n/r$, and $A = n\pi r^2$, where n is the number of turns. Thus $r^3 = 2A/G$ and $\tan \mu = GM/HA$. This was used as the value of GM/HA in the term in (24) in which that quantity occurs.

The telescope and scale (which was straight) were adjusted in the usual manner (see XI. 1). The distance of the scale from the mirror was compared with the scale directly, so that the absolute length of a scale division did not enter in the result. The following were the numbers:

Distance of scale from mirror - - - -	252·28 cm
Correction for glass plate 3·2 mm thick through which mirror was viewed, $3{\cdot}2\left(1 - \frac{1}{\mu}\right)$ - -	0·11 cm
Distance (reduced) - -	252·17 cm

The heights of the centre of the mirror and the centre of the objective above the line of the scale divisions were measured by means of a cathetometer, to obtain the data necessary for finding the angle between the normal to the mirror and the horizontal. For this a correction was applied to the readings.

The torsion of the silk fibre, which was 4 feet long, was also estimated by turning the magnet through 5 complete turns, and observing the deflection of the magnet. It was found that the magnet was shifted 5·6 divisions per turn, or through an angle of ·001107. Opposite turning of the magnet gave ·001117, so that the correction for torsion was obtained by calculating $\tau = {\cdot}00111/2\pi$, and using for A the value $A/(1+\tau)$.

A correction for level of the coil was also applied, as it was found that the upper end of the axis was inclined towards the north by an angle ·0003 radian. The component of force at right angles to the axis was thus, if I be the intensity of the field, and D the dip,

$$I \cos (D - {\cdot}0003) = H(1 + {\cdot}0003 \tan D) \text{ nearly.}$$

Thus for A was used finally the value

$$A(1 + {\cdot}0003 \tan D)/(1+\tau).$$

24. Mode of driving the coil and regulating speed. The spins were taken in sets of four at each speed. The coil was driven by a long cord from a water motor acting by the impulse of water on metal cups.

To insure a constant pressure the motor was driven by water from a small cistern, which gave a head of 50 feet. The regulation of the motor was effected by observing that the work done by the motor is proportional to the difference between the speed of the jet and that of the cups, and to the speed of the cups. For, if the water is just reduced to rest the momentum of unit mass of water destroyed is v, the speed of the jet, and the mass of water received per unit of time is $a(v-v_1)$ if v_1 be the speed of the cups, and a the area of the jet. Thus the rate at which momentum is given by the jet to the cups is $av(v-v_1)$. The rate at which the motor works is therefore $av(v-v_1)v_1$. Thus at zero speed, and at the speed of the jet the water motor does no work. At half the latter speed the motor does work at the maximum rate. Thus the diagram of activity is a parabola with vertex upwards if speeds of the motor be taken as abscissae.

Drawing on this diagram the curve of work done against resistances, we obtain from the points of intersection of the two curves the possible uniform speeds of running, and these speeds are more sharply defined the more nearly the curves are at right angles. Now the activity spent in overcoming resistance to the motion of the coil is a function of the speed v_1 of the form $Av_1 + Bv_1^2 + Cv_1^3 +$ etc., since there are included constant or frictional resistances, which give the first term, resistances such as viscous resistances which are proportional to the speed, which give rise to the second term, and resistances which vary as higher powers of the speed, such as resistance due to air set in motion by the cups, etc.

The curve of activity against resistance is therefore convex downwards, and at high speeds in the experiments there is no difficulty in obtaining definite enough intersection, but at low speeds this is not the case. It was necessary therefore at low speeds of the coil to run the motor fast and use a reducing pulley, in order to enable the curve of resistances to intersect at a suitable place.

The speed of rotation was observed by the stroboscopic method, in which a card marked with circles of alternately black and white spaces (or "teeth") is viewed through narrow slits in thin plates of metal attached in the plane of vibration to the prongs of a tuning-fork. The slits overlap when the fork is at rest, so that to an observer looking through them the card is visible; when the fork is in vibration vision is possible through the slits twice only in every complete vibration. (See Fig. 212 below.)

The fork was electrically maintained, and had a frequency of about $63\frac{1}{2}$ (more nearly 63·69). Thus the card could be seen 127 times a second through the slits. Hence if a circle on the card marked with alternate black and white teeth was carried round at such a speed that the number of black teeth which passed the mean position of the slits in each second was equal to twice the frequency of the fork, the circle appeared to be at rest.

The card was graduated with five circles containing 60, 32, 24, 20, 16 black teeth respectively, to enable a variety of speeds to be observed without any change in the frequency of the fork. Looked at over one end of one of the vibrating plates the card could be seen only once in each complete vibration, and thus the 60 teeth circle could be used for the lower speeds.

The contacts of the fork were made and broken with a platinum point and mercury cup covered with pure alcohol. The arrangement worked exceedingly well, and went for hours without requiring the smallest attention. A comparison was made, by means of beats, between the pitch of the fork and that of a standard fork.

It was found that the speed of the disk could be regulated by the observer by applying slight friction to the driving cord, when the teeth showed any tendency to pass. He therefore allowed the cord to run lightly through his fingers, and after a little practice it was possible so perfectly to regulate the speed that a tooth never passed the pointer except perhaps by inadvertence, when he at once brought it back by slightly retarding the cord. The passage of one tooth in each second meant of course only a variation of 1 in 127 in the speed.

25. Various corrections. In the course of the observations note was taken of the changes of magnetic declination by means of an auxiliary magnetometer set up near enough the revolving coil to be practically in the same magnetic field with it, but at the same time so far away as to be unaffected by the induced currents produced by the spinning. The scale was read by means of a telescope, and the distance from mirror to scale, 2½ metres, was the same as that of the mirror of the magnet in the coil from its scale, so that the corrections could be made by simple comparison of readings.

Some trouble was caused by air currents in the box containing the magnet; these currents caused change of zero during a set of spinnings. They were mainly due to radiation of the lamp and gas jets, and precautions were taken to diminish the effect by covering the magnet box with gold-leaf to reflect the heat as much as possible. The error from this cause, however, was not greater than that which necessarily affected the determinations of the mean radius of the coil, and the distance of the mirror from the scale.

If ϕ be the deflection of the mirror, d the observed reading, and D the distance of the mirror from the scale, δ the distance of the zero position of the spot of light from the zero of the scale, then, approximately,

$$2D\tan\phi = d - \delta - (d - \delta)\frac{(d-\delta)^2}{4D^2} + \frac{(d-\delta)^5}{8D^4}. \quad \text{............(31)}$$

This formula was used for calculating $\tan\phi$, δ being taken positive when in the same direction as d. Irregularities in the scale were allowed for, and, as stated above, a correction applied for the slight non-horizontality of the normal to the mirror. The vertical distance between the centre of the objective and the point in which the normal intersected the scale being denoted by p, the angle between the normal and the horizontal by a, the correction was dpa/D, which amounted to $d \times 0{\cdot}00014$.

The resistance comparisons generally showed a rise of resistance during each set of experiments. This was corrected for on the supposition that the rise of temperature was uniform during the time elapsing between two successive measurements of resistance. The error arising from uncertainty of temperature did not amount to more than ·05 per cent.

26. Specimen set of readings. The following is one set of readings in which C denotes the resistance of the coil, S the resistance of the standard:

Time.		
9 h. 17 m.	Resistance compared	$C = S + \cdot 0225$
9 h. 32 m.	Reading of Auxiliary Magnetometer for change of magnetic declination	26·9
	Position of rest of needle	766·48

Time.	Direction of Rotation.	Deflected Reading.	Auxiliary Magnetometer Reading.
9 h. 37 m.	Negative	367·60	27·55
9 h. 42 m.	Positive	1166·40	28·24
9 h. 47 m.	Negative	366·23	28·50
9 h. 53 m.	Positive	1166·09	28·30

Time.		
9 h. 57 m.	Reading of Auxiliary Magnetometer	27·2
	Position of rest of needle	767·08
10 h. 0 m.	Resistance compared	$C = S + \cdot 0272$

From these the following table of corrected readings and deflections was found :

Position of Rest.	Deflection observed.	Deflection corrected for Scale Errors and Temperature.
766·28	− 398·61	− 396·55
765·59	+ 400·81	+ 397·93
765·33	− 399·10	− 397·23
765·53	+ 400·56	+ 397·23
		Mean 397·42

$$C = S + \cdot 0248.$$

27. Value of resistance of B.A. unit. The value of R was calculated directly from the solution of the quadratic (24) above. If A' be put for $A(1+\cdot 0003 \tan D)/(1+\tau)$, the value of the area of the coil when it is made to include the correction for torsion and level, and $\tan\mu$ denote GM/HA as determined in 23 above, this solution may be written

$$R = n\pi GA' \cot\phi \,\{1+\tan\mu\sec\phi+\sqrt{(1+\tan\mu\sec\phi)^2 - U\tan^2\phi}\}, \quad (32)$$

where $U=(2L/GA')/(2L/GA'-1)$, and n denotes the number of turns of the coil per second

$=2$ frequency of fork / number of teeth in stationary circle.

The following table gives the result of all the experiments. Column 1 gives the date of the experiment, 2 the speed in terms of the number of teeth on the apparently stationary circle, 3 the deflection corrected for scale errors and variation of temperature during the set of experiments, 4 the absolute resistance of the revolving coil on the assumption that the inductance of the coil was $4\cdot 5\times 10^7$ cm, and 5 the absolute resistance of the standard German silver coil at 11°·5 C. as given by the different experiments, subject to a correction for the copper rods connecting the rotating coil with the resistance bridge.

Date.	Teeth on Card.	Deflection.	$R\times 10^{-9}$	$R\times 10^{-9}$ for Standard Coil.	Mean.
Dec. 7	120	110·42	4·5486	4·5419	4·5364
10		110·22	4·5568	4·5309	
Dec. 2	60	218·61	4·5580	4·5487	4·5467
6		218·30	4·5620	4·5471	
10		218·72	4·5531	4·5422	
Dec. 2	32	397·75	4·5639	4·5417	4·5427
6		397·39	4·5672	4·5415	
10		397·26	4·5687	4·5448	
Dec. 2	24	513·73	4·5719	4·5446	4·5442
6		513·58	4·5734	4·5438	

Mean $R=4\cdot 5427\times 10^9$, in cm per sec.

The value of L here used was slightly less than that found by Lord Rayleigh, and agreed very closely with a value ($4\cdot 5130\times 10^7$ cm) deduced by the method of least squares from the results for different speeds.

The German silver standard was then compared with the original standards prepared by the B.A. Committee. The standard was found to be 4·595 B.A. units at 11°·5 C., and the resistance of the copper rods connecting the rotating coil with the bridge was found to be ·003 unit. Thus 4·592 B.A. units were found to be equivalent to $4·5427 \times 10^9$ in cm per second, or

$$1 \text{ B.A. unit} = ·9893 \times 10^9, \text{ in cm per second.}$$

28. Lord Rayleigh's further experiments with the revolving coil method. The investigation just described was repeated by Lord Rayleigh with improved apparatus, with the assistance of Dr. Arthur Schuster and Mrs. H. Sidgwick. The coil was made more massive to remove risk of deformation by the winding, and its dimensions were increased in the ratio of about 3 to 2. The ring was in two halves, joined along the horizontal diameter by projecting flanges, and insulated from one another by a layer of ebonite. Its construction with driving arrangements, etc., is shown in Fig. 192.

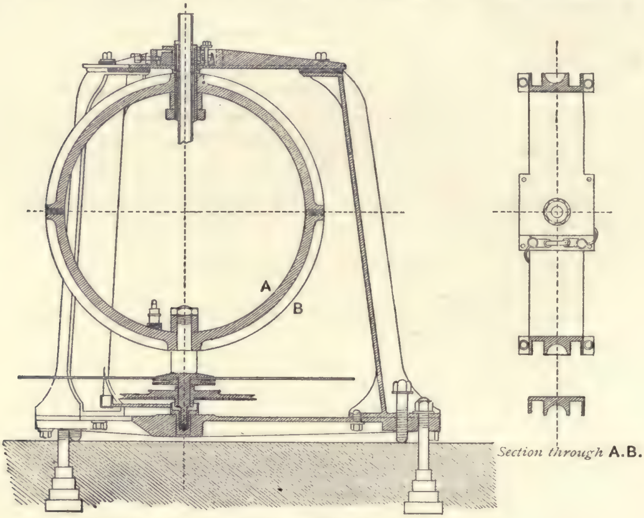

FIG. 192.

The ring having been wound was spun with its circuit open, and it was found that a perceptible effect on the magnet was produced. This was traced to currents circulating in the parts of the ring adjacent to the ebonite layer, where there was sufficient body of metal to give currents in circuits at right angles to the windings. These currents were afterwards allowed for.

To obviate air disturbances of the needle caused by rotation of the coil, the magnet box was screwed air-tight to the lower end of a brass tube which passed through the upper part of the axis of rotation. By unscrewing the box and pulling up the brass tube the magnet could be withdrawn with the fibre intact. The level of the needle was adjustable by means of a sliding piece, to which the upper end of the fibre was attached. The whole arrangement was so rigid that no disturbance was produced by the air even at the highest speeds.

The needle was on the same plan as before. Its moment was however six or seven times as great, with, on account of the greater dimensions of the coil, a value of ·0042 for MG/AH, ($\tan \mu$), or only about twice the former value. (This was determined in a manner similar to that already described.) The horizontal breadth of the mirror was diminished, and thus with greater magnetic moment and smaller mirror the disturbance from air currents inside the box was brought down to about 1/15 of what it was in the former apparatus. The period of oscillation was brought up to a convenient amount by an inertia ring $\frac{3}{4}$ inch in diameter added to the magnet. The weight of the whole was so small that it was easily borne by a single fibre of silk.

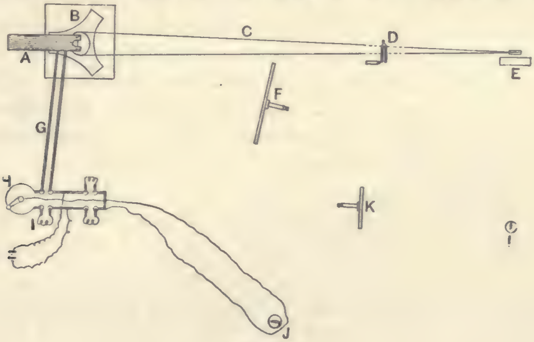

FIG. 193.

A. Stand for suspended parts.
B Frame of revolving coil.
C. Driving cord.
D. Fork and telescope.
E. Water-motor.
F. Telescope and scale for observing inflections.
G. Copper bars connecting to bridge.
H. Fleming's bridge.
I. Platinum silver standard.
J. Bridge galvanometer.
K. Telescope and scale of auxiliary magnetometer.
L. Auxiliary magnetometer.

The coil was driven and its speed determined as in the former experiments.

The resistance of the coil being 23 units as compared with the former value 4·6 units, arrangements were made to add resistances to the copper circuit when the variation of resistance passed beyond the range of the slide wire, and a platinum-silver standard of about 24 units was employed.

The general arrangement of the apparatus is shown in Fig. 193.

A first set of spinnings gave less accurate results than were expected, and the cause was traced to the paper scales. These were then replaced by scales engraved on glass. Some trouble was also caused by an imperfect mercury contact at the junction of the copper coil with the bridge connections; but when this was remedied the arrangements worked satisfactorily.

29. Dimensions and windings of coil. Self-inductance. The dimensions of the coil were as follows :

	Mean circumf. cm.	Mean radius. cm.	Axial breadth. cm.	Radial breadth. cm.
Coil *A*	148·53	23·639	1·99	1·59
Coil *B*	148·35	23·611	1·99	1·54
	Mean 148·44	Mean 23·625		

Each coil was wound with sixteen layers of eighteen turns in each layer, except the eleventh layer of *A*, which had seventeen turns. An extra turn was laid on *A* outside the sixteenth layer.

Each layer was measured during winding, and again on unwinding after the experiments had been made. Thus the effect of the pressure of the layers in diminishing their radii was estimated. The mean of the mean radii of the two coils was then 23·616. Weights of two and one were given to the last result and the former respectively, so that a mean of 23·619 cm was adopted.

GA was calculated from the formulae (27), (28), above, multiplied together, and it was found that $\log(GA) = 8{\cdot}17682$. The correction for level and torsion, it was found, increased this number only to 8·17686.

The value of L for the coil was found by calculating L_1, L_2, and M for the two coils as explained above (p. 615), L_1, L_2 were found by (29), and M by the formula of approximation given at p. 434 above [see also XIII. 31]. Thus

$$L_1 \text{ (for } A) = 1029{\cdot}3 \times 16^2 \times 18^2 \text{ cm},$$
$$L_2 \text{ (for } B) = 1031{\cdot}9 \times 16^2 \times 18^2 \text{ cm},$$
$$2M = 832{\cdot}88 \times 16^2 \times 18^2 \text{ cm},$$

so that

$$L = L_1 + L_2 + 2M = 2{\cdot}4004 \times 10^8 \text{ cm},$$

subject to a very small correction for curvature.

L was also determined experimentally. A full account of the determination is given in XIV. 39 ... 41 above. The final result thus found was $L = 2{\cdot}4052 \times 10^8$ cm.

The currents in the ring were allowed for as follows. Putting $\tan\mu$ for MG/AH as at p. 621 above, and A', G', L', R, for the quantities depending on the ring and corresponding to A, G, L, R, we have from (24)

$$\tan\phi + \tau\frac{\phi}{\cos\phi} = \tfrac{1}{2}\,\frac{GA\omega}{R^2+\omega^2L^2}\,(R + L\omega\tan\phi + R\tan\mu\sec\phi)$$
$$+ \tfrac{1}{2}\,\frac{G'A'\omega}{R'^2+\omega^2L'^2}\,(R' + L'\omega\tan\phi + R'\tan\mu\sec\phi) \quad \text{............(33)}$$

if the wire circuit is closed. If the wire circuit is open and the speed is the same

$$\tan\phi_0 + \tau\frac{\phi_0}{\cos\phi_0} = \tfrac{1}{2}\frac{G'A'\omega}{R'^2+\omega^2L'^2}(R' + L'\omega\tan\phi_0 + R'\tan\theta\sec\phi_0). \quad\ldots(34)$$

Putting $\tau\tan\phi$ for $\tau\phi/\cos\phi$, and $\tau\tan\phi_0$ for $\tau\phi_0/\cos\phi_0$, neglecting the terms multiplied by $R'\tan\mu$, and subtracting, we get after reduction

$$\tan\phi - \tan\phi_0 = \tfrac{1}{2}\frac{GA\omega}{(R^2+\omega^2L^2)(1+\tau)}(R + L\omega\tan\phi$$

$$+ R\tan\mu\sec\phi)\left(1 + \frac{L'\omega}{R'}\tan\phi_0\right). \ldots\ldots\ldots\ldots(35)$$

Thus the effect of L' would be to increase the deflections at high speeds beyond their proper values, whereas that of L is to diminish them. The value of L/R for the wire circuit was ·01 second: for the ring L'/R^2 was no doubt much less, and further $\omega\tan\phi_0$ at the highest speed was only 1/26. The last factor of the expression on the right of (35) may be taken as unity. Hence R is given by (32) above with $\tan\phi - \tan\phi_0$ used instead of $\tan\phi$, (but $\sec\phi$ left unchanged), and

$$U = (2L/G_1A)\{2L/G_1A - \tan\phi/\tan\phi - \tan\phi_0)\},$$

where G_1 denotes $G/(1+\tau)$.

30. Mode of carrying out observations. With regard to the observations, the general mode of carrying out the work and correcting the results was the same as in the former investigation. An auxiliary magnetometer was used as before to trace changes of declination; and the speed and deflections were read off as formerly. For the highest speed it was found that

$$\tan\phi_0/\tan\phi = 7{\cdot}81/439{\cdot}41,$$

and this with the value of G_1A stated above gave $\log_{10} U = {\cdot}84325$.

The standard coil was kept immersed in water the temperature of which was observed, and the temperatures of the air were also observed in the neighbourhood of the copper coil, and near the standard tuning fork by which the frequency of the speed-measuring fork was determined.

Comparisons of the resistance of the copper coil with the platinum-silver coil were made before and after each set of spinnings. The resistance of the copper circuit was equal to that of the standard coil + or − the resistance of the bridge wire required for balance.

A specimen set of readings is here given with the necessary corrections. The first set of six were made with the wire circuit open, the second set with it closed. The spins were successively in opposite directions, as indicated by the signs −, +.

	No. of spinning.	Time.	Magnet reading corrected by auxiliary magnetometer.	Diff.	Mean deflections.
		H. M.			
Wire circuit open	1 –	8 16	593·38	10·48	5·29
	2 +	8 18	603·86	10·45	
	3 –	8 20	593·41	10·69	
	4 +	—	604·10	10·65	
	5 –	8 23	593·45	10·60	
	6 +	8 25	604·05		
Wire circuit closed	7 +	8 45	901·58	605·47	302·56
	8 –	8 47	296·11	605·43	
	9 +	8 50	901·54	605·12	
	10 –	8 52	296·42	604·91	
	11 +	8 55	901·33	604·77	
	12 –	8 58	296·56		

31. Correction of results and value of B.A. unit deduced. The resistance of the standard – the resistance of the copper circuit expressed in terms of the resistance of one division of the bridge wire as unit, was 212 at the beginning of the second six observations, and – 316·5 at the end, giving a mean of – 52 during the interval. But each division of the bridge wire was about 1/480000 of the whole resistance of twenty-four ohms, so that if balance had been obtained on the average at the middle of the bridge wire the deflection would have been 302·59.

Again the temperature of the standard during the experiments had a mean value of 10°·025, so that the resistance of the standard which for this series was taken as normal at 13°, was below its normal value, and the deflections were too large. The variation of resistance of the standard per degree was 3 parts in 10000, so that the deflection fell to be diminished by about 2·8 parts in 3000 or by ·27.

The standard number of beats per minute between the standard fork and the electrically-maintained fork (at 17° C.) was taken as 59 during the series of observations, and in the set of observations here taken as a specimen the number of beats was 56½ per minute, so that the electrically-maintained fork was too sharp by 2½ parts in 60 × 127, 127 being very nearly twice the frequency of the latter fork, that is the speed was too great by this amount. This gives as the correction of the deflection for excess of speed – ·10.

But the standard fork which was at normal frequency at 17° was at 13°·05, and therefore vibrated more quickly than the normal rate. The amount of quickening was about 1 in 10000 per degree of difference of temperature. Thus there was a further temperature correction on the deflection of −·12.

Adding together and applying these three negative corrections, we get for the deflection which would have been obtained if everything had been in its normal state as specified 302·10.

From the series of experiments made at different speeds, it was seen that there was a tendency for the value of the resistance to rise with the speed. This would have been the effect of an under-estimate of the value of L, but as the error to account for the discrepancies at the different speeds would have had to be about 1 per cent., it was taken as more probable that there were ring currents generated which were not conjugate to those in the wire circuit. There was no doubt, however, that the true value would be obtained, no matter which of these views was taken, by applying a correction proportional to the square of the speed. This correction was calculated from two extreme speeds and applied to the results. Thus the principal series of experiments, consisting of many different sets of spinnings, gave the numbers in the following table as their final corrected result :

Speed in teeth on card.	Uncorrected resistance of standard at 13° (unit 10^9 c.g.s.).	Correction proportional to square of speed.	Corrected resistance of standard (unit 10^9 c.g.s.).
60	23·619	·006	23·613
45	23·621	·011	23·610
35	23·630	·018	23·612
30	23·638	·025	23·613
	Mean 23·627		Mean 23·612

The result of this set of experiments was taken as that with which the B.A. standards should be compared. Another series made, however, gave practically the same result, viz. $23{\cdot}618 \times 10^9$ c.g.s. units as the resistance of the standard coil at 13°.

A careful comparison of the resistance of the standard coil with the B.A. unit gave

$$23{\cdot}612 \times 10^9 \text{ c.g.s. units of resistance} = 23{\cdot}9348 \text{ B.A. units},$$

or

$$1 \text{ B.A. unit} = {\cdot}98651 \times 10^9 \text{ c.g.s. units}.$$

32. Method of Lorenz. Lord Rayleigh and Mrs. Sidgwick made a determination of the value of the B.A. unit of resistance by the method of Lorenz [*Pogg. Ann.* 149, 1873]. A disk of metal touched near its centre and at its circumference by the terminals of a conductor was spun round its axis of figure at a uniform observed speed, in the magnetic field of a coaxial coil carrying a current. The electromotive force produced in the circuit thus formed was balanced by the difference of potential between the terminals of a resistance through which flowed the current, or a known fraction of the current producing the magnetic field.

The theory of the method is exceedingly simple. If the disk be touched at its centre, the total change in the flux of induction through the circuit in one turn is equal to the induction produced by the coil through the circular edge of the disk, or if M denote the mutual inductance of the coil and this circle, and γ the current, it is $M\gamma$. If n revolutions of the disk be made per second the electromotive force is $nM\gamma$. This is balanced by the difference of potential $R\gamma$ between the terminals of a conductor of resistance, R, and so we have

$$R = nM. \qquad \text{...(36)}$$

M is calculated from the known data of the coil and thus R is found.

If the disk is touched, not at the centre, but at a distance a from the centre, the induction M for the annular space between the edge and the circle of radius a is to be taken.

33. Lord Rayleigh and Mrs. Sidgwick's experiments. In no practical case can nM be large, and therefore R must be small, and a difficulty arises on this account in the carrying out of the method. This was overcome in Lord Rayleigh and Mrs. Sidgwick's experiments by arranging that the main current should flow along AC (Fig. 194), through a resistance a small compared with the resistance c between A and B, while at the two points B and C, including a resistance b also small compared with a, the terminals connected with the revolving disk were applied. Thus b was the resistance which was evaluated by the experiment. The connections at A, B, C were made by means of mercury cups.

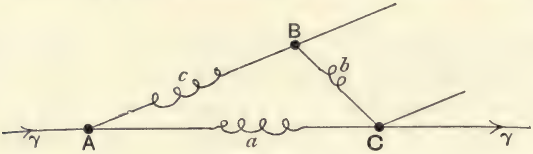

FIG. 194.

The main current being γ, and no current flowing in the disk circuit applied at BC, the current through ABC was $\gamma a/(a+b+c)$. Hence the difference of potential between B and C was $\gamma ab/(a+b+c)$. This was therefore the electromotive force generated by the motion of the disk. It will be convenient to regard it as the difference of potential produced by the current γ between the ends of a conductor of resistance $ab/(a+b+c)$.

34. Arrangement of apparatus. The pair of coils used by Glazebrook in his determination of the ohm (see above, p. 596) were employed, and were at first placed close together with the disk between them, so as to give a maximum inductive effect. The axle was mounted vertically in the frame already used for the spinning coil determinations, so that the arrangements then used for driving and measuring the speed were available also in the present case.

The diameter of the disk was about ·6 of that of the coils. This size was chosen as, on the one hand, it was not desirable to have any part of the disk near the wire, on account of the more rapid variation there of the magnetic induction, and the consequent greater importance of errors in the estimation of the radius of the coils or disk, and on the other hand too small a radius rendered the arrangement insensitive.

After some trials it was decided to make the edge cylindrical, and to make the edge contact by a brush of fine copper wires placed tangentially to the edge and amalgamated with mercury.

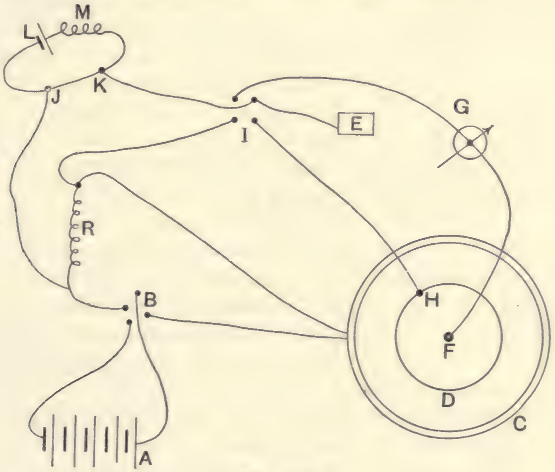

FIG. 195.

The arrangement of the apparatus is shown diagrammatically in Fig. 195. The battery *A* is connected with a mercury cup commutator *B*, by which the current can be sent in either direction through *R*. *R* is here taken as a simple conductor, but the shunt arrangement was of course used, and *R* may be taken as standing for the resistance $ab/(a+b+c)$.

The terminals *F* and *H* attached to the centre and circumference of the disk were connected with a mercury reversing key *I*, and in one of them was included a reflecting galvanometer *G*. From *I* the wires of the disk circuit proceeded to the terminals of *R*, one of them however having included in it a portion, *JK*, of a circuit containing a sawdust Daniell *L*, and a resistance coil of 100 ohms.

The latter circuit was designed to balance the effect of thermoelectric force at the sliding contacts of the brush on the disk, and the inductive effect of the earth's magnetic field in which the disk rotated, which would have given a current through the sliding contacts, thereby bringing these resistances into the account. The function of the galvanometer G was to test this balance, and that required when the disk was rotated in the field of the coil.

The battery and frame carrying the disk were insulated from the ground, and the coils insulated by ebonite supports, and for definiteness one point of the galvanometer was connected to earth at E. It was found that there was no error from leakage.

In the carrying out of the experiments the test of perfect balance of the electromotive force of the disk, together with the thermoelectric force and inductive action of the earth's field, above referred to, was absence of deflection of the galvanometer needle when the battery current was reversed. It was not however thought desirable to seek accurate balance, but to make observations of the effect on the galvanometer reading of reversal of the battery current with a resistance R_1, very little different from that (R) needed for balance. After a series of readings had been taken, R_1 was changed to R_2, which was such that the same reversal of the current was accompanied by a galvanometer deflection of opposite sign to the former. The two series of results gave R by interpolation.

To eliminate progressive change in the battery electromotive force, the observations for R_1 were interspersed with those for R_2. As soon as each series of results had been obtained for one direction of driving, the driving cord was reversed and a similar series of observations made. The speed of rotation was found by the stroboscopic method [24 above].

Preliminary trials proved that the shunt arrangement represented in Fig. 194 was faulty. The pieces dipping into the cup C were moved from day to day to verify the contacts, and the fact was overlooked that as the main current also traversed C, a small change in the positions of the contacts might make a considerable difference. For any uncertainty, even of very small absolute amount, would affect both a and b, which were small, and therefore seriously also $ab/(a+b+c)$.

35. Shunt arrangement for balancing e.m.f. of disk. The arrangement shown in Fig. 196 was accordingly adopted. Two cups, A, D, were connected by two 1 unit coils, through which the main current flowed, while two other mercury cups, B, C, received the galvanometer terminals of the disk circuit. C was connected with D by a stout rod of copper. A resistance box E was placed as a shunt across A to enable the resistance of the shunt to be adjusted.

Two series of results were taken with the coils close together, and a third series with the coils separated to a position in which the disk, midway between them, was so situated that the induction through it was as nearly as possible independent of variations of the mean radius of the coils. That there was such a position is clear from the fact that, for given values of the radius of the disk and the distance of the plane of the disk from the mean plane of either coil, the induction is zero, both when the mean radius of the coil is 0 and when it is infinite. Hence there was some value of the mean radius of the coils for which the induction was a maximum, and at which therefore the rate of variation of M with change of mean radius was zero.

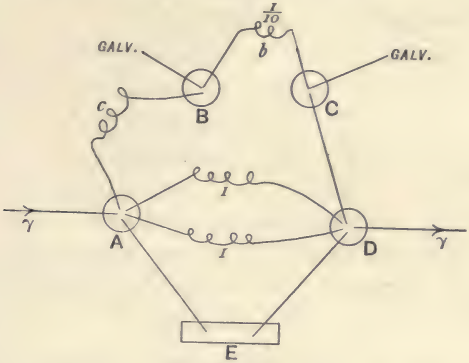

Fig. 196.

For this purpose the coils were separated by distance-pieces of proper size; and to eliminate uncertainty as to the position of the mean planes relative to the bobbins, after one set of observations had been completed, the bobbins were reversed on the distance-pieces, and another set of observations taken.

The dimensions of the coils are given above (p. 596), and the distance of their mean planes apart in the close position was 3·275 cm. In the separated positions the distances apart of the mean planes were 30·681 cm and 30·710 cm respectively.

The diameter of the disk was measured by callipers, and its circumference by a steel tape. It was found that the edge was slightly conical, and it was estimated that the mean diameter at the contact of the brush was 31·072 cm. The other contact was made at the shaft, and the diameter of the circle of contact there was 2·096 cm.

36. Calculation of mutual inductance of coils and disk. The coefficient of mutual induction was calculated first by the elliptic integral formula (by aid of the tables given in the Appendix) for two circles of radius equal to the mean radii of either coil and disk, and at a distance apart equal to the distance of the mean plane of the coil from that of the disk. Then the cross-section of the coil was taken into account by the formula of quadratures given above (p. 434).

If a, a', be put for the radii of the coils and disk respectively, and x for the distance apart of the mean plane of the coil and of contact on the disk, $2b$ and $2d$ the axial breadth and radial depth of the coils, and $M(a, a', x)$ the result for the two circles, the results per turn of wire were as follows :

COILS NEAR TOGETHER.

$a = 25{\cdot}760$ cm $\qquad a' = 15{\cdot}536$ cm $\qquad x = 1{\cdot}637$ cm

$b = {\cdot}948$ cm $\qquad d = {\cdot}955$ cm

$$\begin{aligned} M(a, a', x) &= 215{\cdot}4674 \\ M(a+d, a', x) &= 205{\cdot}1917 \\ M(a-d, a', x) &= 226{\cdot}9835 \\ M(a, a', x+b) &= 211{\cdot}7246 \\ M(a, a', x-b) &= 217{\cdot}5972. \end{aligned}$$

Adding to twice the first of these values the sum of the others, and taking $\frac{1}{6}$ of the result, the average value of M for one turn of wire was given by

$$M = 215{\cdot}405.$$

When the coils were separated by the insertion of distance-pieces, so that $x = 15{\cdot}3472$ cm, without change of the other data, the corresponding values found were

$$\begin{aligned} M(a, a', x) &= 110{\cdot}9240 \\ M(a+d, a', x) &= 111{\cdot}2573 \\ M(a-d, a', x) &= 110{\cdot}2442 \\ M(a, a', x+b) &= 104{\cdot}5571 \\ M(a, a', x-b) &= 117{\cdot}6579, \end{aligned}$$

which gave (again for one turn)

$$M = 110{\cdot}926.$$

The effect of errors in the measurement of a, a', and x can be estimated by the formula

$$dM = \frac{\partial M}{\partial a} da + \frac{\partial M}{\partial a'} da' + \frac{\partial M}{\partial x} dx,$$

conjoined with $$\frac{a}{M}\frac{\partial M}{\partial a} + \frac{a'}{M}\frac{\partial M}{\partial a'} + \frac{x}{M}\frac{\partial M}{\partial x} = 1,$$

which holds because the expression for M is homogeneous in a, a', x. Writing the last equation in the form

$$\lambda + \mu + \nu = 1,$$

we have for the first $$\frac{dM}{M} = \lambda\frac{da}{a} + \mu\frac{da'}{a'} + \nu\frac{dx}{x}.$$

Now we may take it that approximately

$$\lambda = \frac{M(a+d, a', x) - M(a-d, a', x)}{2d} \frac{a}{M},$$

and similarly for μ, ν.

Thus for the case of the coils near together

$$\lambda = -1{\cdot}36, \quad \mu = -{\cdot}02, \quad \nu = 2{\cdot}38,$$

and for that of the separated coils

$$\lambda = {\cdot}123, \quad \mu = -{\cdot}956, \quad \nu = 1{\cdot}833.$$

In the former case an overestimate of the mean radius would lead to an underestimate of M, and *vice versa*, while the reverse would be the case for the coils so far apart as here indicated. There must of course be a distance apart of the coils for which the effect of an overestimate or underestimate of mean radius would be zero to the first order of small quantities.

In the former case here specified the importance of an error in the estimation of a is of rather more than half the importance of an equal proportional error in x, while an error in the estimation of a' is relatively unimportant. On the other hand, by the separation of the coils the importance of an error in a is diminished to about 1/11 of its former amount, while that of an error in a' is enhanced. The numbers show that the separation had been carried rather beyond its proper amount.

From the values of M in both cases had to be subtracted the part, M_0, say, corresponding to the small circle touched by the inner brush. The area of this circle was $\frac{1}{4}\pi \times 2{\cdot}096^2$; and therefore taking the magnetic force at the centre of the disk due to unit current in the coil of mean radius a as a sufficiently near approximation to the average induction over this circle, we get

$$M_0 = \frac{2\pi a^2}{(a^2+x^2)^{\frac{3}{2}}} \times \tfrac{1}{4}\pi \times 2{\cdot}096^2.$$

This was equal to ·836 in the first case, and to ·534 in the other.

37. Comparison of absolute resistance with B.A. unit. The resistances, the arrangement of which is shown in Fig. 196, were the same in all three series of experiments. The coil b was of German silver and had a resistance of $\frac{1}{10}$ unit nearly, the resistance, a, between A and D was made up of two standard single units, and a resistance of 7 or 8 B.A. units from the box and all placed in parallel.

In the first series of experiments c was a [10], in the second [10] + [5] + [1], and in the third series [10] + [5] + [5′]. The resistances of the single units were already known, the others, that is the [10], [5], [5′], [$\frac{1}{10}$], had to be carefully compared with standard B.A. units. The [5]'s were compared by comparing first one of them with 5 units in series, and then the two [5]'s with one another; afterwards the sum of the two [5]'s was compared with the [10], the value of which was found by a special process.

Three German silver coils of about 3 units each wound on the same tube, had their ends arranged so that they could by mercury cups be put either in parallel or in series, and a change made in a very small interval of time from one arrangement to the other. In parallel they were compared with a standard [1], and found to have a resistance $1+\alpha$. The arrangement was now rapidly changed to series, and the resistance became very nearly $9(1+\alpha)$. The standard unit was now added, and the resistance became $10+9\alpha$. This was compared with the [10], the value of which was to be found. If there was a difference β, then $[10]=10+9\alpha+\beta$.

The [1/10] was determined as follows. Two standard units, the [10] and the [1/10], were joined as shown in Fig. 197 as a Wheatstone

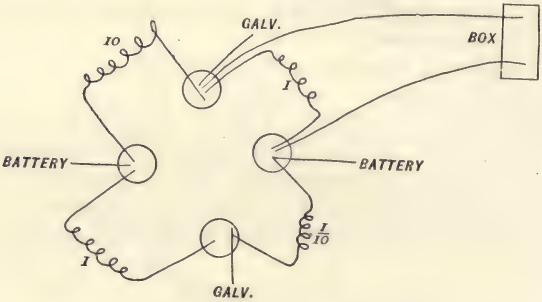

FIG. 197.

bridge, in which the battery and galvanometer terminals were, as shown, brought into direct contact with those of the [1/10] in the mercury cups. A resistance box containing coils up to 10000 was placed in parallel with one of the units to enable the latter to be adjusted to balance with all necessary accuracy. The four coils were so nearly in proportion that a resistance of several hundred units was required from the box to give balance, so that the delicacy of the arrangement was very great.

38. Specimen set of results. As a specimen of the results showing the mode of applying the various corrections the table of results given for the second series of experiments with the coils near together is reproduced on the opposite page.

Coils Near Together.

Speed of Disk about 8 Revolutions per Second.

Approximate Resistances $a=\frac{1}{2}$, $b=\frac{1}{10}$, $c=16$.

Date.	Effective resistance (B.A. units) used.	Difference of reading of galvanometer on reversal of current.		Effective resistance in B.A. units corresponding to zero difference in galvanometer.		Correction for change of speed of fork.				Effective resistance (B.A. units) as finally corrected.		Mean of effective resistance for both directions of rotation.
Aug. 1882.		Rotation +	Rotation −	Rotation +	Rotation −	Beats between forks.	Correction to 72 beats.	Temperature of standard fork.	Correction to 16°.	Rotation +	Rotation −	
7th	·0027827 ·0028126	− 8·2 +20·1	+ 9·4 − 21·5	·0027914	·0027918	73	− ·0000004	17°·6	+ ·0000005	·0027915	·0027919	·00279170
8th	·0027821 ·0028120	− 8·6 +21·0	+ 9·5 − 19·1	·0027908	·0027920	73	− ·0000004	17·2	+ ·0000004	·0027908	·0027920	·00279140
9th	·0027826 ·0028125	− 7·9 +19·5	+ 7·5 − 19·1	·0027912	·0027910	72	0	17·5	+ ·0000005	·0027917	·0027915	·00279160
									Means	·00279133	·00279180	·00279157

39. Final results of experiments. The first series gave

$$R = \cdot 00443407 \times 10^9 \text{ B.A. units};$$

hence the ratio of the B.A. unit to 10^9 c.g.s. units of resistance being x, the absolute value of R was $x \times \cdot 00443407 \times 10^9$ c.g.s. But the value of M was M_1 multiplied by the number of turns in the coil (1588), and n the number of revolutions per second $= 2 \times$ frequency $\div$ number of teeth stationary on the stroboscopic card. Hence by (36) for the first series, since

$$n = 128 \cdot 407/10,$$

$$x \times \cdot 00443407 \times 10^9 = 12 \cdot 8407 \times 214 \cdot 569 \times 1588$$

or

$$x = \cdot 98674.$$

The second series gave, since for it

$$n = 129 \cdot 340/16 \quad \text{and} \quad R = \cdot 00279157 \times 10^9,$$

$$x = \frac{214 \cdot 569 \times 1588 \times 129 \cdot 340}{\cdot 00279157 \times 10^9 \times 16} = \cdot 98669.$$

In the third series $n = 129 \cdot 340/10$, and $R = \cdot 00229762 \times 10^9$, so that from it

$$x = \frac{110 \cdot 392 \times 1588 \times 129 \cdot 340}{\cdot 00229762 \times 10^9 \times 10} = \cdot 98683.$$

Taking the mean of the first two results, and, giving it the same weight as the last, Lord Rayleigh found as the final result of the investigation,

$$1 \text{ B.A. unit} = \cdot 98677 \times 10^9 \text{ c.g.s.}$$

With the value of the specific resistance of mercury in terms of the B.A. unit found by Lord Rayleigh and Mrs. Sidgwick, this gives 1 ohm = resistance at 0° C. of a column of mercury 106·214 cm long and 1 sq. mm in cross-section.

40. Absolute determination of sp. resistance of mercury. A carefully planned and executed determination by Lorenz's method was made in 1891 by Prof. J. V. Jones,* of Cardiff, who used in the construction of his apparatus the most accurate obtainable engineering appliances. The disposition of the apparatus is shown in Figs. 198, 199.

The standard coil consisted of a single layer of double silk-covered wire, ·02 in in diameter, wound on a cylinder of brass about 10·5 in in radius, in a screw thread of pitch ·025 in. This cylinder was very carefully turned, and the screw thread cut on an accurate Whitworth lathe, and great care was taken to test the figure of the cylinder after it was finished. It was found that the cross-section of the cylinder, instead of being circular, was always slightly oval, however many cuts were made over its surface, showing apparently an effect of internal stresses.

41. Adjustment of parts of apparatus to position. After the screw had been cut the mean plane of the coil was determined for the after placing of the disk in the following manner. The slide-rest of the lathe was made to carry a *V* tool, and a microscope, so adjusted that the image of the point of the tool was seen exactly at the centre of the graduated plate in the focal plane of the eyepiece. When the slide-rest was moved along the bed, the tool passed inside the cylinder while the microscope remained outside. The guide-screw of the slide-rest (of pitch ·25 in) was turned by a wheel 9·75 in in radius divided into 360 parts, and it was possible to estimate the position of the wheel

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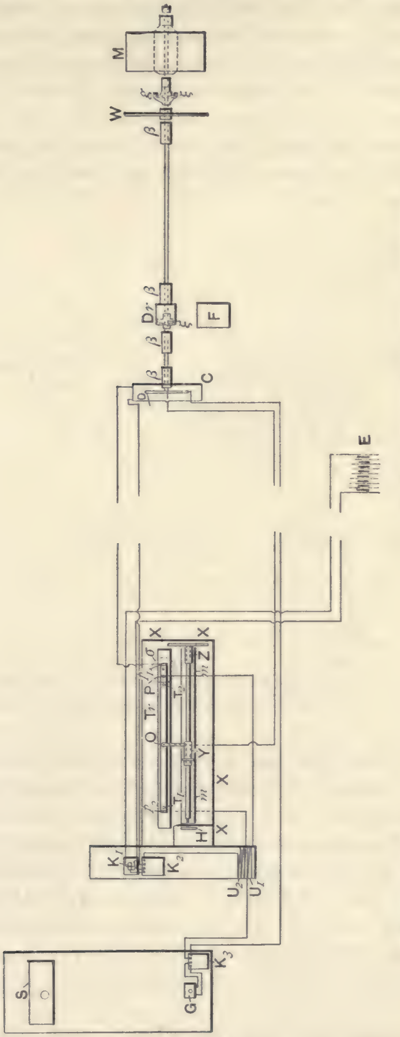
* *Phil. Trans.* 1891, A.



C. Standard coil.
D. Disk.
Dr. Stroboscopic cylinder.
β, β, β. Bearings.
Tr. Mercury trough.
T_1, T_2. Thermometers in mercury.
O. Movable electrode.
P. Fixed electrode.
E. Battery.
G. *S*. Galvanometer, lamp, and scale.
M. Electromotor.
ξ. Flexible coupling.
W. Fly wheel.
m, *m*. Measuring machine.
Y, *Z*. Loose and fixed headstock on measuring machine.
K_1, K_2, K_3. Keys.
XXXX. Box surrounding trough and measuring machine.
U_1, U_2. Adjusting coils.

FIG. 198.

to 1/10 of a division. By drawing, then, a generating line along the cylinder, and reading on this wheel the position of the microscope when the ridges of the first and second threads on this line were focussed in the field of view, then running the microscope along the generating line, and taking in like manner the readings for the last ridge and last ridge but one, the reading for the mean plane could be at once found. The mean of the first two readings subtracted from the mean of the last two gave obviously the distance between the first hollow and the last, and half the sum of these two means therefore gave the required reading. The tool was then moved to this position by the wheel and guide-screw, and a cut made round the inside of the cylinder at the plane thus found.

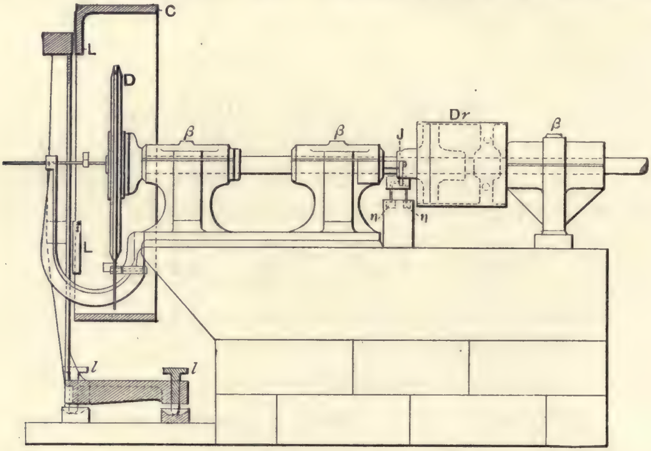


FIG. 199.

At the intersection of the first and last hollows with this generating line small holes were bored radially through the brass of the cylinder, and were bushed with paraffined ebonite to receive the ends of the wire. The wire was secured at one end in the hole there, and was then laid on in the screw-thread by the lathe, under uniform tension given by a weighted pulley. The ends of the wire were secured by melted paraffin run into the bushes, and blocks of ebonite attached to the cylinder at the ends of the generating line, on which the coil began and ended, carried binding screws, to which the ends of the wire were soldered.

The disk was insulated from the axle by ebonite, and was fixed coaxially as described below in the mean plane of the coil. It was driven by an electromotor coupled direct, and was rotated in position and ground true by an emery wheel driven rapidly by an electromotor. Its diameter was measured by a Whitworth measuring-machine. This consisted of a graduated bed carrying two headstocks, one fixed, the other movable along the bed by a guide-screw turned by a divided wheel. The distances used on the bed were compared with a standard scale.

A side view of the coil, disk, stroboscopic cylinder, etc. (for explanation of reference letters see Fig. 198), is given in Fig. 199, and an end view showing the disk and edge-brush, *Q*, in Fig. 200.

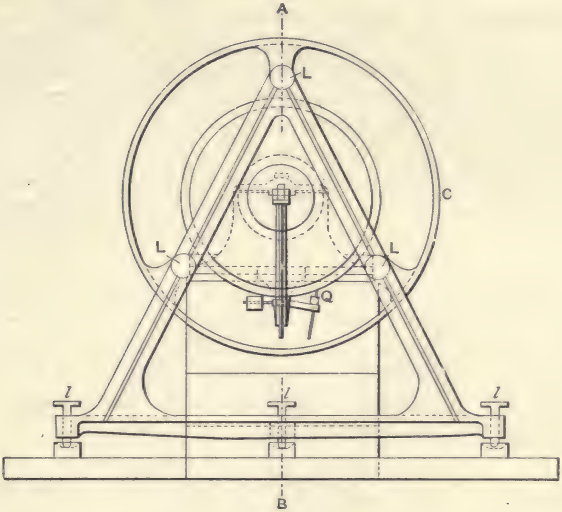


FIG. 200.

42. Contact brushes: measurement of speed. The brush finally adopted for the edge of the disk was a single wire perforated by a channel, through which was supplied a small stream of mercury. A piece of copper an inch long was drilled to a depth of $\frac{3}{4}$ in, to meet another hole at right angles, which received the phosphor wire brush. The perforation drilled along the wire of the brush was connected with that in the copper piece, and an india-rubber tube slipped over the free end of the latter kept up a constant supply of mercury. This gave a constantly fresh surface for contact. The central brush was fed also with mercury but more slowly.

The speed of driving was measured by the stroboscopic method by observing one of a set of rows of teeth, marked round a cylinder, through slits in brass plates attached to the prongs of a tuning fork, which vibrated at right angles to the circles of teeth. The fork was bowed, not electrically maintained: the number of turns per second n was given as in Lord Rayleigh's experiments by $n = 2f/N$, where f is the frequency, and N the number of teeth in the stationary circle.

The pitch of the fork was determined by driving the cylinder; keeping a row of teeth stationary, and causing the cylinder by means of a lever to make and break a battery circuit every revolution, so that for about half the time of revolution the contact was made and for the other half broken. This registered on a telegraph tape a series of alternate dashes and spaces, and on the same tape a mark was made once a second by the laboratory standard clock. The observations being continued over three or four minutes, N and n were obtained with accuracy, and f was deduced by the equation $f = \frac{1}{2}nN$.

43. Arrangement of mercury column. The resistance used for balancing the electromotive force of the disk was a column of mercury, so that the experiment gave the specific resistance of mercury directly. The mercury was placed in a long rectangular trough, Figs. 198 and 201, carefully cut, as described below, in paraffin by machinery, and two electrodes dipped into the mercury at some distance from the ends of the trough. One of these electrodes was kept fixed, the other was attached to the movable headstock of the Whitworth measuring-machine, by which its position was altered by the difference of distance between the electrodes necessary for two different speeds of the disk. Thus the difference only of two distances between the electrodes (and this could be obtained with accuracy) was used in deducing the final result. For if n_1, n_2, be the two speeds of rotation of the disk, ρ the specific resistance of mercury, A the cross-section of the column, and l the distance between the two positions of the electrodes, the resistance of the column between the two positions was

$$\frac{l}{A}\rho = M(n_1 - n_2). \quad \text{..................................} (37)$$

The capillary depression at the sides of the trough was allowed for by taking observations for two different depths of mercury in the trough. For if ΔA be the change in area produced by increasing the depth from h to h', n_1, n_2, l the speeds of rotation and difference of lengths of column in the first case, n'_1, n'_2, l' those in the second, then we have by (37), assuming that the groove is true and the temperature the same in both experiments,

$$A = \frac{l\rho}{M(n_1 - n_2)},$$

$$A + \Delta A = \frac{l'\rho}{M(n'_1 - n'_2)},$$

and therefore

$$\Delta A = \frac{\rho}{M}\left(\frac{l'}{n'_1 - n'_2} - \frac{l}{n_1 - n_2}\right),$$

or since $\Delta A = b(h' - h)$, where b is the breadth of the trough and h', h the two depths,

$$\rho = \frac{Mb(h' - h)}{\dfrac{l'}{n'_1 - n'_2} - \dfrac{l}{n_1 - n_2}}. \quad \text{............................} (38)$$

The trough (shown in section in Fig. 201) was cut in paraffin wax melted in a longitudinal groove left in a strong casting of iron. The wax was melted in the groove and allowed to solidify on the surface, after which melted wax was poured through a hole in the crust to the interior in order to obtain a perfectly homogeneous mass. A channel was then cut and covered with a thin layer of paraffin to fill up air-holes, after which it was recut and scraped true.

A length of 10 in of the trough was used in the experiments, and this was carefully calibrated by internal callipers of special construction.

The position of the surface of the mercury was determined by placing a spherometer in a fixed position over the trough and screwing down the movable point until contact was indicated by the completion of a battery circuit through the mercury and point. The division on the head of the micrometer corresponded to 1/5040 in, and the size of the head allowed of an estimation of tenths of a division. Successive measurements did not differ by more than 1/20000 of an inch when the point was kept clean by being carefully wiped with filter paper, and sparking was prevented as far as possible by including a large resistance in the circuit and breaking the circuit before removing the point from the mercury after a reading.

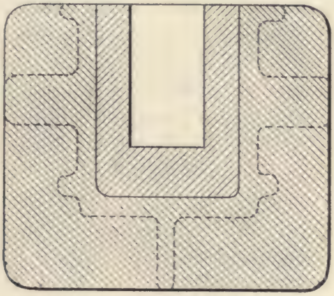


FIG. 201.

The temperature of the mercury in the trough was determined by two thermometers, one at each end of the trough. A third thermometer was placed between the prongs of the speed-measuring fork. These thermometers were corrected by comparison at Kew.

To prevent warping of the trough by change of temperature, and to make as certain as possible that the mercury in contact with the poorly conducting wax should be all at one temperature, the temperature was kept as nearly constant as possible by enclosing the trough, etc., in a wooden box covered with felt paper, and protected round about with felt curtains. The thermometers were read through windows in the box by lifting the curtain.

The galvanometer used to test for balance was a Thomson reflecting galvanometer of 0·968 ohm resistance, the needle of which was carried by a quartz fibre 13 in long.

44. Adjustment of the disk in position : observations. The axis of rotation was placed at right angles to the magnetic meridian, so that the plane of the disk might be in the meridian and thus avoid any current due to earth induction. When the disk was rotated without current in the standard coil any displacement of the light spot could be annulled by a slight movement of a compensating magnet on the table.

The bearings of the disk were made as nearly as possible perfectly true, and were each provided with a sight-feed lubricator. The disk was adjusted in position in the coil by arranging an arm to fit upon the disk so that a carefully scraped face on the arm should be a prolongation of the mean plane of the disk. The coil was then placed in position so that the outside edge of this face should travel round the interior circle cut in the mean plane of the coil as already described.

The mercury trough was carefully levelled and adjusted parallel to the bed of the measuring machine. The last adjustment was made by attaching to the movable headstock a cylinder projecting vertically downwards into the trough, running the headstock from end to end and testing at the extremities the distance from the cylinder to the same side of the groove by pushing a wooden wedge lightly between them. Further, by making the wedge-reading the same on both sides of the cylinder, the headstock was adjusted so that when an electrode was substituted for the cylinder it dipped into the medial plane of the mercury column.

A slight direct effect on the needle produced by the current was observed, and was compensated by placing a coil of three turns of the battery wire close to the needle.

The insulation of the wire of the coil from the bobbin and of the disk from the axle were tested and found satisfactory.

Lord Rayleigh's plan (XV. 34 above) of taking two sets of galvanometer readings for each equilibrium position was followed. One set gave the change of galvanometer reading for reversal of current when the resistance was slightly below that required for balance, the other set the corresponding change when the resistance was a little above the proper current. To eliminate uncertainties owing to variations of speed and of the brush contacts, a number of reversals were quickly taken for each resistance and combined to give a mean result. The readings were taken without waiting for the needle to come to rest, but elongations were observed which with a previously determined damping coefficient enabled the position of rest to be calculated.

45. Reduction of results. The dimensions of the coil and disk, and the calculation from them of the mutual inductance, M, are given in Chapter XII. above, which is devoted to calculation of inductances, and so are not repeated here. It only remains to state the mode of reduction of the observations and the final result.

If ρ_t and A_t be the specific resistance of mercury and the cross-section of the column at temperature t, and ρ, A, the same quantities at 15°·5, to which the results were in the first instance reduced, L the distance between the electrodes in any equilibrium position, then

$$Mn = \frac{L\rho_t}{A_t}.$$

Now if f be the frequency of the fork at the standard temperature 15°·5, and f_θ the frequency at temperature θ, we have

$$n = \frac{2f_\theta}{N} = \frac{2f\{1 + k(\theta - 15^\circ{\cdot}5)\}}{N},$$

where $k(= -{\cdot}00011)$ was a temperature coefficient. Also

$$\rho_t = \rho\{1 + \alpha(t - 15^\circ{\cdot}5)\},$$
$$A_t = A\{1 + \gamma(t - 15{\cdot}^\circ 5)\},$$

where α is the temperature coefficient for the specific resistance, and γ the coefficient of cubical dilatation of mercury. Hence

$$Mn = \frac{2Mf\{1 + k(\theta - 15^\circ{\cdot}5)\}}{N} = \frac{L\rho\{1 + \alpha(t - 15^\circ{\cdot}5)\}}{A\{1 + \gamma(t - 15^\circ{\cdot}5)\}}$$
$$= \frac{L\rho}{A}\{1 + (\alpha - \gamma)(t - 15^\circ{\cdot}5)\},$$

or

$$\frac{L\rho}{A} = 2Mf\nu,$$

where

$$\nu = \frac{1 + k(\theta - 15^\circ{\cdot}5) + (\gamma - \alpha)(t - 15^\circ{\cdot}5)}{N}.$$

If now $\Delta\nu$ be the difference of two values of ν for equilibrium positions separated by an interval l,

$$\rho = 2MfAs,$$

if $s = \Delta\nu/l$.

As stated above, two observations were made with the mercury at different levels h' and h to eliminate error from capillarity. Calling the two values of $\Delta\nu/l$ for these observations s', s, and the areas of cross-section of the trough A', A, we have, if b be the mean breadth of the trough over the length used,

$$A' - A = b(h' - h),$$

so that

$$\rho = \frac{2Mfb(h' - h)}{1/s' - 1/s}, \quad\text{......................................(39)}$$

and from this the specific resistance of mercury at 15°·5 was calculated.

The coefficient α was obtained from the formula

$$R_t = R_0(1 + {\cdot}0008649t + {\cdot}00000112t^2), \quad\text{......................(40)}$$

given by Mascart, de Nerville, and Benoit for the resistance of a column of mercury at t^0 in a glass tube. Thus

$$R_{15{\cdot}5} = R_0 \times 1{\cdot}013675$$

and

$$R_t = R_0\{1 + (\alpha - \beta)t\}, \quad\text{..................................(41)}$$

where β is the coefficient of cubical expansion of glass $(=\cdot 000008)$.

Thus $$(\alpha-\beta)\times 15\cdot 5=\cdot 013675,$$

or $$\alpha\times 15\cdot 5=\cdot 013799.$$

This gave the mean value of α from 0 to 15°·5 which was used to obtain the specific resistance of mercury at 0° from its value at 15°·5. The equation of reduction was thus

$$\rho_{15\cdot 5}=\rho_0\times 1\cdot 013. \qquad\qquad (42)$$

The value of α at 15°·5 or $\alpha_{15\cdot 5}$ was obtained by calculating from (40) above

$$\begin{aligned}\frac{dR_t}{dt}&=R_0\times(\cdot 0008649+\cdot 00000224\times 15\cdot 5)\\&=R_0\times\cdot 00089962.\end{aligned}$$

But by (41) $$\frac{dR_t}{dt}=R_0(\alpha_{15\cdot 5}-\beta),$$

and therefore $$\alpha_{15\cdot 5}=\cdot 0009076,$$

which was used to correct the experimental results for the small differences between 15°·5 and the observed temperatures.

The final result of five sets of experiments gave

$$\rho=94067 \text{ c.g.s.}$$

as the resistance at 0° C. of a column of mercury one square centimetre in cross-section and one centimetre in length.

According to this result the ohm is equal to the resistance at 0° of a column of mercury 106·307 centimetres long and one square millimetre in cross-section.

46. Final result of Jones's determination. The extreme variation from the mean result was about 4 parts in 10,000. Much of this Professor Jones deemed to be due to the paraffin trough, which varied slightly in temperature. Accordingly he made a little later,[*] by this apparatus, a determination of the values in terms of the true ohm of a resistance the value of which was known in terms of the Board of Trade or International Ohm. It was made up according to Lord Rayleigh's arrangement from coils which had been carefully compared and tested by Glazebrook. The ratio of the true to the international ohm came out as a result of a very concordant series of determinations to be 106·326/106·30. Working direct in mercury Professor Jones had, as stated above, obtained in 1890 a somewhat lower result, 106·307 with a probable error of ±·011.

47. Ayrton and Jones's determination by method of Lorenz. An apparatus similar to that constructed by Viriamu Jones at Cardiff was built by Messrs. Nalder Brothers of London for the Physical Institute of the McGill University, Montreal, and advantage was taken of the experience gained with the former apparatus to effect various improvements. An account of a determination made with this apparatus was given at the Toronto Meeting of the British Association, in 1897, by Professors Ayrton and Jones.

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[*] *B.A. Rep.* 1894.

The field coil was a single layer of wire laid on in a helical groove cut on an accurately turned marble cylinder or ring, of 21 in outside diameter and 15 in inside diameter. The groove consisted of 201 turns of step 0·025 in, so that the axial length of the helix was 5·025 in. At first bare wire of mean thickness 0·02136 in was used, and the outside diameter of the coil was measured for 18 diameters inclined at successive angles of 10°, for (*a*) the front face (next the disk), (*b*) the middle section, and (*c*) the back face. The means for these were, in inches, 21·04797, 21·04784, 21·04872, giving a general mean of 21·04818 in, at the mean temperature 20°·4 C.

As the insulation was not quite satisfactory the bare wire was replaced by double silk-covered wire of mean thickness ·01914 in, which gave by the measurements already made a mean outside diameter of 21·04488 in at 20°·4 C. It was found that the coefficient of expansion of the marble was about 0·000004 per 1° C. A direct measurement of the outside diameter of the coil as rewound gave 21·04687 in as its mean value. The difference was probably to be attributed to the fact that after the rewinding the coil had been brushed over with melted paraffin wax, then sewed over with silk ribbon, and finally covered with a wide silk ribbon that had been passed through the melted wax, so that the silk covering of the wire had probably swelled a little. [The covering was carefully removed down to the silk covering of the wire to allow the measurement to be made for two diameters at right angles.]

The thickness of the wire gave 21·02773 in at 20°·4 C. as the diameter of the coil from axis to axis of the wire.

The disk was of phosphor bronze, and was ground in position so as to be exactly coaxial with the coil, and its diameter was then measured and found to be 13·01435 in at 19°·5 C. Its linear expansion coefficient had previously been found to be 0·0000125 per 1° C., so that, at 20°·4 C., its diameter was 13·01451 in. The disk was placed in the medial plane of the coil with all proper precautions as in the Cardiff instrument.

48. Mutual induction of coil and disk. The mutual induction of the coil and disk had been calculated by Mr. W. G. Rhodes for the bare wire winding and the diameter 13·01997 in of the disk, and had been found to be, in cm, 45862·3. This was corrected to 18037·51 in, or 45814·45 cm for the values 21·02773 in and 13·0451 in for the new diameters of the coil and disk. The equation of correction was found to be

$$\frac{dM}{M} = 1{\cdot}246\frac{dA}{A} + 2{\cdot}346\frac{da}{a} + 0{\cdot}0997\frac{dx}{x}$$

where $2A$ was the radius of the coil, $2a$ that of the disk, and $2x$

the axial length of the coil, all taken in inches. The data finally were thus:

Diameter of coil, $2A$ $= 21{\cdot}02773$ in.
Diameter of disk, $2a$ $= 13{\cdot}0451$,,
Axial length of coil, $2x = 5{\cdot}025$,,
Number of convolutions 201

$$M = 18037{\cdot}51 \text{ in.}$$
$$= 45814{\cdot}45 \text{ cm.}$$

49. Arrangement of brushes: result obtained. The contact with the edge of the disk was made by three small phosphor bronze tubes, 120° apart, placed tangential to the edge and lightly pressed on it, and conveying a small stream of mercury by which the contact was made continuous and certain; the contact with the centre was a single tube (also supplying a small stream of mercury) which projected into an axial hole of 0·144 in in diameter. The mercury which dropped from the central brush was kept off the disk by an ebonite bar cemented to the face of the disk. The value of M was thus reduced to 45809·95 cm.

The electromotive force of the revolving disk was balanced by a derived current from the coil circuit as in Lord Rayleigh's determination. The coils used were those which had been used in the Cardiff determination, and had been carefully compared with standards by Glazebrook in 1894 and again in 1896 at the Board of Trade Standardizing Laboratory.

The result of nine determinations made after everything had been got into working order was to give the result

$$1 \text{ Board of Trade Ohm} = 1{\cdot}00026 \text{ true ohm.}$$

It may be recalled that the ohm [10^9 cm/sec] was, by an Order in Council of date August 23, 1894, taken to be represented by the resistance of a uniform column of mercury 14·4521 grammes in mass and 106·3 cm in length. This received the name Board of Trade Ohm; the specification was based in the main on at least seven closely concordant determinations by Mascart, Rowland, Kohlrausch, Glazebrook, Rayleigh, Wuilleumeier, and Duncan and Wilkes, and Mrs. Sidgwick's determination.

50. Apparatus of National Physical Laboratory. In his paper to the British Association in 1893 Viriamu Jones made various suggestions as to the corrections to be made in discussing the results of a determination of resistance by the method of Lorenz, and stated that in his opinion a well-constructed Lorenz apparatus, in a national laboratory, would prove to be the best ultimate standard of electrical resistance. In 1900 the Drapers' Company of London expressed to him their willingness to undertake the construction of such a standard. Professor Jones, however, died in 1901, and thereafter the Company made a grant of £700 in his memory, to enable the apparatus to be made at the National Physical Laboratory, under the superintendence of Professor Ayrton and Dr. Glazebrook.

Professor Ayrton died in 1908, before the design of the national instrument had been completed; he had, however, given much attention to the subject in connection with the Montreal instrument, and had spent much time in the work of constructing and testing the Ayrton Jones Current Balance (see Chapter XII. above), which was not finished until 1907.

A full account of the new instrument is given in a paper by Mr. F. E. Smith in the *R. S. Transactions* 214 (1914). The design as finally adopted differed considerably from the Cardiff and Montreal instruments. The arrangement is shown diagrammatically in Fig. 202. It was decided to use two disks, *D*, *D*, on two aligned shafts connected by a coupling as there shown. These were arranged to run in the fields of two double coils, and to be driven by a motor and fly-wheel as shown to the left of the diagram. The diameter of a disk was 53 cm, that of the shaft 5 cm.

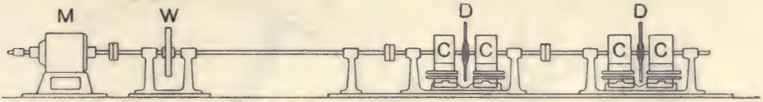

FIG. 202.

M, motor; *W*, fly-wheel; *DD*, discs; *CCCC*, coils on cylinders.

51. Modification of method of Lorenz. Round the edge of each disk were ten circular segments of phosphor bronze mounted on the insulating substance called stabilit [Fig. 203]. To each segment was attached one end of a copper wire which travelled round a groove cut in the stabilit, until a radial channel was reached by which it was led to the shaft. In this passage were collected thus all the wires, to be carried to the shaft, which they entered by a brass tube and radial hole, to be then carried along an axial boring to the coupling midway between the disks, on which were arranged ebonite blocks furnished with terminals for the wires. Each wire was of No. 26 gauge, double silk covered, insulated with a coating of shellac, within a tube of silk.

Ten wires were thus arranged for each disk, and brought to terminals at the coupling by which they can be made into ten wires passing from the edge of one disk to the edge of the other.

The segments are touched by brushes of phosphor bronze wire, five on each disk. These brushes were formed somewhat like violin bows, with the hair replaced by the bronze wire, wound round two screw-threaded rods across the bow near the ends, and capable of being tightened by screws after the manner of an ordinary bow. Their form and arrangement are indicated in Fig. 203. The ten brushes (five on each disk) can be joined in two ways (*a*) so as to put the successive wires from disk (1) to disk (2), and from disk (2) to disk (1), and so on alternately, in series, (*b*) so that the brushes are arranged in two sets of five each in parallel. They were lubricated with paraffin oil, without which the contacts were not satisfactory.

In the first case when each brush is in contact with a single segment, the differences of potential are added together for five rotating conductors; when each brush touches two segments five rotating conductors are joined in parallel, but each now consists of the wires from two neighbouring segments in parallel. The differences of potential for five disk-to-disk-conductors are thus added together, whether the brush contact be with one segment or with two.

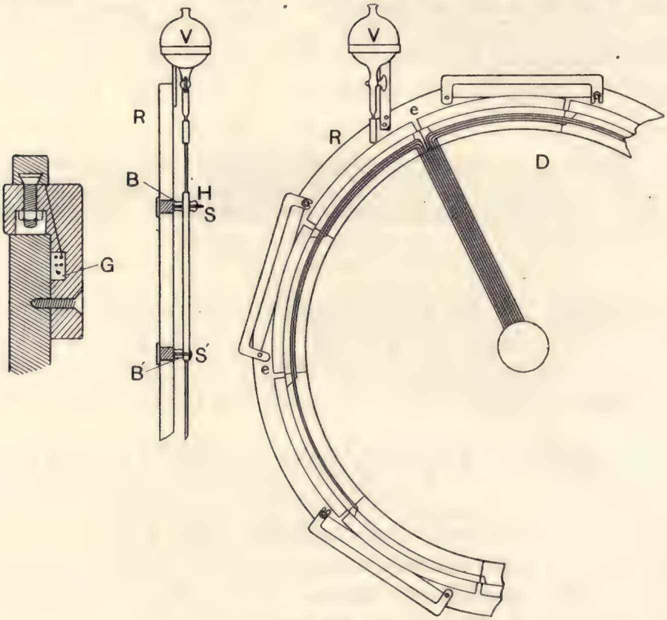

FIG. 203.

In the other case the difference of potential is simply that for a single conductor.

The function of the disks is simply to support the wires and carry them round in the rotation. The wires thus cut the lines of force of the fields of the coils, which are joined so that the two fields are oppositely directed, as shown in the diagram of Fig. 204. As stated above, the wires come radially along a passage in the side of each of the disks to the shaft; but it will be seen that a wire passing anyhow from *A* to *B* will have the same electromotive force produced in it by the rotation, inasmuch as it cuts across the same lines however it may be situated.

52. Construction and tests of materials, etc. The apparatus has been very strongly and exactly made, and is supported on massive foundations. All the materials were very carefully tested for strength, and for absence of magnetic matter, and only specimens which passed the tests satisfactorily were used.

The coils were single layers of bare wire wound in screw threads cut on hollow cylinders of what is called "First Statuary" Carrara marble, which was found to insulate well and to have negligible magnetic susceptibility. The cylinders were selected carefully, and are free from flaws and cavities. The coefficients of linear expansion of the marble was found by comparison with an invar bar to be $5{\cdot}0 \times 10^{-6}$.

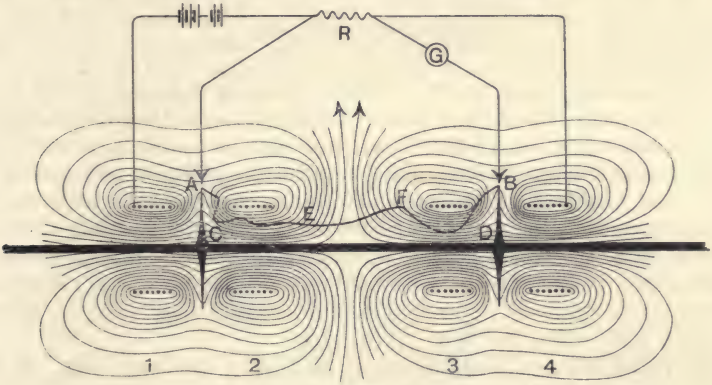

FIG. 204.

Each coil was about 36 cm in diameter, and had an axial length very slightly over 16 cm. For the exact dimensions and other details, the reader should refer to Mr. Smith's paper (*loc. cit. supra*) for a full record of all the measurements and adjustments, which were very elaborate, and were made with all necessary precautions. The windings were tested for insulation and for variation of pitch.

53. Calculation of M and final result. The value of M was calculated by the equation (26) of Chapter VI. above, namely

$$M = \Theta p(A+a)\beta\gamma\left\{\frac{G-H}{\gamma^2}+\frac{1-\beta^2}{\beta^2}\right\}(G-\Pi),$$

where Θ is the total angle of the helix, A the radius of the helix, a the radius of the contact circle on the disk, $\gamma^2 = 4Aa/\{(A+a)^2+x^2\}$, $\beta^2 = 4Aa/(A+a)^2$, so that γ was the modulus of the elliptic integrals G, H, Π of the first, second, and third kinds, p the pitch of the helix, and β^2 the parameter of the elliptic integral of the third kind. Details of the method of computation will be found in the Chapter referred to.

For further particulars of this investigation we have no space. The final result is given in the following statement.

The ohm (10^9 cm/sec) *is represented by the resistance at* 0° C. *of a column of mercury* $14{\cdot}4446 \pm 0{\cdot}0006$ *gramme in mass, of a constant cross-sectional area* (*the same as for the international ohm*) *and having a length of* $106{\cdot}245 \pm 0{\cdot}004$ cm. Thus the result is claimed to be correct to 4 parts in 100,000.

54. Method of Joule. Joule's method is in principle very simple. Supposing a current of strength γ to flow through a wire of resistance R for a time t, a quantity of energy $\gamma R^2 t$ is spent in the conductor. This is expressed in ergs if γ and R are taken in c.g.s. units, and t in mean solar seconds of time. If H be the heat generated in the conductor in that time, and J be the work equivalent of heat, we have

$$\gamma^2 Rt = JH \quad \text{and} \quad R = \frac{JH}{\gamma^2 t}.$$

The absolute measurement of the current might be made with sufficient accuracy, though it is of very nearly the same order of difficulty as the determination of the ohm; but there are also involved exact calorimetric determinations which require the greatest care and skill. Over and above all these is the determination of J with an accuracy equal or superior to that to which it is required to find the ohm say to 1 in 10,000. That would be a research of difficulty far transcending that of the measurement of absolute resistance by most other methods.

For descriptions of other methods the reader may refer to Wiedemann's *Elektricität*, Bd. 4, 2$^{\text{te}}$ Abtheilung.

55. Carey Foster's modification of method of revolving coil. It has been proposed by Carey Foster to modify the method of the revolving coil by rotating the coil on open circuit, and applying to its terminals, at the instant when the inductive electromotive force is a maximum, a difference of potential equal and opposite to that then existing at the terminals of the coil. This will not be exactly the instant at which the coil passes through the meridian, as on account of the capacity of the conductors a certain retardation of phase will exist.

This applied difference of potential may be that existing between the terminals of a conductor in which a current γ is flowing. The current is measured by a tangent galvanometer of principal constant G, and therefore has for absolute value $H \tan \alpha / G$; so that the applied difference of potential is $RH \tan \alpha / G$. The induced difference of potential has the value $AH\omega$ only. Assuming H to be the same for the revolving coil and the galvanometer, we have therefore

$$R = GA\omega \cot \alpha = 2\pi^2 nn'\omega \frac{a^2}{a'} \cot \alpha,$$

if a be the mean radius of the revolving coil, a' that of the galvanometer, n, n', the numbers of turns in the coils.

Thus error of measurement of the mean radius a is of twice the importance of equal proportional error in a'.

Addition to Results Collected on Page 651.

The table of collected results of the absolute measurement of resistance given on p. 651 gives the value of the ohm in cms of mercury, that is, the length of a column of mercury 1 sq. mm in cross-section which at the temperature 0° C. would have one true ohm resistance. The values quoted in the table are those given by the experimenters. If the results are given in terms of the National Physical Laboratory mercury standards of resistance, the numbers are slightly altered. They are:

1882. Rayleigh - - - - -	106·26
1883. Rayleigh and Sidgwick - -	106·24
1882. Glazebrook - - - - - -	106·29

Professor J. V. Jones repeated his determination by the method of Lorenz, using the same apparatus, but with certain coils which had been carefully measured by Glazebrook in terms of the international ohm. On the assumption that the international ohm is the resistance at 0° C. of a column of mercury of 1 sq. mm sectional area, and 106·30 cm long, the result obtained by this determination gave the true ohm as the resistance at 0° C., of a column of mercury of the same cross-section and 106·326 cms in length.

A determination was made by Albert Campbell in 1912, by comparing a resistance with a mutual inductance. Two nearly equal sine-wave alternating currents $C_1 \cos nt$, $C_2 \sin nt$ (so that they are in quadrature) flow respectively through a resistance R (known in terms of the international ohm), and the primary circuit of a variable mutual inductance. The voltages produced across the resistance R and across the secondary circuit of the mutual inductance, were balanced against one another by varying the mutual inductance and testing the balance with a vibration galvanometer. The value M of the mutual inductance was determined by comparing it with a standard mutual inductance. The resistance R is then found from this value of M, and the known value of the frequency of the alternations.

The result obtained was 106·273 cms of mercury.

Mr. F. E. Smith concludes that

The international ohm is equal to 1·00052 ± 0·00004 ohm (10^9 cm/sec), where the probable error 0·00004 is approximately the sum of those of the realization of the ohm and the international ohm.

His results are summed up in the statement:

The ohm 10^9 cm/sec is represented by the resistance at 0° C. of a column of mercury 14·4446 ± 0·0006 grammes in mass, of a constant cross-sectional area of 1 sq. mm and having a length of 106·245 ± 0·004 cm.

The numerical table of values of the quantities X_1, X_2, etc., defined in (78), XIII. 25, promised on p. 501, has been inadvertently omitted. It is given in vol. 8, No. 1, of the *Bulletin of the Bureau of Standards*, Washington, which is either in the possession of most laboratories or in an accessible library of reference.

Messrs. Grüneisen and Giebe, of the Reichanstalt, Berlin, now give as the result of researches on the value of the ohm

$$1 \text{ international ohm} = (1{\cdot}00051 \pm 0{\cdot}00003)10^9 \text{ c.g.s.}$$

The most probable value of the e.m.f. of the Weston normal cell is now given as 1·0188 volts at 20° C.

The main advantage of this method lies in the elimination of self-induction, as the current is almost zero at each instant. In its practical use error from thermo-electric force at the rubbing surfaces, and from mutual induction between the wire circuit and secondary circuits in the ring currents would have to be guarded against.

The method does not seem to have been applied to a complete determination of absolute resistance.

56. Table of collected results. The following table, mainly derived from a Report on the "Absolute Resistance of Mercury" by R. T. Glazebrook (*Brit. Assn. Report*, 1891), contains the principal results obtained since 1881 :

Date.	Observer.	Method.	Value of B.A. unit in Ohms.	Value of Ohm in Cms of Mercury.
1882	Lord Rayleigh and Schuster	Revolving Coil	·98651	106·24
1883	Lord Rayleigh and Mrs. Sidgwick	Method of Lorenz	·98677	106·21
1884	G. Wiedemann	Earth Inductor	—	106·19
1884	Mascart, de Nerville, and Benoit	Induced Currents	·98611	106·33
1887	Rowland	Mean of Several Methods	·98644	106·32
1887	Kohlrausch	Damping of Magnet	·98660	106·32
1882 and 1888	Glazebrook	Induced Currents	·98665	106·29
1890	Wuilleumeier	Induced Currents	·98686	106·27
1890	Duncan and Wilkes	Method of Lorenz	·98634	106·34
		Mean	·98653	
1891	J. V. Jones	Method of Lorenz	—	106·307
1897	Ayrton and Jones	Method of Lorenz	—	106·27
1913	F. E. Smith and others	Method of Lorenz (modified)	—	106·245
1884	*H. F. Weber	Induced Currents	—	105·37
—	*H. F. Weber	Rotating Coil	—	106·16
1884	*Roiti	Induced Currents	—	105·89
1885	*Himstedt	——	—	105·98
1883	*Wild	Damping of a Magnet	—	106·03
1889	*Dorn	Damping of a Magnet	—	106·24
1885	*Lorenz	Method of Lorenz	—	105·93

It was decided in 1892 by the British Association Committee on Electrical Standards to define the ohm for practical purposes as the resistance at 0° of a uniform column of mercury weighing 14·4521 grammes, in a tube 106·3 cm long. This corresponds to cross-section 1 sq. mm, and density of mercury 13·5956. It will be seen that there is a slight discrepance between this statement and that given in 12 above as expressing the result of the N.P.L. determination.

* The absolute measurements here referred to were compared with standards of German silver by Siemens or Strecker. The values in mercury units of these standards were certified by the makers.

CHAPTER XVI.

COMPARISON OF UNITS.

1. Ratio of units. Relation to speed of propagation of electromagnetic action. The experimental comparison of the ordinary electrostatic and electromagnetic units of an electrical quantity is of great importance in the electromagnetic theory of light, as it enables the velocity of propagation, according to that theory, of an electromagnetic disturbance to be determined numerically, and compared with the observed velocity of light. That the ratio of the two units of the same quantity gives the speed of propagation of electromagnetic action is an important proposition of Clerk Maxwell's electromagnetic Theory of Light, which is set forth in his celebrated essay on that subject. We begin the present chapter with one or two illustrations of this relation, modifying however the mode of applying them in accordance with the more general theory of dimensions adopted in Chapter I. above.

It has been shown (V. 34) that the electromagnetic force acting on an element ds of a conductor carrying a current γ in a magnetic field is $\mathbf{B}\gamma \sin\theta\, ds$, if $\mathbf{B}$ be the magnetic induction at the element, and θ the angle between the element and the direction of the magnetic induction.

If the field be produced by a current γ' in a straight conductor parallel to ds at distance b from it, and infinitely extended both ways, we get, by integration of the expression $\gamma' \sin\theta' ds'/r^2$ (p. 178 above) along the conductor, the expression $2\gamma'/b$ for the field intensity at ds due to the current γ'. Hence, if μ be the magnetic inductivity of the medium, the electromagnetic force on ds is $2\mu\gamma\gamma' ds/b$; and if the first conductor be straight the force on a length $b/2$ is $\mu\gamma\gamma'$.

Now let the quantities of electricity γt, $\gamma' t$, conveyed by the currents in time t, be used to charge two spheres whose centres are at a distance r apart great in comparison with the radius of either. The electrostatic repulsion between the spheres would then be $\gamma\gamma' t^2/\kappa r^2$, if κ denote the electric inductivity of the medium. If r be chosen so that this force is the same as the attraction between the conductors exerted on a length equal to half the distance between them, we have

$$\mu\gamma\gamma' = \frac{\gamma\gamma' t^2}{\kappa r^2},$$

or

$$\frac{1}{\mu\kappa} = \frac{r^2}{t^2}, \qquad \text{...}(1)$$

that is, $1/\sqrt{\mu\kappa}$ may be expressed as a speed. This is true whatever hypothesis as to dimensions is adopted for μ and κ, and all such hypotheses which may be framed must fulfil this condition.

This speed, moreover, is perfectly definite. For, if t^2/r^2 remain constant, the electrostatic force of repulsion between the spheres will remain unchanged, while their charges are increased at the time-rates γ, γ', respectively; and, therefore, $1/\sqrt{\mu\kappa}$ is equal to the speed r/t with which the spheres must be separated in order that their mutual repulsion may be kept equal to the force of attraction on a length of either of the parallel conductors equal to half the distance between them. It is proved in the electromagnetic theory of light that $1/\sqrt{\kappa\mu}$ is the speed of propagation of an electromagnetic wave in an isotropic insulating medium. See Maxwell, *Electricity and Magnetism*, or Gray, *Treatise on Magnetism and Electricity*.

2. Ratio of the units of quantity considered as a speed. If now we denote by v the ratio of the electromagnetic to the electrostatic unit of quantity, the charges on the spheres expressed in ordinary electrostatic units are, if γ, γ', now denote the ordinary electromagnetic measure of the currents, $v\gamma t$, $v\gamma' t$. Hence the force between the two spheres is

$$\frac{v^2\gamma\gamma' t^2}{K_s r^2},$$

where K_s denotes the specific inductive capacity of the medium, defined in the ordinary way as the ratio of the electric inductive capacity to that of the medium of reference (air or vacuum for example). But if ϖ denote the ordinary electromagnetic value of the permeability,

$$\varpi\gamma\gamma' = v^2\frac{\gamma\gamma' t^2}{K_s r^2},$$

that is

$$v^2 = \varpi K_s \frac{r^2}{t^2},$$

or by (1)

$$v^2 = \frac{\varpi K_s}{\mu\kappa}. \qquad (2)$$

If the medium be air, for which $K_s = 1$ $\varpi = 1$, we have

$$v = \frac{1}{\sqrt{\mu\kappa}}, \qquad (3)$$

or v is equal to the speed of propagation of an electromagnetic disturbance in air.

3. A moving electrified surface regarded as a current. The following illustration, also due to Maxwell, gives a remarkable physical meaning to the velocity $1/\sqrt{\mu\kappa}$ of propagation of an electromagnetic disturbance. In the first place it is assumed that an electrified surface in motion may be regarded as equivalent to a current.

This assumption is justified by the experiments of Rowland, who found that a statically electrified surface set into rapid motion affects a magnet properly placed in its vicinity, and has made measurements of the magnitude of the effect produced.

Considering then a plane surface of indefinitely great extent electrified to a surface density σ taken in any chosen system of units, we have $u\sigma$ as the measure of the convection current across unit breadth at right angles to the direction of motion, if u be the velocity. Let now another indefinitely extended surface parallel to the first and at a distance b from it be electrified to a uniform density σ', and move with velocity u', in the same direction as in the former case. A current in this case of strength $u'\sigma'$, per unit of breadth of the electrified surface, may be regarded as flowing parallel to the former current.

The two surfaces will repel one another electrostatically and attract one another electromagnetically. The electrostatic repulsion between two elements of surface dS, dS', at distance r is $\sigma\, dS \,.\, \sigma' dS'/\kappa r^2$, and integrating over the first surface we get $2\pi\sigma\sigma' dS'/\kappa$ for the resultant force on an element dS' of the second surface. Hence the force over unit area is $2\pi\sigma\sigma'/\kappa$.

The electromagnetic force between the two plane current sheets can be found as follows. Consider two narrow strips of the two planes with their lengths in the direction of motion. Let dz, dz', be their breadths, and z' the distance of the second strip from a plane coinciding with the first strip, and cutting the two moving plane surfaces at right angles. The distance between the two strips is $\sqrt{b^2+z'^2}$. The attraction between them is $\mu u\sigma\, dz \,.\, 2u'\alpha' dz'/\sqrt{b^2+z'^2}$ per unit of length of either. The total attraction, F say, per unit of length on the strip of breadth dz, is at right angles to the planes, and can be found by resolving the attraction just found in that direction, and integrating from $z'=-\infty$ to $z'=+\infty$. Thus

$$\begin{aligned}F &= 2\mu uu'\sigma\sigma' b\, dz\int_{-\infty}^{+\infty}\frac{dz'}{b^2+z'^2}\\ &= 2\pi\mu uu'\sigma\sigma'\, dz.\end{aligned}$$

Thus the electromagnetic attraction on unit area of either plane is $2\pi\mu uu'\sigma\sigma'$.

If the electrostatic repulsion be supposed to balance the electromagnetic attraction and u be taken equal to u', we get

$$\frac{2\pi\sigma\sigma'}{\kappa} = 2\pi\mu u^2\sigma\sigma'$$

or

$$u^2 = \frac{1}{\mu\kappa}. \qquad\qquad (4)$$

Thus the speed of propagation of an electromagnetic disturbance in the medium is equal to the speed with which the two electrified planes must move relatively to the medium in order that there may be no mutual force between them.

4. Methods of determining v. It has been shown above (p. 34) that v may be obtained from the ratio of the electrostatic and electromagnetic measures of any electric or magnetic quantity. It has been found experimentally in at least six of the following different ways:

I. By measuring electrostatically and electromagnetically a given quantity of electricity.

II. By measuring electrostatically and electromagnetically a given difference of potential.

III. By comparing the value of the electrostatic capacity of a given standard condenser, obtained by calculation from its dimensions and arrangement, with its capacity in electromagnetic measure as given by experiment.

IV. By comparing an electrostatic capacity, obtained by calculation as in III., with the self-inductance of a coil.

V. By determining (in either system of units) the product CL of the capacity of a given condenser, and the self-inductance of a given coil, and comparing this with the product of the electrostatic value C_s of the capacity and the electromagnetic value L_m of the self-inductance. [The product CL is the same in both systems of units.]

VI. By measuring electrostatically and electromagnetically a given resistance.

VII. By observation of the period of oscillatory discharge of a condenser of known capacity (in electrostatic units), through a circuit of known self-inductance.

5. Experiments of Weber and Kohlrausch: Rowland's experiments. The first attempt to determine v was made by Weber and Kohlrausch, who employed method I.[*] A Leyden jar was charged to a potential measured electrostatically by means of an electrometer, and was then discharged through a ballistic galvanometer, which measured by the throw of the needle the quantity of electricity with which the jar was charged. This quantity was known in electrostatic measure from the measured potential and the capacity of the jar, which was obtained by comparison with that of a sphere insulated at a distance from other conductors. The value obtained for v was 31,074,000,000 cm per second.

This determination cannot be regarded as one of high accuracy, chiefly on account of the unsuitableness of a condenser with a solid dielectric for exact experiment. The construction also of absolute electrometers for exact work had not then been brought to so high a pitch of excellence as has since been reached.

An accurate determination by this method was carried out at a more recent date by the late Professor Rowland at Baltimore,[†] and of this we give here a more detailed account.

The electrometer employed was an absolute instrument made on the guard-ring principle. The protected disk was circular and 10·18 cm in diameter, and was suspended in an aperture in the guard-ring of 1 mm greater radius.

[*] *Abh. d. Königl. Sächs. Ges. d. Wissens,* 1856.

[†] *Phil. Mag.* Oct. 1889.

The diameters of the guard-plate and attracting-plate were each 330 cm. The surfaces were all nickel plated, and worked true, so that the distance between the surfaces could be accurately found. The disk could be adjusted in the plane of the guard-ring, and the attracting-plate and disk to parallelism, to $\frac{1}{40}$ mm. External action was screened from the disks by a case of sheet brass.

The protected disk was hung from one arm of a sensitive balance, and the exact position of the beam was observed by means of a hair moving in front of a scale in the manner described at p. 695 below.

In the actual use of the electrometer, since the suspended disk could not be in stable equilibrium under the action of electrostatic attraction, its swing was limited to a range of $\frac{1}{10}$ mm on each side of the sighted position; and the attracting-plate was then placed successively at two near positions, for one of which the plate rose above the sighted position, for the other fell below it. The mean of these was taken as the reading for the position of the attracting-plate.

If d be the distance of the electrometer plates apart, w the weight on the balance, and S the area of the disk, we have for the electrostatic measure, V_s, of the difference of potential between them

$$V_s^2 = \frac{8\pi d^2 wg}{S}. \quad \text{.....................................(5)}$$

For the energy of the charge, Q say, on S is $\frac{1}{2}V_sQ$, which since the electrostatic capacity of the disk is $S/4\pi d$, has the value $V_s^2 S/8\pi d$. But if F be the attraction on the charged plate, this is Fd. Hence if $F = wg$, we get (5).

By a formula given by Maxwell * for the effective area of a protected disk of radius R, in an opening of radius R',

$$S = \tfrac{1}{2}\pi\left\{R'^2 + R^2 - (R'^2 - R^2)\frac{a}{d+a}\right\}, \quad \text{.......................(6)}$$

where $a = (R' - R)(\log 2)/\pi = \cdot 221\ (R' - R)$ nearly.

Thus the working equation for V_s was

$$V_s = 17221\, d\sqrt{w}\left(1 + \frac{\cdot 0002}{d}\right). \quad \text{..........................(7)}$$

[A more exact correction of the allowance for the circular gap, made (according to a note in the 3rd edition of the *El. and Mag.*) for the effect of curvature, brings down the charge per unit potential by the amount $(R' - R)^2/16(d + a')$, that is, diminishes the value of S in (6) by $\frac{1}{4}\pi(R' - R)^2 d/(d + a')$. We leave, however, the working equation as it was used by Rowland.]

6. Standard condenser and galvanometer, etc. The standard condenser consisted of two concentric spheres. The spheres were very accurately constructed, and the inner was hung concentrically within the outer by a silk cord (see also p. 672 below). A section of the condenser is shown in Fig. 205, which also indicates the arrangement of charging wires, etc., shown in Fig. 206. Two balls of different diameters were provided for use as inner spheres. The electrostatic capacity was obtained by determining the diameters of the balls by weighing in water, and was 50·069 c.g.s. or 29·556 c.g.s. according as the larger or smaller inner sphere was used.

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* *El. and Mag.* vol. i. Art. 201.

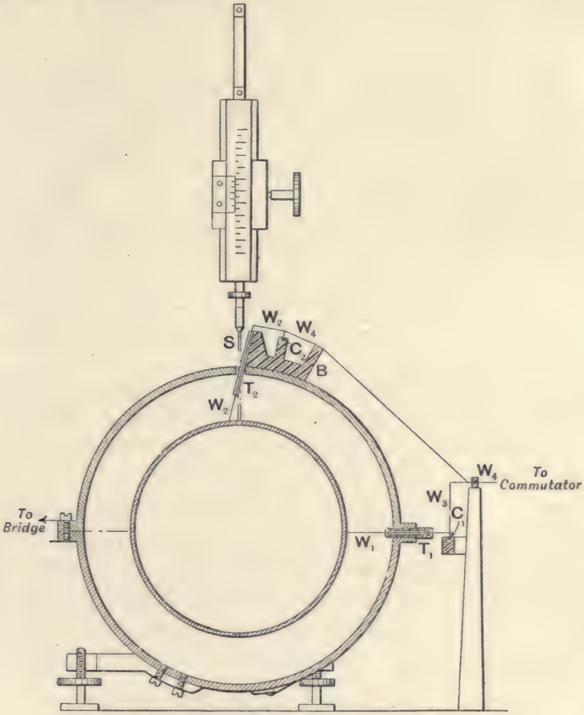

FIG. 205.—Section of Spherical Condenser.

The inner sphere is suspended by the silk cord S at the centre of the shell. The two charging wires W_1 and W_2 are guided by the fixed tubes T_1 and T_2. The outside ends of W_1 and W_2 dip into the small mercury cups C_1 and C_2. These mercury cups were joined by the wires W_3 and W_4 to the rotating commutator.

The galvanometer used for the discharges was a specially constructed and carefully insulated instrument. It had two coils, each of about 5600 turns of No. 36 silk-covered copper wire. These were fixed on the two sides of a plate of vulcanite. The needle was surrounded by a metal box to screen off possible electrostatic action of the coils from the needle.

The constant of this galvanometer was determined by comparison with the galvanometer described above (p. 435). The constant of this had been slightly altered, and was now found to be by measurement of its coils 1833·24, by comparison with an electrodynamometer 1833·67, and by comparison with a single circle (p. 386) 1832·56, giving a mean of 1832·82 instead of 1833·19 as before. The ratio of the constant of the new galvanometer to this was found to be 10·4141, so that for the ballistic galvanometer used

$$G=19087,$$

including the factor for the number of turns.

An absolute electrodynamometer on Helmholtz's double-coil principle, almost an exact copy of the instrument described at p. 396 above, but on a smaller scale, was used to find the directive force H at the ballistic galvanometer, at any instant during the progress of the experiment, so as to eliminate magnetic changes which were continually going on in the building used for the investigation; changes which were all the more important as H was only about $\frac{1}{3}$ of the horizontal component of the earth's field at the place. The suspension of the instrument was a bifilar one, and it was found that, for the small angles used, no correction was necessary for the torsion of the wire.

The electrodynamometer gave the absolute value of a steady current, which was made to flow also through the ballistic galvanometer, and thus enabled H to be found from the deflection of the latter, and the galvanometer constant.

It follows from p. 398 above, that if c be a constant depending on the coils, and the electrodynamometer be set up so that H does not affect it, or readings be taken so as to eliminute it, and the same current pass through both coils, we may write

$$\gamma=c\sqrt{F}\sqrt{\sin\beta}, \quad\quad (8)$$

where F is the coefficient of $\sin\beta$, in the couple applied by the bifilar, β being the angle through which the suspension head is turned to bring the suspended coil back to parallelism with the fixed coil. But it is clear that, if mk^2 be the moment of inertia of the coil, by the theory of simple harmonic motion we have $\ddot{\beta}/\beta=-4\pi^2/T^2$, and

$$\ddot{\beta}=-F\sin\beta/mk^2,$$

so that, if β be a small angle,

$$\sqrt{F}=\frac{2\pi}{T}\sqrt{mk^2}.$$

Thus, including 2π in the constant c, we have

$$\gamma=\frac{c\sqrt{mk^2}}{T}\sqrt{\sin\beta} \quad\quad (9)$$

for the electrodynamometer.

The value of c was calculated from the particulars of the coils, which were

	Large Coils.	Suspended Coils.
Mean radius - - - - -	13·741 cm	2·760 cm
Mean distance - - - - -	13·786 ,,	2·707 ,,
Radial depth - - - - -	·84 ,,	·41 ,,
Axial width - - - - -	·86 ,,	·38 ,,
No. of turns - - - - -	240	126

From which by (10), p. 397 above, and the values of G_1, g_1, given at p. 229,

$$c = \cdot 012914.*$$

To verify this constant a circle 80 cm in diameter was made and used as the coil of a tangent galvanometer. The ballistic galvanometer was set up so that its needle was at the centre of this circle, and acted, when required, as the suspended needle of the tangent galvanometer of which the circle was the coil. The current from the electrodynamometer was passed through the circle, and the horizontal field intensity H deduced from the galvanometer deflection and the current as given by the electrodynamometer. The value of H was found also by the magnetic method, and the two results were found to differ by only about 1 in 1000. Thus the tangent galvanometer gave $\frac{1}{2}c = \cdot 006451$, and the mean ·006454 of this and the former result was used.

The moment of inertia mk^2 was found by placing weights at different distances along a tube passed through the centre of the suspended coil, and observing the period of free swing of the coil. It was thus found that $mk^2 = 826 \cdot 6$ in gramme-centimetre units.

The value of H at the needle of the ballistic galvanometer was found, when required, by sending the same current through the dynamometer and the galvanometer, observing the deflections in the two cases, calculating the value of the current from the deflection from the former, and hence deducing H by the tangent galvanometer formula.

7. Method of experimenting. The condenser was charged by being connected to a large charged battery of Leyden jars. This battery was kept connected to the electrometer. The potential reading was first observed, then the battery connected to the condenser for an instant, after which the condenser was disconnected from the Leyden jar battery and discharged through the ballistic galvanometer. This was repeated 1, 2, 3, 4, or 5 times in succession, so that the galvanometer received that number of very nearly equal impulses in the same direction before it had moved far from the position of rest. The reading of the position of the electrometer attracting disk was again taken after the series of impulses, on disconnection of the battery from the condenser, and was slightly less than before of course. Corrections for the displacements of the needle from zero at the times of the successive impulses were calculated and applied.

The mean of the electrometer readings before and after a single discharge was, with a correction, taken as the potential of that discharge. This correction arose from the fact that the first reading was higher than that for the potential of discharge by a certain small amount depending on the capacities of the battery of jars and the condenser. It was obtained by multiplying the mean reading d of distance between the plates by a factor $1 - \cdot 0013$, when the larger sphere was used in the condenser, and by the factor $1-\cdot0008$ when the smaller sphere was used. The other series were similarly corrected.

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* This is double the value given by Prof. Rowland in his paper. The full period of vibration appears in equation (7), whereas Prof. Rowland used the half period.

The values of d thus corrected and the corresponding values of δ, the swing caused by the discharge of the condenser, gave for each set of observations, in number from ten to twenty, a series of values of d/δ, for the observations of the set. The mean value of d/δ was taken as the result for the set.

Before and after each series of observations the times of vibration of the suspended coil of the electrodynamometer and of the ballistic galvanometer needle were observed. Also the logarithmic decrement of the needle deflection was measured almost daily.

A correction was applied for the time occupied in producing the series of impulses. This was calculated approximately on the supposition that the time between one impulse and the next was $\frac{1}{5}$ of a second, and without taking into account the altered position of the magnet relative to the coil, or the induced magnetism of the needle. The inclination, however, of the magnet to the plane of the coil would cause the impulsive couple on the needle to be less for impulses later than the first, while the induced magnetization of the needle brought about by the same inclination would have an opposite effect. Prof. Rowland came to the conclusion by experiment that no sensible error from neglect of these refinements of correction could result.

8. Reduction of results. The principal equations used in reducing the results were (7) above, and others obtained as follows :

First, the ballistic galvanometer equation for the quantity, Q (in electromagnetic units), of electricity discharged, is

$$Q=\frac{HT}{\pi G}(1+\tfrac{1}{2}\lambda)\sin\tfrac{1}{2}\theta, \quad\ldots\ldots\ldots\ldots(10)$$

where θ is the ballistic deflection, corrected for everything except damping.

But if C_s be the capacity of the condenser in electrostatic units, and N the number of discharges,

$$Q=N\frac{V_s C_s}{v}. \quad\ldots\ldots\ldots\ldots(11)$$

Also H was obtained from the constant current measured by the dynamometer while it flowed round the 80 cm circle, at the centre of which the ballistic galvanometer needle was situated. Thus denoting by ϕ the deflection of the needle produced by the constant current, by r the radius of the large circle, and by b the distance of its plane from the centre of the ballistic needle, we have by (9) and the elementary theory of the tangent galvanometer

$$\gamma=\frac{c\sqrt{mk^2}}{T}\sqrt{\sin\beta}=\frac{(r^2+b^2)^{\frac{3}{2}}}{2\pi r^2}H\tan\phi,$$

so that

$$H=\frac{2\pi r^2 c\sqrt{mk^2\sin\beta}}{T(r^2+b^2)^{\frac{3}{2}}\tan\phi}. \quad\ldots\ldots\ldots\ldots(12)$$

Using this in (10), equating to (11), and solving for v we find

$$v = \frac{(r^2+b^2)^{\frac{3}{2}} GNV_s C_s \tan\phi}{r^2 c\sqrt{mk^2}\sin\beta(1+\frac{1}{2}\lambda)\sin\frac{1}{2}\theta}, \quad \text{.........................(13)}$$

where V_s is given by (7), and C_s by the dimensions of the condenser.

The approximate equation

$$2\sin\frac{1}{2}\theta = \frac{1}{2}\frac{\delta}{D}\left(1-\frac{1}{3}\frac{\delta^2}{D^2}\right) \quad \text{..............................(14)}$$

was used to find the value of $\sin\frac{1}{2}\theta$ from the observed deflection δ and the scale distance D. This approximation is easily obtained as follows: since

$$\delta/D = \tan 2\theta = 2\sin\theta\cos\theta/(1-2\sin^2\theta),$$

or

$$\sin\theta = \frac{1}{2}\frac{\delta}{D}\frac{1-2\sin^2\theta}{\sqrt{1-\sin^2\theta}}.$$

Putting $\sin\theta = \frac{1}{2}\delta/D$ on the right the equation becomes approximately

$$\sin\theta = \frac{1}{2}\frac{\delta}{D}\left(1-\frac{3}{8}\frac{\delta^2}{D^2}\right)$$

or

$$2\sin\frac{1}{2}\theta = \frac{1}{2}\frac{\delta}{D}\left(1-\frac{3}{8}\frac{\delta^2}{D^2}\right)\frac{1}{\sqrt{1-\sin^2\frac{1}{2}\theta}}.$$

In the last factor on the right, which is not very different from unity, $\sin\frac{1}{2}\theta$ may be put equal to $\delta/4D$. The equation then becomes

$$2\sin\frac{1}{2}\theta = \frac{1}{2}\frac{\delta}{D}\left(1-\frac{3}{8}\frac{\delta^2}{D^2}\right)\left(1+\frac{1}{32}\frac{\delta^2}{D^2}\right) = \frac{1}{2}\frac{\delta}{D}\left(1-\frac{11}{32}\frac{\delta^2}{D^2}\right)$$

$$= \frac{1}{2}\frac{\delta}{D}\left(1-\frac{1}{3}\frac{\delta^2}{D^2}\right), \text{ nearly.}$$

The value of $\tan\phi$ was calculated, by successive approximation, from the value of $\tan 2\phi$ given by δ_1 and the distance D_1 of the scale from the mirror, so that

$$\tan\phi = \frac{1}{2}\frac{\delta_1}{D_1}\left(1-\frac{1}{4}\frac{\delta_1{}^2}{D_1{}^2}+\frac{1}{8}\frac{\delta_1{}^4}{D_1{}^4}\right). \quad \text{......................(15)}$$

The following are the results obtained:

Number of discharges.	Mean result in cm per second	Number of results of which mean was taken.
1	$298{\cdot}80\times10^8$	9
2	$298{\cdot}48\times10^8$	5
3	$297{\cdot}26\times10^8$	5
4	$297{\cdot}45\times10^8$	5
5	$296{\cdot}69\times10^8$	5

To these were given weights inversely as the number of discharges, except in the case of the first, which was given twice the weight of the second, on account of the larger number of observations. Thus the final result obtained was $v = 2{\cdot}9815 \times 10^{10}$ in cm per second.

9. Lord Kelvin's method. Determinations by method II., which is due to Lord Kelvin, were made by Lord Kelvin himself,* Mr. D. McKichan,† F. Exner,‡ and Mr. R. Shida.§

A current is made to flow through a coil the absolute value R of the resistance of which is known, and the current is measured electromagnetically by an absolute current-meter, while the difference of potential between the extremities of the coil is measured by an absolute electrometer. If V be the difference of potential in electrostatic measure, the work done in the passage of one electrostatic unit of electricity is V. But one electrostatic unit of electricity is $1/v$ of an electromagnetic unit; and if γ be the measured current, the time t taken for a quantity $1/v$ of electricity to pass is $1/v\gamma$. Hence the work done in the conductor, or $\gamma^2 Rt$, is $\gamma R/v$. Thus

$$v = \frac{\gamma R}{V}. \qquad \text{.................................(16)}$$

The result therefore involves the absolute value of a resistance R in electromagnetic units. Now in the earlier experiments by this method the resistance of a conductor was not known with accuracy, and the results are unreliable, unless some means exists of correcting the values of R which were used.

Lord Kelvin's first result (corrected for the value of the B.A. unit) was $2{\cdot}808 \times 10^{10}$ cm per second, Mr. D. McKichan's $2{\cdot}896 \times 10^{10}$ cm per second.

Shida's determination was made later and gave $v = 2{\cdot}955 \times 10^{10}$ cm per second. The difference of potential at the terminals of a battery of large tray Daniell cells was measured by a Thomson's absolute electrometer, while the current maintained by the battery through a tangent galvanometer was measured.

In reducing his results Mr. Shida multiplied both numerator and denominator of (16) (unnecessarily) by the factor $(R+r)/R$, where r was the resistance of the battery and connections. On this account the accuracy of the result was mistakenly called in question. For though the factor $R+r$ was of uncertain value, its introduction in both numerator and denominator could in no way affect the value of the ratio $\gamma R/V$. The real ground for uncertainty lay in the construction of the tangent galvanometer, which could hardly work up to the degree of accuracy required. Its coil was wound on a wooden bobbin, and was said to have been made, many years before, in Manchester under the superintendence of Dr. J. P. Joule. It required reconstruction and redetermination of its constant.

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* *Phil. Trans. R.S.* 1868.

† *Ibid.* 1879.

‡ *Wien. Ber.* 86, 1882.

§ *Phil. Mag.* 10, 1880.

A measurement of v was made by this method again in 1889 by Lord Kelvin, by means of the electrostatic voltmeters of his own invention; but the details of the investigation do not seem to have been published. The result obtained was

$$v = 3{\cdot}004 \times 10^{10} \text{ cm per second.}$$

Two of the electrostatic voltmeters described in XVII. 20, below, were carefully compared at Glasgow. One of them (A) was sent to London, where the values of its scale readings were determined by Professors Ayrton and Perry, with an absolute electrometer. A difference of potentials of about 80 volts between the terminals of a coil of resistance 600 ohms was determined electromagnetically by measurement of the current, 133 milliamperes, through it, by means of a Kelvin centiampere balance. The graduation of this balance, if carried out in the ordinary way, must have depended on the electrolysis of copper sulphate.

This difference of potential was multiplied 16 times by the step up method of XVII. 21, below, and measured by voltmeter B.

From a comparison of all these measurements the result stated above was derived.

Exner's result obtained by a modification of this method was, with the value ·941 ohm for one Siemens' unit, $2{\cdot}92 \times 10^9$ cm per second.

10. Maxwell's method. Another form of this method has been given by Maxwell,* and used by him in a determination of v. The electromagnetic repulsion between two parallel coils produced by the same current flowing in opposite directions through them, was balanced by the attraction between two disks to the backs of which the coils were attached, and between which a difference of potential was produced by another current the ratio of which to the former current was known. One of the disks was the protected disk of a Thomson's guard-ring condenser, and to the back of this one of the coils was attached directly: the other coil was carefully insulated from the attracting disk by a plate of glass and a layer of insulating material.

The apparatus is shown in Fig. 206, and shortly described in the list of references attached. The small disk (diameter four inches) and attached coil were carried at one end of a torsion balance suspended by a No. 20 copper wire from a graduated torsion head movable by a tangent screw. The disk and coil were protected by a cylindrical brass box 7 inches in diameter, one end of which formed the guard-ring. The disk carried on the side towards the interior of the box a glass scale divided to $\frac{1}{100}$ of an inch, which was viewed by a reading microscope fixed on the outside of the box.

To eliminate the turning couple due to the earth's field a coil was attached to the other end of the balance, and connected with the first coil in such a way that the current flowed through the coils in opposite directions.

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* *Phil. Trans. R.S.* 158 (1868), or *Rep. of Papers*, vol. ii. p. 125.

The attracting disk (which was 6 inches in diameter) was, with its attached coil, on a slide worked by a micrometer so that the distance of the disks could be varied and measured. The plane of this disk was adjusted parallel to the guard-ring, which was placed exactly vertical by means of adjusting screws.

The graduations of the glass scale and the micrometer were compared by pressing the suspended disk forward by a light spring against the large disk, and then working the screw so as to send the small disk back towards the plane of the guard-ring, while readings of the micrometer were taken for successive divisions of the glass scale. This motion was quite regular until the large disk came into contact with the guard-ring at one point. It was found then that a motion of about $\frac{1}{1000}$ of an inch sufficed to bring the whole of the guard-ring into contact with the large disk.

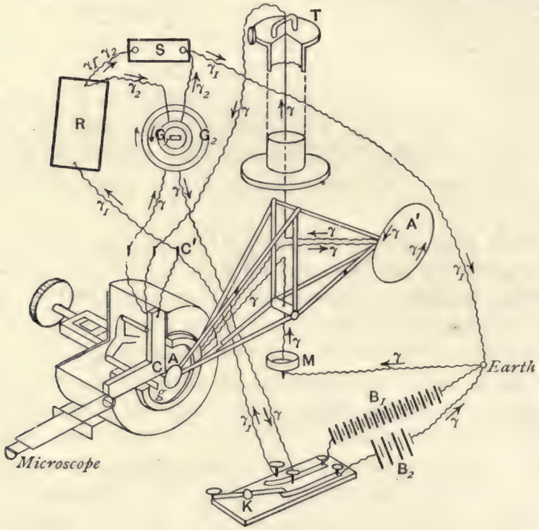

FIG. 206.

A.	Suspended disk and coil.
A'.	Counterpoise disk and coil.
C.	Fixed disk and coil.
B_1.	Great battery.
B_2.	Small battery.
G_1.	First coil of Galvanometer.
G_2.	Second coil of Galvanometer.
R.	Great resistance.
T.	Torsion head and tangent screw.
K.	Double key.
g.	Graduated glass scale.
C'.	Electrode of fixed disk.
γ.	Current through the three coils and G_2.
γ_1.	Current through *R*.
γ_2.	Current through G_1.
$\gamma_1-\gamma_2$.	Current through *S*.

When the small disk had thus been brought into the plane of the guard-ring, the reading microscope had its cross-wires focussed on a known division of the glass-scale, and two pieces of silvered glass were fixed, one to the back of the guard-ring, the other to the back of the suspended disk, so that when the disk and guard-ring were in one plane these mirrors were also, and gave a continuous image of objects in front of them. This arrangement gave a test of coplanarity of the surfaces to $\frac{1}{1000}$ of an inch.

The torsion wire, which was of soft copper stretched to straightness, seemed in great measure free from imperfectness of elasticity. The torsion balance could be adjusted by moving the supporting pillar, which could be adjusted and clamped in position by screws at its base. The balance itself could be raised or lowered, turned about any horizontal axis by sliding weights attached to it, and about the axis of suspension by the torsion head.

A large battery, the property of Mr. Gassiot, containing 2600 cells charged with bichloride of mercury, was used to electrify the disks. One terminal of the battery was connected through a key with the large disk, the other with the case of the instrument, and the circuit between was composed of a large resistance of over a megohm, in series with one (hereafter called the first) coil of a standard galvanometer shunted by a coil of resistance S.

A current was sent from another battery through a second coil of the tangent galvanometer (in the direction opposed to the other coil), through the coil behind the large disk, and thence to the suspended coils by the suspension wires. A common connection was given to earth, the case, and the other electrode of the battery, by a copper wire hanging from the centre of the torsion balance, and dipping into a mercury cup M.

When the suspended disk was at rest at zero the battery contacts were made simultaneously, and, according as the suspended disk was attracted or repelled, the other was moved farther from or nearer to the suspended one. It was necessary, on account of the instability of the small disk, when at the zero position under the action of the electric forces, to work the micrometer disk gradually up by successive trials from a distance initially too great, making contacts as zero was approached, so as if possible to bring the suspended disk to rest under the action of the opposing forces due to the disks and coils. An observer at the galvanometer altered the shunt S, while the contacts were being made, so as to bring the needle to zero.

To compare the magnetic effects produced by the two galvanometer coils at the needle, a current was sent through the second coil of the galvanometer, then through a divided circuit, consisting of a resistance of 31 B.A. units placed across a branch made up of the first coil of the galvanometer and an added resistance S'. The latter resistance was varied until the effects on the needle balanced one another.

11. Theory of Maxwell's method. Result. If V denote in electrostatic units the difference of potential between the disks, a the radius of the small one, and b their distance apart, the attraction between them was, clearly,

$$\frac{1}{2}\frac{V}{b}\frac{V}{4\pi b}\pi a^2 = \frac{1}{8}V^2\frac{a^2}{b^2}.$$

The repulsion between the two coils is $\gamma^2 dM/dx$, if γ be the current in each, x the distance apart of their mean planes, and M their mutual inductance. Thus we have

$$\frac{1}{8}V^2\frac{a^2}{b^2}=\gamma^2\frac{dM}{dx}. \qquad\text{.............................(17)}$$

But the difference of potential, V, between the disks is produced by the large battery, which sends a current γ_1 through the resistance R, and a current $\gamma_1 S/(G+S)$, $(=\gamma'$, say), through the first coil of the galvanometer, if G denote the resistance of that coil. Hence if E be the electromagnetic measure of this difference of potential

$$E=\left(R+\frac{GS}{G+S}\right)\gamma_1. \qquad\text{.............................(18)}$$

Again, if F_1, F_2, be the magnetic forces produced at the needle by unit current in the two coils, we have

$$F_1\gamma'=F_2\gamma, \text{ or } F_1\gamma_1 S/(G+S)=F_2\gamma.$$

But if in the comparison of the magnetic forces which was made γ'_1, γ'_2, denote the currents in the two coils, $F_1\gamma'_1=F_2\gamma'_2$, and by the arrangement of the circuits $(G+S')\gamma'_1=31/(\gamma'_2-\gamma'_1)$, so that $F_2/F_1=31/(G+S'+31)$. This substituted in the former equation gives

$$\gamma_1=\frac{G+S}{S}\,\frac{31}{G+S+31}\gamma,$$

and (18) becomes, with this value of γ_1,

$$E=\left(\frac{RG}{S}+R+G\right)\frac{31}{G+S'+31}\gamma. \qquad\text{......................(19)}$$

But if γ_m, γ_s, denote the electromagnetic and electrostatic values of the same current, $E\gamma_m=V\gamma_s$, since they denote the same rate of working: and we have $v\gamma_m=\gamma_s$. Hence $V=E/v$. Substituting this value of V_m in (17) with that of E given by (19), and solving for v, we get

$$v=\frac{1}{2\sqrt{2}}\left(\frac{RG}{S}+R+G\right)\frac{31}{G+S'+31}\,\frac{a}{b}\,\frac{1}{\sqrt{\dfrac{dM}{dx}}}. \qquad\text{.............(20)}$$

The value of dM/dx given in terms of elliptic integrals in XII. 48 (63) above was used in the calculation of v by this formula. The numbers of turns in the coils were 144 and 121, and their mean radius was 1·934 in.

The mean of 17 experiments gave

$$v=2{\cdot}8798\times10^{10}, \text{ in cm per second,}$$

on the assumption that 1 B.A. unit was 10^9 c.g.s. The corrected result is

$$v=2{\cdot}841\times10^{10}, \text{ in cm per second,}$$

if 1 B.A. unit be taken as ·98674 ohm. [This requires further correction. It is the result as given: in the table at the end of this chapter, the more recent results are in accordance with our knowledge of the value of resistances.]

12. Third method of determining v. Method III. has been used by Professors Ayrton and Perry, J. J. Thomson, E. B. Rosa, and others (see also 17 below).

If C_m be the capacity of the condenser in electromagnetic units determined by any process, and C_s its capacity in electrostatic units as given by measurement, then if Q_m and Q_s denote the electromagnetic and electrostatic values of the same charge, we have $Q^2{}_m/C_m = Q^2{}_s/C_s$, since each denotes the same quantity of electric energy. Thus

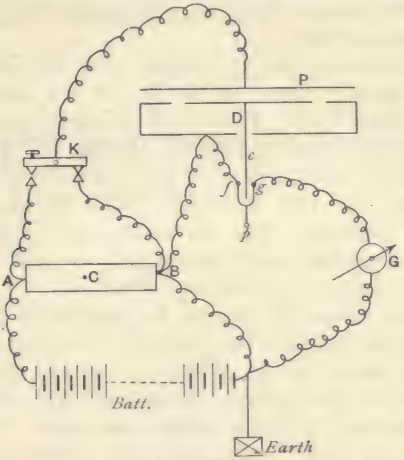

FIG. 207.

$$\frac{C_m}{C_s} = \frac{Q^2{}_m}{Q^2{}_s} = \frac{1}{v^2},$$

or

$$v = \sqrt{\frac{C_s}{C_m}}. \qquad \text{......................................(21)}$$

The arrangement of Ayrton and Perry's apparatus* is shown in Fig. 207. The attracting plate P of a guard-ring condenser was connected to a key K, by which it would be put in contact with either terminal, A or B, of a resistance of about 10,000 ohms. Unless the key was depressed it was kept in contact with B by means of a spring. The resistance was in circuit with a battery of 382 Daniell's cells, and the point B was connected with the earth and with the guard-ring as shown. A fork-shaped connecting piece turning round a pivot was used to connect the guard-ring to the projecting electrode of the protected disk, or the latter to earth through the galvanometer G.

The protected disk, D, of the condenser was a square of area of 1325·14 sq. cm, and was separated from the guard-ring by a gap 2·5 mm wide. The distance between the plates was ·7728 cm. The plates were supported on well paraffined levelling screws of ebonite, and were strengthened by diagonal ribs on the upper side of the plate *P*, and the under side of the disk *D*.

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* *Journ. Soc. Tel. Eng.* 1879.

The galvanometer was a Thomson's astatic instrument of about 20 ohms resistance. The ordinary needles were however replaced by small spheres each built up of a number of tiny magnets having their like poles all turned the same way, the spheres being completed with pieces of lead. The period *T* of the needle was 39·5 seconds, and its logarithmic decrement ·1565.

The mode of operating was as follows. The key *K* was depressed, and the plate *P* thereby connected to *A*; at the same time the electrode *e* was connected to *f*. Thus the condenser was charged to the difference of potential existing between *A* and *B*. Then the contact was broken between *e* and *f*, and the key released so as to make contact between *P* and *B*. This connected *P* and the guard-ring to earth while *D* was left insulated. The electrode *e* was then connected to *g* by the pivoted connector, and discharged the disk *D* through the galvanometer, the reading of which was observed.

The difference of potential *E* given by the battery between *A* and *B* was measured in the following manner. A very high resistance *R* was put in the circuit of the galvanometer, and its terminals were then connected to *A* and another point *C* in *AB*, enclosing between them a known fraction *k* of the whole resistance. The difference of potential between *A* and *C* was thus *kE*. The galvanometer was shunted through a resistance *S*, so that *G* being the resistance of the coil a current $kES/\{R(G+S)+GS\}$ was sent through the instrument. The deflection thus produced was observed.

Now if θ and α denote the angular deflections given by the transient and the steady current respectively, and C_m the capacity in electromagnetic units of the protected disk *D*, we have by the ballistic and tangent galvanometer formulae

$$\frac{C_m E}{kES/\{R(G+S)+GS\}} = \frac{T}{\pi}\frac{\sin\frac{1}{2}\theta}{\tan\alpha},$$

or

$$C_m = \frac{T}{\pi}\frac{kS}{R(G+S)+GS}\frac{\sin\frac{1}{2}\theta}{\tan\alpha}.$$

Thus C_s denoting the calculated capacity, we find

$$v^2 = \frac{C_s}{C_m} = C_s\frac{\pi}{T}\frac{R(G+S)+GS}{kS}\frac{\tan\alpha}{\sin\frac{1}{2}\theta}. \quad \text{......................(22)}$$

Three series of experiments were made consisting of 39, 41, and 18 discharges for *T*, 25·3, 39·5, 42·2 seconds respectively. The mean result obtained was

$$v = 2{\cdot}98 \times 10^{10} \text{ in cm per second.}$$

This, however, must be corrected for the value of the B.A. unit, and becomes

$$v = 2{\cdot}955 \times 10^{10} \text{ in cm per second.}$$

This method was used by Klemenčič * with the modification that a rapid succession of discharges was sent through the galvanometer so that a constant deflection was produced. The mean result of two different researches by this method was

$$v = 3{\cdot}041 \times 10^{10}$$

in cm per second.

Similar experiments by Stoletow † gave

$$v \begin{matrix} > 2{\cdot}98 \times 10^{10} \\ < 3{\cdot}00 \times 10^{10} \end{matrix}$$

in cm per second.

13. Maxwell's bridge form of method III. The following form of the method, due to Maxwell,‡ has the advantage over that just described of being a null method, and therefore of not requiring any correction for torsion, damping, etc., while it shares with the former the advantage of involving the square root only of $\sqrt{C_s/C_m}$, and therefore only half of any error made in determining C_m or C_s. A Wheatstone bridge (Fig. 208) has a gap in one of the arms at p, q, and a contact piece or tongue, u, is made to vibrate across the gap so as to connect one plate of a condenser alternately to p and to q, while the other plate is kept permanently in contact with the point C. The resistances of the wires Cp, qB are made inappreciable, so that the plates of the condenser are alternately brought to the same potential, and charged to the potential existing between C and B.

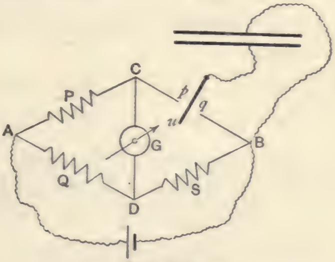

FIG. 208.

A succession of transient currents are thus produced in the same direction through the galvanometer, and if P, Q, S are properly adjusted, are prevented by a steady current in the opposite direction from producing any deflection. From the condition, (29) below, fulfilled by the resistances of the bridge, the value of C_m can be found, and compared as before with the value C_s of the capacity in electrostatic units.

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* *Wien. Ber.* 83, 1881.

† *Soc. Franc. de Phys.* Nov. 4, 1881.

‡ *El. and Mag.* vol. ii. arts. 775, 776.

So far as C_s is concerned the error of this method (and of others which require the capacity of a standard condenser) is only that involved in the measurement of the dimensions of the condenser, and reduces finally to that of the measurement of a length. Proper allowances can easily be made for want of accurate adjustment of the parts of the condenser.

The determination of C_m is limited in accuracy only by the error involved in the use of the galvanometer, which must be so sensitive as to detect a sufficiently small variation of resistance. This error in the experiments described below was well within the limits of accuracy aimed at. In Thomson and Searle's investigations below it was estimated that the error from the galvanometer was not more than 1 in 2500 in the value of v.

14. Theory of Maxwell's bridge method. Calling the resistances P, Q, S as marked on the figure, and denoting the currents from C to p, C to D, and B to A, by $\dot{x}$, $\dot{z}$, $\dot{u}$, the resistance and self-induction of the galvanometer by G and L, we have from the circuits $ACDA$, $ADBA$, the equations of currents, supposing all the branches, except CD, devoid of inductance,

$$\left.\begin{aligned} P(\dot{x}+\dot{z})-Q(\dot{u}-\dot{x}-\dot{z})+L\ddot{z}+G\dot{z}=0,\\ Q(\dot{u}-\dot{x}-\dot{z})+S(\dot{u}-\dot{x})+B\dot{u}-E=0.\end{aligned}\right\} \quad\text{..................(23)}$$

At the beginning and end of the charging of the condenser the currents have their steady values, and therefore these equations become

$$P\dot{z}_s-Q(\dot{u}_s-\dot{z}_s)+G\dot{z}_s=0,$$
$$Q(\dot{u}_s-\dot{z}_s)+(B+S)\dot{u}_s-E=0,$$

where the suffixes indicate the steady values of the currents.

Subtracting these last equations from the corresponding equations (23) for the variable state, and putting $\dot{u}_1$, $\dot{z}_1$, for $\dot{u}-\dot{u}_s$, $\dot{z}-\dot{z}_s$, we find

$$\left.\begin{aligned} P(\dot{z}_1+\dot{x})-Q(\dot{u}_1-\dot{z}_1-\dot{x})+L\ddot{z}+G\dot{z}_1=0,\\ Q(\dot{u}_1-\dot{z}_1-\dot{x})+S(\dot{u}_1-\dot{x})+B\dot{u}_1=0.\end{aligned}\right\} \quad\text{................(24)}$$

The quantities $\dot{u}_1$, $\dot{z}_1$, it is to be noticed, denote the excess in each case of the current flowing at any instant above the steady current, in consequence of the charging of the condenser, while $\dot{x}$ is the charging current.

Integrating, from the beginning of the charging to the end, the equations just found, remembering that $\dot{z}$ has the value $\dot{z}_s$ at both limits, and rearranging, we get

$$\left.\begin{aligned} (P+Q)x+(G+P+Q)z_1-Qu_1=0,\\ -(Q+S)x-Qz_1+(Q+S+B)u_1=0,\end{aligned}\right\} \quad\text{..................(25)}$$

where x denotes the whole charge of the condenser, and u_1, z_1, the excess in each case of quantity of the electricity conducted by the currents $\dot{u}$, $\dot{z}$, above that which would have flowed in the same time if the current had remained constant.

Eliminating u_1 from (25), we find

$$x=\frac{-(Q+S+B)(G+P+Q)+Q^2}{(P+Q)(Q+S+B)-Q(Q+S)}z_1. \quad \text{.....................(26)}$$

But when the condenser is fully charged the difference of potential between its coatings is x/C_m, and this is $G\dot{z}_s+S\dot{u}_s$, so that

$$x=C_m(G\dot{z}_s+S\dot{u}_s).$$

Also clearly $(G+P)\dot{z}_s=Q(\dot{u}_s-\dot{z}_s)$, and therefore

$$\dot{u}_s=\frac{G+P+Q}{Q}\dot{z}_s$$

and

$$x=C_m\left(G+S\frac{G+P+Q}{Q}\right)\dot{z}_s. \quad \text{...........................(27)}$$

If the condenser is charged and discharged n times a second, the quantity of electricity which passes through the galvanometer over and above that which passes in the steady current is nz_1. Hence, if there is no deflection, we must have $\dot{z}_s+nz_1=0$, or $\dot{z}_s=-nz_1$. Thus (27) becomes

$$x=-nC_m\left(G+S\frac{G+P+Q}{Q}\right)z_1. \quad \text{.......................(28)}$$

This value of x used in (26) gives

$$nC_m=\frac{Q\{(Q+S+B)(G+P+Q)-Q^2\}}{\{P(Q+S+B)+QB\}\{S(G+P+Q)+GQ\}}. \quad \text{...........(29)}$$

If P and S are very great in comparison with the other resistances, this reduces to the approximate solution

$$nC_m=\frac{Q}{PS}. \quad \text{.......................................(30)}$$

The electromagnetic value of the capacity of the condenser having thus been found, that of v is of course obtained as before from the ratio $\sqrt{C_s/C_m}$. The result thus depends on the exactness of our knowledge of the absolute value of a resistance, that is, of the ohm.

15. Experiments of Rosa. The method has been carried out with this mode of determining C_m by Prof. J. J. Thomson * in a very careful series of experiments, giving the result

$$v=2{\cdot}963\times10^{10} \text{ in cm per second,}$$

by Mr. E. B. Rosa † at Baltimore, and again by Prof. J. J. Thomson and Mr. G. F. C. Searle ‡ at Cambridge in an elaborate research made with improved apparatus.

We shall describe here Mr. Rosa's experiments and the later investigation of Thomson and Searle.

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* *Phil. Trans. R.S.* 1883.

† *Phil. Mag.* Oct. 1889.

‡ *Phil. Trans. R.S.* vol. 181 (1890).

Mr. Rosa used the standard spherical condenser described in 6 above as made for Prof. Rowland's experiments on this subject. See Fig. 205.

The vibrating tongue u (Fig. 208) was operated by one or other of two forks made by Kœnig, of Paris, of frequencies 32 and 130 per second. These were maintained in vibration in the ordinary way by an electromagnet between the prongs worked by the current from three or four Bunsen cells.

With the slower fork a commutator was used, but with the faster fork a different arrangement was adopted. A double branch wire led from the inner coating of the condenser, and a branch was connected by wax to the end face of each prong of the tuning-fork. The plane of vibration was vertical, and each wire was turned so as to dip into two mercury cups cut in fixed pieces of vulcanite, at a vertical distance apart equal to that between the prongs of the fork. The upper cup was connected with the point C of Fig. 208, the lower cup to B. Thus when the prongs moved apart the lower wire dipped into the mercury, connecting the inner ball of the condenser to B, while the upper broke contact; when the prongs approached one another the upper contact was made and the lower broken, and the two plates of the condenser were put into direct contact. Thus in the former case the condenser was charged, in the latter discharged.

The galvanometer used was a very sensitive Thomson's astatic instrument.

The battery consisted of about 40 cells of a storage battery, giving an electromotive force of about 80 volts.

The resistances Q and S were taken from two resistance boxes by Elliott, containing 12,000 ohms and 100,000 ohms respectively.

The resistance P, which was very great, was made by ruling pencil lines on ground glass, and protecting the surface of glass and graphite with a thick coat of shellac varnish. Connection was made at the ends by tinfoil pressed against the graphite by rubber packing. Ten such resistances were made and mounted in cylindrical cases, so that their temperatures might be maintained as nearly constant as possible. Their values were determined by a comparison (made by the method of Wheatstone's bridge with a ratio of about 100) with the resistances of the boxes used for Q and S, and proved very constant and reliable.

The capacity of the vibrating piece and the connecting wires was determined experimentally by separating them from the condenser. Special attention was given to the question as to whether the capacity of the charging wire might be taken as the same when the wire was in contact as when detached, and no appreciable difference was found.

The inner sphere was adjusted by lifting off the upper half of the outer shell, and adjusting the position of the ball relatively to the equatorial circumference of the shell, then replacing the hemisphere, and moving the ball vertically from contact at top to contact at bottom of the shell, and causing the contact in each case to be indicated by the closing of an electric circuit. The readings of a sliding vernier gave the top and bottom positions, and the mean of these readings the central position. It was estimated that the ball was centred to 0·1 mm vertically and 0·2 mm horizontally, or to an error of less than 1 per cent. of the distance between ball and shell.

Now, for an eccentric cylinder, theory shows * that a similar displacement of 1 per cent. from centrality would give an error of capacity of 1/200 per cent., and a smaller error for a spherical condenser. A displacement of four per cent., it was found by trial, caused a quite inappreciable change in capacity.

The dimensions of the outer shell were determined by filling it with water and weighing, and of the inner ball by weighing it sunk in water by an attached mass, and making all necessary corrections for displaced air, etc. The results were checked by measurements made by callipers, compared with a standard metre bar. The results were :

	Radius.	
	By weighing.	By direct measurement.
Shell - - -	12·6805 cm	12·6791 cm
Ball A - - -	10·1180 ,,	10·1183 ,,
Ball B - - -	8·8735 ,,	8·8736 ,,

The experiments were made with the larger ball, and four series were made, the first, second, and fourth with both forks, the third with the slow fork alone.

It was found that the results for the fast fork were slightly lower than those for the slow fork, coming out according to the weights given to the observations.

$$v = 2{\cdot}9994 \times 10^{10} \text{ in cm per second for the fast fork, and}$$

$$v = 3{\cdot}0023 \times 10^{10} \text{ in cm per second for the slow fork.}$$

The results for the fast fork were the more uniform and it was thought the more accurate, and were given double weight in striking the final mean. Thus the final result of all the experiments was

$$v = 3{\cdot}0004 \times 10^{10} \text{ in cm per second.}$$

The results of Series II. and III. were greater than those of I. and IV., and it was thought possible that the halves of the outer shell had been very slightly separated in the former case by an obstruction in the flange of junction. It is to be noticed that the results with the slow fork are the greater, indicating too *small* a value of C_m. This is the kind of result which the *fast* fork might be expected to give if the period was not long enough to allow the condenser to be fully charged. The rejection of the observations of Series II. and III. would give

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* J. J. Thomson, " On the Determination of v," *Phil. Trans. R.S.* 1883.

$$v = 2{\cdot}9993 \times 10^{10} \frac{\text{cm}}{\text{sec}},$$

which only differs from the former value by $\frac{1}{30}$ per cent.

16. Determinations of Rosa and Dorsey. This investigation was repeated in the period 1904 ... 1907 with the same spherical condenser, and also with other condensers (cylindrical condensers and a new parallel plate guard-ring condenser) by Messrs. E. B. Rosa and N. E. Dorsey. The work was done at the laboratory of the Bureau of Standards at Washington, and a full account of it appears in the *Bulletin* of the Bureau, 3, 1907. The memoir is very elaborate, for the investigation was really at least three independent and mutually corroborative determinations, made with the utmost attention to all details of observations and corrections. Only a very slight statement of methods and results is possible here, with some details of the apparatus employed.

Figs. 205, 206 show the spherical condenser, of which the dimensions, at 20° C., were as follows, when measured in 1905 :

Radius of shell (in cm) - - - -	12·67140	
Internal ball (A) ,, - - - -	10·11790	
Internal ball (B) ,, - - - -	8·87380	
Capacity with ball A - - - -	50·2095	at 20° C.
Capacity with ball B - - - -	29·6092	

Some trouble was caused by uncertainty as to the effect of variation of capacity of the charging wire which passed, as the diagram shows, through a small hole in the top of the outer shell to touch the inner sphere, and, of course, changed in capacity when the latter contact was broken. This difficulty was met by the employment of the other condensers, and, independently, by taking the capacity with two charging wires, one at the pole and one at the equator. The difference in electromagnetic capacity caused by the withdrawal of the polar wire gave the capacity of the wire itself. Then replacement of the polar wire and withdrawal of the equatorial gave the capacity of the latter.

With regard to the cylindrical condensers, consisting of two coaxial circular cylinders of radii R and r, and of length l, the capacity C is given by

$$C = \frac{l + \delta l}{2 \log \frac{R}{r}}, \quad \text{...(31)}$$

where δl is a correction of the length for end effect. The correction δl could be obtained in two ways, (1) by using guard cylinders to prolong the inner cylinders both ways ; (2) by varying the length l, taking care to make the end conditions identical in the two cases. The equation for C can be written in the approximate form

$$C=\frac{l}{2\log\frac{R}{r}}=\frac{lr}{2(R-r)}, \qquad\text{(31')}$$

so that it was necessary to obtain the value of $R-r$ with all the precision required for r, which made it undesirable that $R-r$ should be very small.

The cylinders were ground on their ends in pairs, so that the two ends of the space between them could be closed water tightly by a pair of truly plane glass plates, slightly greased. The volumes of the outer cylinder, and the coaxial space between the two were determined by a process similar to that used for the spheres, and indicated above (p. 673).

The length of each cylinder along each of four generating lines was measured by means of a comparator. The following dimensions are here stated to give an idea of the size of the apparatus.

CONDENSER NO. 2 AT 20° C.

(Dimensions in centimetres.)

Outer cylinder, mean radius	7·24411
Inner ,, ,, ,,	6·25760
Length	19·99946

CONDENSER NO. 3 AT 20° C.

(Dimensions in centimetres.)

Outer cylinder, mean radius	7·23831
Inner ,, ,, ,,	6·25740
Length	20·00718

The capacities (in centimetres) without corrections for the guard ring gap, etc., were as follows:

No. 2, 68·3080, at 20° C.
No. 3, 68·6965, ,,

17. Comparison of methods. We cannot afford space for a description of the parallel plate guard-ring condenser, of the rotating contact breaker, the special chronograph used for determining the speed of charge and discharge, or the details of corrections. The arrangement of the guard-ring and other parts of the plate condenser is shown in Fig. 209. The contact-making pieces and the observing microscope will be easily made out. We can only state the final result, which was

$$v=2{\cdot}9963\times10^{10}\,\frac{\text{cm}}{\text{sec}}\quad\text{[Int. Ohm]},$$

with an uncertainty of not more than 1 in 10,000.

Assuming the dielectric constant of air at pressure 760 mm and temperature 20° C. to be 1·00055, this gives for the value of v in vacuo

$$v = 2{\cdot}9971 \times 10^{10} \, \frac{\text{cm}}{\text{sec}}.$$

Some instructive critical remarks are given in this paper on the advantages of the different methods enumerated above (p. 655) for the determination of v. As the authors point out, method I. was favoured by the earlier experimenters, as all the other methods really required exact knowledge of the absolute value of a resistance, a knowledge which these experimenters did not possess. The method II. requires no condenser, and has thus an advantage over I., but involves a knowledge of the resistance between the terminals of which the electromagnetic difference of potentials is measured, and also the exact measurement of that difference of potential by an absolute electrometer, a measurement of considerable difficulty.

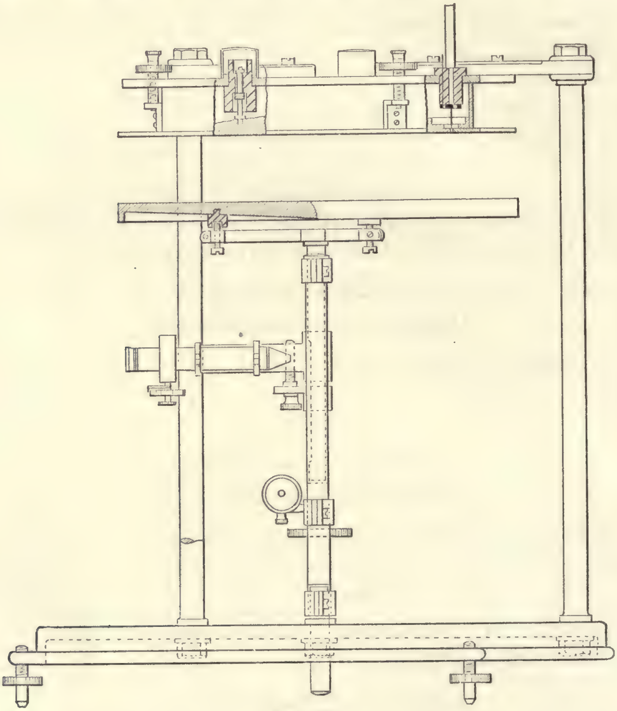

FIG. 209.

Plate condenser showing manner in which the guard-ring and plates are supported and adjusted, switch for connecting the collector plate to either guard-ring or charging wire, and the microscope for determining the position of the lower plate.

Method VI. gives $$v^2 = \frac{R_m}{R_s}, \qquad (32)$$

where R_m is the electromagnetic value of a resistance and R_s the electrostatic value. Here any error or uncertainty in the value of the ohm is halved in taking the square root of the ratio, R_m/R_s. But there is the difficulty of measuring R_s. As we shall see, the difficulty resolves itself into one of finding the time in which the difference of potential between the plates of an exceedingly well constructed air condenser falls from an initial V to a value V_0, when it is discharged through a very high resistance. This air condenser must have a very considerable capacity, and so can hardly itself be capable of having its capacity determined by direct computation from its form and dimensions. Its capacity can thus only be found by comparison with an absolute air condenser of very *much* smaller capacity, an operation difficult of accurate performance.

Of the four methods I., II., III., VI., the authors very much prefer III. There are several ways of determining C_m which are all capable of considerable accuracy when a resistance is accurately known in absolute value. We mention only two or three of these which experience has found most practical and accurate.

At first a condenser of known electrostatic capacity was charged by means of a battery and discharged through a ballistic galvanometer. The same battery was used for charging the condenser and for calibrating the galvanometer, so that the galvanometer really measured $Q_m/E_m = C_m$. The elimination in this way of E, the electromotive force of the battery, introduced a resistance which had to be known in absolute value.

Next a fork or rotating commutator was used to charge and discharge the condenser a certain ascertained number of times per second; the average value of this discontinuous current was then compared with a steady current produced by the same battery through a known resistance. This, of course, was a great improvement over a single discharge sent through a ballistic galvanometer.

But the best way of carrying out the method is the Maxwell bridge arrangement described above. Condenser and commutator are placed in one arm of the bridge, and balance is obtained by variation either of a resistance or of the speed of the fork or commutator. The method has the advantage of being a null one, and sensibility can be obtained by using high voltages and high frequencies. It was used by J. J. Thomson at Cambridge in 1883, by Himstedt in 1887, by Rosa at Baltimore in 1888, by Thomson and Searle at Cambridge in 1890, and by Rosa and Dorsey in their researches as described in the memoirs now referred to.

The interrupted current from the condenser and the steady current from the battery may be made to pass at the same time through the coils of a differential galvanometer and balance one another. This method has practically the same advantages as that of Maxwell's bridge. It has been employed by Klemenčič (Vienna, 1884), Himstedt

(Freiburg, 1886; Darmstadt, 1888), Abraham (Paris, 1892), and Rosa and Dorsey (Washington, 1905-1907).

18. Thomson and Searle's experiments. In Thomson and Searle's investigation the condenser used was cylindrical, and was provided with a guard-ring at top and bottom, so that the effect of the ends was in great measure avoided. The condenser is shown in section in Fig. 210. The dimensions of the inner cylinder were measured by accurate callipers in the most careful manner. It was found that the cylinder was slightly elliptic in section, as shown in the following statement of results of measurement:

Top end : maximum diameter	-	23·5302 cm.
,, minimum ,,	-	23·5161 ,,
Bottom end : maximum diameter	-	23·5348 ,,
,, minimum ,,	-	23·5169 ,,
	Mean	23·5245

The internal diameter of the outer cylinder was measured by callipers specially provided for this purpose with projecting steel pieces on their jaws. The results obtained for two diameters at right angles to one another at each end of the cylinder gave a mean diameter of 25·4114 cm.

The internal cylinder was supported on pieces of ebonite placed on the lower ring, and the upper ring on similar pieces on the internal cylinder. The outer cylinder was also in three parts, two ring pieces for top and bottom, and a long central piece corresponding to the internal cylinder.

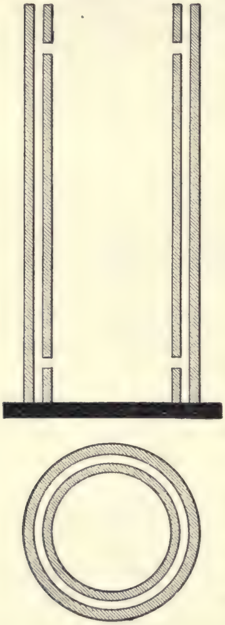

FIG. 210.

The length of the internal cylinder was measured by applying the jaws of a beam compass to its ends and measuring under microscopes first the distance between two marks, one on each jaw, then the distance between these marks when the jaws were put close.

The length of the cylinder was found to be 60·9784 cm. The correction for want of equality in the distribution caused by the two equal air spaces was calculated and found to amount, within 1 part in 2000, to a lengthening of the internal cylinder by the breadth of one air-space. The mean allowance for the gaps at the guard-ring was thus found to be ·2907 cm, so that the total effective length of the internal cylinder was 61·2691 cm.

The distance between the inner and outer cylinders was determined by fastening down the internal cylinder, and the outer cylinder of the same length, in co-axial position on a glass plate with cement, and fixing a glass cover on top; then filling, by means of two openings left in the cover, the annular space between the cylinders with water. The water was taken from a flask containing a known weight of water, and so by a second weighing of the flask the weight of water used was obtained. The weighings were all corrected to vacuum, and for error in weights, effect of temperature, etc.

The volume was found to be 4412·08 cubic cm, so that the mean distance d between the cylinders was, with the radii given above, ·94128 cm. The ratio of external and internal radii a/b used was thus $1+{\cdot}94128/11{\cdot}76225=1{\cdot}0800262$. Thus

$$C_s=\frac{l}{2\log\dfrac{a}{b}}\,\frac{61{\cdot}2691}{{\cdot}15397063}=397{\cdot}927$$

in centimetres.

The measurement of capacity in electromagnetic units was made by the method already described, somewhat modified on account of the existence of the guard-ring. The arrangement of apparatus is shown in Fig. 211. The condenser plate is shown connected as before to a contact-making piece u, which makes contact alternately with p and q, while one guard-ring is connected with a second contact-piece v, which makes contact alternately with r and s. The pieces p and q represent the contact-plates of a commutator which alternately came into contact with a spring or brush, u, connected with the inner coating of the condenser; r and s represent the contact-plates of another commutator, v a brush which alternately connected them with the guard-ring.

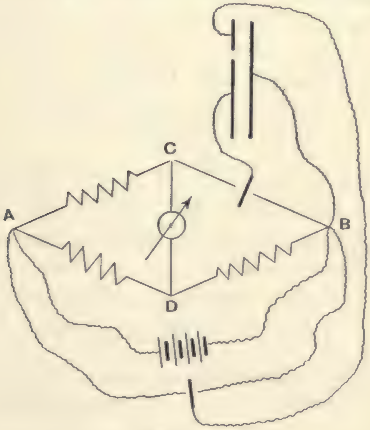

FIG. 211.

The two commutators were mounted on the same axis, so that they were kept always in the same relative position. When the commutators were worked the following contacts were made in the order indicated by the numbers. V_A, V_B, V_C, denote the potentials of the points A, B, C, respectively.

1. $\begin{cases} u \text{ on } q: \text{ condenser discharged.} \\ v \text{ on } s: \text{ guard-ring discharged.} \end{cases}$
2. $\begin{cases} u \text{ on } p: \text{ condenser begins to charge.} \\ v \text{ on } s. \end{cases}$
3. $\begin{cases} u \text{ on } p: \text{ condenser charged to potential } V_A - V_B. \\ v \text{ on } r: \text{ guard-ring charged to potential } V_C - V_B. \end{cases}$
4. $\begin{cases} u \text{ on } q: \text{ condenser begins to discharge.} \\ v \text{ on } r. \end{cases}$
5. $\begin{cases} u \text{ on } q: \text{ condenser discharged.} \\ v \text{ on } s: \text{ guard-ring discharged.} \end{cases}$

19. Theory of method. The theory of the method has been given in 14 above. We repeat it here, however, for clearness on account of the guard-ring correction. According to the notation already adopted we denote the currents in Cp, CD, BA, by $\dot{x}$, $\dot{z}$, $\dot{u}$; in addition, in the present case we have, when v is in contact with r, a current in Ar. Let this be denoted by $\dot{w}$. The circuits $ACDA$, $ADBA$, give the equations

$$-Q(\dot{u}-\dot{x}-\dot{z}-\dot{w})+P(\dot{x}+\dot{z})+L\ddot{z}+G\dot{z}=0,$$
$$Q(\dot{u}-\dot{x}-\dot{z}-\dot{w})+S(\dot{u}-\dot{x}-\dot{w})+B\dot{u}-E=0.$$

At the beginning and end of the charging the currents have their steady values, and then

$$-Q(\dot{u}_s-\dot{z}_s)+P\dot{z}_s+G\dot{z}_s=0,$$
$$Q(\dot{u}_s-\dot{z}_s)+(B+S)\dot{u}_s-E=0.$$

These subtracted from the preceding pair of equations for the varying state, give, if $\dot{u}_1$, $\dot{z}_1$ denote $\dot{u}-\dot{u}_s$, $\dot{z}-\dot{z}_s$, respectively,

$$\left.\begin{aligned} -Q(\dot{u}_1-\dot{z}_1-\dot{x}-\dot{w})+P(\dot{x}+\dot{z}_1)+L\ddot{z}+G\dot{z}_1=0, \\ Q(\dot{u}_1-\dot{z}_1-\dot{x}-\dot{w})+S(\dot{u}_1-\dot{x}-\dot{w})+B\dot{u}_1=0. \end{aligned}\right\} \quad \text{.........(33)}$$

These integrated from the beginning of the charging to the end yield

$$\left.\begin{aligned} (P+Q)x+(G+P+Q)z_1+Qw-Qu_1=0, \\ -(Q+S)x-Qz_1-(Q+S)w+(Q+S+B)u_1=0, \end{aligned}\right\} \quad \text{.......(34)}$$

where x, as before, denotes the whole charge of the inner coating of the condenser, while w denotes that of the guard-ring.

Elimination of u_1 from (34) gives

$$\{P(Q+S+B)+BQ\}x+BQw$$
$$=-\{(G+P+Q)(Q+S+B)-Q^2\}z_1 \text{..................(35)}$$

This differs from the former equation (26) only in having the term BQw on the left.

When the condenser is fully charged we have as before

$$x = C_m\left(G + S\frac{G+P+Q}{Q}\right)\dot{z}_s, \quad \text{.........................(36)}$$

and further, if C'_m be the capacity of the guard-ring,

$$w = C'_m\left(G + P + S\frac{G+P+Q}{Q}\right)\dot{z}_s, \quad \text{.....................(37)}$$

since the multiplier of C'_m on the right is the final difference of potential between A and B.

Again, if there be no galvanometer deflection $\dot{z}_s + nz_1 = 0$, or $\dot{z}_s = -nz_1$, so that (36) and (37) become

$$\left.\begin{aligned} x &= -nC_m\left(G + S\frac{G+P+Q}{Q}\right)z_1, \\ w &= -nC'_m\left(G + P + S\frac{G+P+Q}{Q}\right)z_1. \end{aligned}\right\} \quad \text{...............(38)}$$

These substituted in (35) give

$$\begin{aligned} nC_m\{P(Q+S+B)+BQ\}\{S(G+P+Q)+GQ\} \\ +nC'_m BQ\{(G+P)Q+S(G-P-Q)\} \\ =Q\{(G+P+Q)(Q+S+B)-Q^2\}. \quad \text{......(39)} \end{aligned}$$

The second term on the left was negligible in the experiments made, inasmuch as the resistance B of the battery was small in comparison with the other resistances. Thus the value of C_m was given as before by (29). It was necessary to apply a correction for the small difference of potential δV between the guard-ring and the inner cylinder after charging, which prevented the distribution on the inner cylinder from being so nearly uniform as it otherwise would have been. It is shown in the paper that this correction could be made by adding to the internal cylinder a strip of breadth

$$h\left(\frac{t}{c} - \frac{2}{\pi}\log\frac{4c}{he}\right)\frac{\delta V}{V},$$

where V is the difference of potential between the cylinders, t the thickness of the guard-ring, c the half thickness of the pieces of ebonite supporting the guard-ring, h the distance between the cylinders, and e the base of the Naperian system of logarithms. The coefficient of $\delta V/V$ was approximately 7·5, and from the values given above

$$\delta V = -\left\{G + S\frac{G+P+Q}{Q} - G - P - S\frac{G+P+Q}{Q}\right\}\dot{z}_s = P\dot{z}_s,$$

$$V = \left(G + S\frac{G+P+Q}{Q}\right)\dot{z}_s;$$

so that

$$\frac{\delta V}{V} = \frac{1}{183} \text{ nearly.}$$

Thus the correction was a strip of breadth 7·5/183 cm, or about 1 part in 1800 of the whole.

20. Description of commutators. Each commutator consisted of two rings with projecting semi-cylindrical pieces overlapping, as shown in Fig. 212, mounted on an ebonite casing round the common axis.

Two springs, shown in Fig. 212, made permanent contact with grooves in the ring portions of the contact-pieces, and formed the connections to the points CA and AB of the bridge [Fig. 211]. The charging contacts on the commutator were made with a brush of fine brass wire. On the axle are fixed the driving pulleys and a stroboscopic disk for the observation of the speed, by means of a maintained fork in the manner already sufficiently described in 15 above. A side view of the stroboscopic disk is shown on the right in Fig. 212, and of the arrangement of contact wires (with disk removed) in Fig. 213.

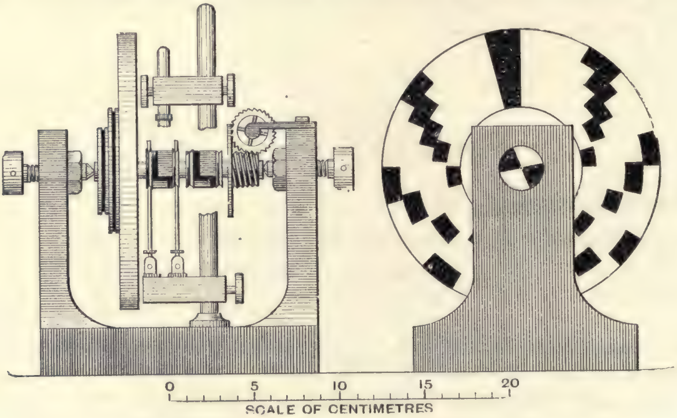

FIG. 212.

The worm-wheel and endless screw were used to make a contact with a spring at every revolution of the wheel, that is every 30 turns of the commutator, to excite one of the electromagnets of the recording apparatus referred to below. The commutator was driven by a water-motor and long cord made of fishing-line joined in a long splice to prevent inequalities in speed. The speed was regulated by letting the cord run through the fingers.

The stroboscopic disk, Fig. 212, had, as shown, five circles containing 4, 5, 6, 7, 8 black spots at equal intervals; the fork making 64 complete vibrations per second, and the commutator not running much faster than 80 revolutions, the speeds of the disk from 16 revolutions per second upwards when a stationary pattern was visible were the fractions of 64 revolutions per second:

$$\tfrac{1}{4},\ \tfrac{2}{7},\ \tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{1}{2},\ \tfrac{4}{7},\ \tfrac{2}{3},\ \tfrac{3}{4},\ \tfrac{4}{5},\ \tfrac{6}{7},\ 1,\ \tfrac{8}{7},\ \tfrac{6}{5},\ \tfrac{5}{4},\ \tfrac{4}{5},\ \tfrac{4}{3},$$

The electrically driven fork maintained another of about twice its frequency, and the latter gave beats with Lord Rayleigh's standard fork, so that the speed of the observing fork was obtained.

The frequency of the standard fork was redetermined by causing the worm-wheel driven by the commutator to make a mark on a running tape every 30 revolutions of the commutator. This was effected by the completion of a circuit which excited an electromagnet, and thereby caused an armature to descend slightly, and bring an inked roller down on the paper. A mark was similarly made on the tape every second by the completion of a circuit by the laboratory clock. Fig. 214 shows the electromagnets, armature, and marking roller, with an inking drum above, on which the roller made contact when the armature was not pulled down.

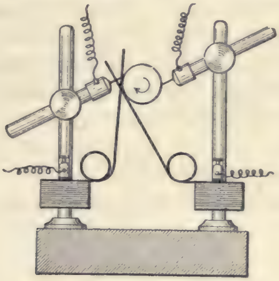

FIG. 213.

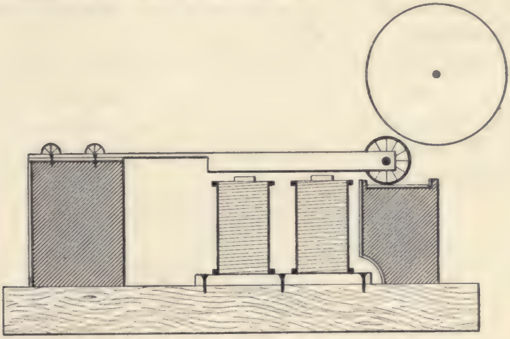

FIG. 214.

21. Mode of experimenting. The method of experimenting was as follows.

The beats between standard and auxiliary forks were counted. The motor was then started and the commutator kept at a constant speed by the disk, and after the apparatus was stopped the beats were again counted. Thus the speed of the observing fork was directly measured, and that of the standard obtained from the beats. Three observations gave a mean of 128·1045 for the frequency at 16° C., a slightly smaller frequency than that found by Lord Rayleigh. The difference was attributed to secular softening of the steel in the intervening six or seven years.

The resistances were taken from resistance boxes which were carefully compared with standard coils.

The galvanometer had a resistance of 17380 ohms, and had two coils of about 16000 turns each. The coils were very carefully insulated, and showed no leakage when tested by a gold-leaf electroscope.

The current was produced with 36 small storage cells, arranged in two parallels of 18 cells each. It was also carefully insulated.

All the quantities observed were corrected with great care for temperature variations, and the capacity of the connecting wires to the condenser was taken into account.

Three sets of experiments 7, 10, and 6 in number were taken, and gave as mean values of C_m,

$$443{\cdot}471 \times 10^{-21}, \quad 443{\cdot}417 \times 10^{-21}, \quad 443{\cdot}569 \times 10^{-21} \text{ c.g.s.};$$

or as mean of all $C_m = 443{\cdot}486 \times 10^{-21}$ c.g.s. electromagnetic units. Thus

$$v = \sqrt{\frac{397{\cdot}927}{443{\cdot}486 \times 10^{-21}}} = 2{\cdot}9955 \times 10^{10},$$

in cm per second.

22. Methods IV. and V. Comparison of a capacity and an inductance. Determination of the product of a capacity and an inductance. Methods of comparing the capacity C_m of a condenser with the self-inductance L of a coil have been given above, Chapter XIV. 48 *et seq.* If then the capacity of a condenser has been thus found, in terms of a self-inductance L which can be exactly calculated, the value C_s in electrostatic units can be found either directly by calculation for the condenser, or, if that is not possible, by comparison with the accurately known capacity of a standard condenser.

Thus if
$$C = \frac{L}{QR},$$
we have
$$v = \sqrt{\frac{C_s}{C_m}} = \sqrt{\frac{C_s QR}{L}}. \quad \text{......................(40)}$$

The next two methods are mainly of theoretical interest. According to Method V., which was suggested by the author, but has never been carried out, a magnet is rotated within a coil suspended with its plane vertical by a bifilar. The current induced in the coil causes it to turn round a vertical axis, and, if the period of rotation be constant and small in comparison with the period of vibration, to take up a constant deflection. The coil is in circuit with a fixed coil of considerable self-inductance, so that the whole inductance of the circuit is L, and with a condenser of capacity C. The value of CL can be found by observing the deflections D_1, D_2, D_3, for three different angular velocities n_1, n_2, n_3, of the magnet. Then

$$C^2L^2 = \frac{1}{n_1^2 n_2^2 n_3^2} \frac{\sum \frac{n_1^3}{D_1}(n_2^2 - n_3^2)}{\sum \frac{n_1}{D_1}(n_2^2 - n_3^2)}. \quad \text{..................(41)}$$

If the induction through the coil due to the magnet when its axis is parallel to that of the coil be M, then when the magnet has turned through the angle θ from that position the induction is $M \cos\theta$, or $M \cos nt$, if n denote the angular velocity, and t be reckoned from the instant at which $\theta = 0$.

23. Theory of Method V. If x be the whole quantity of electricity which has flowed through the circuit from the era of reckoning, the current is $\dot{x}$, and the induction through the circuit due to the current in it is $L\dot{x}$. Thus if E denote the difference of potential between the plates of the condenser, the electromotive force producing the current is $E + d(L\dot{x} + M \cos nt)\, dt$, and the equation of currents is

$$R\dot{x} + \frac{d}{dt}(L\dot{x} + M \cos nt) + E = 0.$$

But $CE = x$, so that this equation becomes

$$CL\frac{d^2x}{dt^2} + CR\frac{dx}{dt} + x = CMn \sin nt. \quad \text{..................(42)}$$

From (42) it is clear that the values of CL and CR are the same whether the electromagnetic or the electrostatic system of units is used.

This differential equation is one of forced oscillation, so that for x we have the equation

$$x = \frac{MCn}{\sqrt{R^2C^2n^2 + (1 - CLn^2)^2}} \cos(nt - e), \quad \text{................(43)}$$

where
$$\tan e = -\frac{1 - CLn^2}{RCn}.$$

The couple on the suspended coil produced by electromagnetic action is at time t

$$\Theta = \dot{x}nM \sin nt,$$

and the mean value Θ of this over one revolution is, since $2\pi/n$ is the period,

$$\overline{\Theta} = -\frac{n}{2\pi} \frac{M^2Cn^2}{\sqrt{R^2C^2n^2 + (1 - CLn^2)^2}} \int_0^{2\pi/n} \sin(nt - e) \sin nt \,.\, dt$$

$$= -\frac{1}{2} \frac{n^3M^2C^2R}{R^2C^2n^2 + (1 - CLn^2)^2}. \quad \text{......................................(44)}$$

If the coil have a sufficiently great moment of inertia the variations of the couple acting on it will not cause it to oscillate sensibly, but it will take up a position of equilibrium depending on the mean couple $\overline{\Theta}$.

The mean deflection D is proportional to $\overline{\Theta}$, and so

$$P\frac{n}{D} = R^2C^2 + \left(\frac{1}{n} - CLn\right)^2, \qquad\qquad (45)$$

where P is a constant. By means of three different angular velocities three equations of this form are obtained, which give (41) by elimination of P and R.

If the experiment were carried out it would be desirable to take say n_2 as that for which n/D is a minimum, that is $n_2{}^2 = 1/CL$, and n_1, n_3, one greater, the other less than n_2.

Since $v^2 = C_s/C_m$, we have, if L_s, L_m denote the electrostatic and electromagnetic values of L,

$$C_sL_m = v^2C_sL_s = v^2C_mL_m.$$

Hence
$$v^2 = \frac{C_sL_m}{C_mL_m}. \qquad\qquad (46)$$

The denominator of the expression on the right is determined experimentally, as explained above, and the numerator is obtained by direct calculation of C_s and L_m, or by comparison of the condenser and circuit with proper standards.

24. Method VI. Electrostatic measurement of a high resistance. Method VI. involves the determination of the electrostatic value, R_s, of a high resistance, through which a condenser of capacity C_s is discharged. This can be done by measuring the rate of fall of difference of potential between the plates of the condenser by means of an electrometer connected with them. If V be the electrostatic value of the difference of potential at any time t we have

$$C_s\frac{dV}{dt} + \frac{V}{R_s} = 0,$$

and therefore
$$\log V + \frac{t}{C_s} = A,$$

where A is a constant. If V be the difference of potential t seconds after it was V_0, we get from this equation

$$\frac{t}{C_sR_s} = \log\frac{V_0}{V}$$

or
$$R_s = \frac{t}{C_s}\,\frac{1}{\log\dfrac{V_0}{V}}.$$

If $V = \frac{1}{2}V_0$, $R_s = t/C_s \log 2$.

If now R_m is known we have, since $C_sR_s = C_mR_m$, $R_m/R_s = C_s/C_m = v^2$, and therefore

$$v^2 = \frac{R_mC_s \log 2}{t}. \qquad\qquad (47)$$

25. Method of electrical oscillations. The method of electrical oscillations has been used by Lodge and Glazebrook.* An air condenser was made to discharge through a coil of measurable inductance across a spark-gap between a pair of knobs about a millimetre apart. The condenser consisted of 11 squares (each 2 feet in side) of plate glass silvered on both sides, set up parallel to one another with a distance of 5 mm between each pair of opposed silvered surfaces, and the silvered surfaces of the alternate plates joined metallically to form the coatings of the condenser. It had thus a capacity of about 600 metres in electrostatic measure. The coil was composed of about three miles of india-rubber covered wire of No. 22 gauge, and had diameters 19 in and 11 in, and thickness 4 in. Its inductance was about $4{\cdot}5 \times 10^9$ cm in electromagnetic measure.

The condenser was charged by a Voss machine arranged to give a brush discharge across half an inch of air to the inner coating, while the other coating (that is, the two outer plates and the four alternate interior plates) were connected to earth.

The sparks were photographed on a revolving sensitive plate on which the knobs were focussed by a quartz lens. The plate was driven by a water motor at a speed of about 64 turns per second, and its speed measured as in Lord Rayleigh's determination of the ohm, by observation of a stroboscopic disk through a slit alternately opened and closed by the vibration of an electrically maintained tuning-fork. The result was that a pattern was produced on the plate consisting of a long band, with a bead-like broadening for each half-oscillation. The period of vibration was thus measured with great exactness.

The resistance and inductance of the circuit could also be obtained with very considerable accuracy, as the resistance of the spark-gap was inappreciable. The value of L for the coil could also be obtained by direct calculation or by comparison with another coil.

The value of the period given above [VIII. 14 (53)] furnishes for these data the electromagnetic value of the capacity of the condenser. Also C_s can be found from an exact comparison with a standard condenser, and thus v can be obtained by (21) above. There is left, however, the important question of the distributed capacity of the coil, which involves certain points of theory which so far have not yet been satisfactorily dealt with. The method certainly involves very considerable difficulties.

The final results of the experiment do not seem to have been published.

26. General table of results. The following table gives the values of v obtained by different experimenters, and for comparing the velocity of light as determined experimentally by the methods of Fizeau and Foucault. It is in great part taken from Mr. E. B. Rosa's 1889 paper already referred to. The various results given were corrected by Rosa where necessary to the value ·98664 ohm for the B.A. unit. Since 1889 much work has been done, and a table as given by Rosa is extended so as to comprise the chief more recent results corrected to date.

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* *B.A. Report*, 1889, or *Electrician*, vol. 23 (1889), p. 544.

Ratio of Units.		
Date.	Experimenter.	v in cm per sec.
1856[1]	Weber and Kohlrausch	3·107 × 10^{10}
1868[2]	Maxwell	2·842 × 10^{10}
1869[3]	W. Thomson and King	2·808 × 10^{10}
1872[4]	McKichan	2·896 × 10^{10}
1879[5]	Ayrton and Perry	2·960 × 10^{10}
1880[6]	Shida	2·955 × 10^{10}
1881[7]	Stoletow	2·99 × 10^{10}
1881[8]	Klemenčič	3·019 × 10^{10}
1882[9]	F. Exner	2·92 × 10^{10}
1883[10]	J. J. Thomson	2·963 × 10^{10}
1889[11]	Rowland	2·9815 × 10^{10}
1889[12]	E. B. Rosa	3·0004 × 10^{10}
1889	W. Thomson	3·004 × 10^{10}
Later results.	Himstedt	3·0057 × 10^{10}
	Rosa	3·000 × 10^{10}
	Thomson and Searle	2·9960 × 10^{10}
	H. Abraham	2·9913 × 10^{10}
	Pellat	3·0092 × 10^{10}
	Hurmuzescu	3 0010 × 10^{10}
	Perot and Fabry	2·9978 × 10^{10}
	Mean	3·0001 × 10^{10}

Velocity of Light.		
Date.	Experimenter.	Vel. of Light in cm per sec.
1874	Cornu— (Method of Fizeau)	2·9850 × 10^{10}
1874	,, ,,	3·0040 × 10^{10}
1880–1	Young and Forbes— (Method of Fizeau, modified by Forbes)	3·0138 × 10^{10}
1879	Michelson— (Method of Foucault)	2·9991 × 10^{10}
1882	,, ,,	2·9985 × 10^{10}
1882	Newcomb— (Method of Foucault)	2·9986 × 10^{10} 2·9981 × 10^{10}
1902	Perrotin— (Method of Fizeau)	2·9986 × 10^{10}

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[1] *Pogg. Ann.* 1856.

[2] *Phil. Trans. R.S.* 68.
or *Rep. of Papers*, 2, p. 125.

[3] *B.A. Report*, 1869.

[4] *Phil. Trans. R.S.* 1872.

[5] *Jour. Soc. Tel. Eng.* 1879.

[6] *Phil. Mag.* 10, 1880.

[7] *Soc. Franc. de Phys.* Nov. 1881.

[8] *Wien. Ber.* 89, 1884.

[9] *Wien. Ber.* 86, 1882.

[10] *Phil. Trans. R.S.* 1883.

[11] [12] *Phil. Mag.* Oct. 1889.

CHAPTER XVII.

ELECTROSTATIC MEASUREMENTS.

1. Electrometers. The subject of electrostatic measurements has become much more important since the discovery of radio-activity. Before this discovery the manifold applications of electricity in the industries, involving as they did for the most part the utilization of electromagnetic and electrolytic action, had concentrated the attention of electricians on the phenomena of electric currents, and caused the relegation of electrostatic phenomena to a comparatively subordinate place. At first the use of the gold-leaf electroscope as an indicator of electrostatic potentials was made very general; but, later, various forms of sensitive electrometers were constructed for more exact measurement of such potentials.

In the discussions of electrostatic measurements which follow, the usual theorems of the action and of the energy of charged conductors will in general be assumed. For proofs of the various theorems stated and used, the reader may refer to Maxwell's *Electricity and Magnetism*, Webster's *Electricity and Magnetism*, or Gray's *Magnetism and Electricity*.

2. Attracted disk electrometers. The first accurate electrometer devised was Coulomb's torsion balance, which gave good results in the hands of Coulomb himself and of Faraday, and its action is very instructive. It has, however, been almost entirely superseded by much more delicate and convenient electrometers, chiefly belonging to two classes:

I. Attracted-disk electrometers.
II. Symmetrical electrometers.

We give here some account of these two classes of electrometers.

The first electrometer of the first class seems to have been made by Sir William Snow Harris[*] about the year 1834. At the time of its construction there was little general appreciation of the exact mode of distribution of electricity on conductors in different circumstances. It was observed that when disks were placed parallel and near to one another the attraction between them was independent, or nearly so, of the unopposed surfaces—the backs of the disks—but from this, and other observations, no general law of electric action was deduced. It is shown in Fig. 215. A disk *d* is suspended as one scale of a balance above a similar disk *a*, connected with the interior coating of a Leyden jar *J*, the potential of which is to be tested. The other scale of the balance is weighted, so as to equilibrate *d* when there is no electrification. When *a* is charged *d* is attracted, and equilibrium is restored by placing weights in *P*.

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[*] "On the Elementary Laws of Electricity," *Phil. Trans.* 1834.

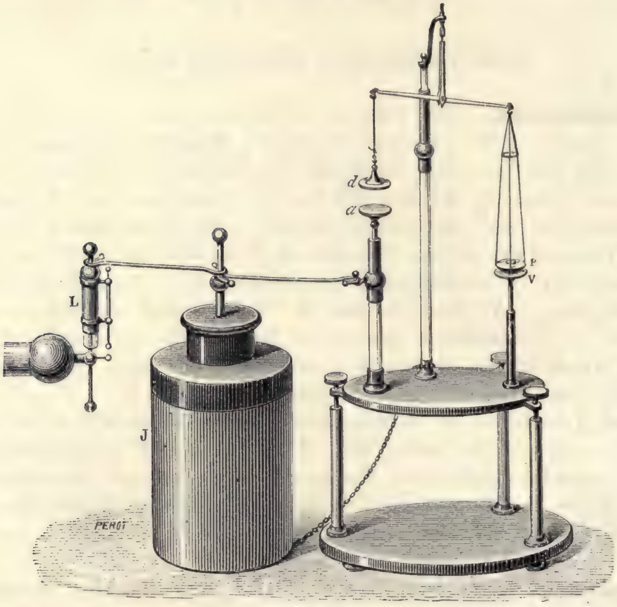

FIG. 215.

The downward pull on *d* in a definite position of equilibrium is thus obtained in absolute units of force from the known force of gravity on the mass placed in *P*. The arrangement marked *L* is a " unit-jar " which was used in the experiments of Snow Harris to give a rough approximation in arbitrary units to the charge of the jar *J*. For when a certain difference of potential, which can be regulated by the length of the spark-gap shown between two small knobs in the figure, was attained by the prime conductor of the machine, the unit-jar discharged itself to the inner coating of the large jar *J*, the charge of which was thus said to contain as many units as there had been discharges of the small jar.

This form of electrometer is exceedingly defective in many respects, but contains in a rudimentary form the principle of an attracted-disk electrometer.

One serious imperfection of the electrometer devised by Snow Harris was that non-uniformity of the distribution on the opposed disks prevented any accurate expression of the difference of potential between them in terms of the force of attraction. This can be remedied to a high degree of accuracy by surrounding the attracted disk *C* by what is called a "guard-ring," as shown in Fig. 216. When the disk and ring surfaces opposed to the attracting disk *A* are in plane, it may be assumed that distribution of electricity on the disk is approximately uniform. The disk acts as the inner part of a large disk, the outer edge of which is that of the ring.

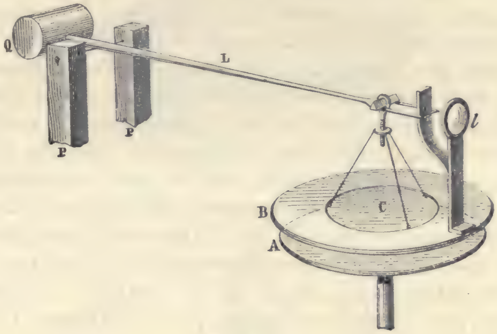

FIG. 216.

The disk *C* is hung by wires from one end of a metal beam pivoted on a horizontal wire stretched between the pillars *PP*, and twisted so that the torsion and the counterpoise *Q* tend to raise *C*. A downward force is applied to *C* to bring it into the plane of the guard-ring against this action, and this is obtained from weights placed on the upper surface, so as to bring the lever, from which the disk is hung, into a sighted position. The lever is forked at the outer end as shown, and across the fork is stretched a horizontal black hair, which, when the lever turns, moves in front of a white plate carried by the stand of the lens *l*. When the lever is in the sighted position it lies between two black dots on the white surface, which can be viewed through the lens. The lens is placed at a distance from the hair slightly less than the focal distance, and the eye is 20 cm. or more from the lens. Parallax is avoided by placing the lens with its convex side towards the hair, and moving the eye up or down until the hair seems straight in the middle, and to widen out at the ends equally above and below. A very slight deviation of the hair from the position of no parallax is possible with this arrangement. Lord Kelvin and Dr. J. P. Joule corrected in this way so small a deviation as 1/50,000 of an inch. The disk and guard-ring are electrically connected by a wire which joins the guard-ring with the metal pillars *PP*.

The disk nearly fills the aperture in the guard-ring, and its effective area, reckoned as uniformly charged on the side turned towards the disk B, is approximately the mean of the areas of the aperture and the plate A. (See Maxwell, *Electricity and Magnetism*, i. p. 308, 3rd ed., where also a closer approximation will be found.)

The attracting plate is carried by an insulating pillar attached to a micrometer screw, by which the plate can be moved upward or downward through measured distances.

3. Method of use and theory of an attracted disk electrometer. The method of using the electrometer is as follows: A constant difference of potentials is maintained between one of the plates, say the disk and guard-ring, and the earth, and the other plate is connected to earth. The latter is then raised or lowered until the attracted disk is brought into the sighted position, and the micrometer screw is read. The plate A is then connected to the body to be tested, and the attracted disk brought once more to the sighted position, and the micrometer again read. Then the value of V, the potential of the plate, can be found.

Let d be the distance between the plates and S the effective area of the part of the attracted plate surrounded by the ring. Then the field intensity between the plates is $V/d=4\pi Q/S$. The whole force of attraction on the charged area S is $\frac{1}{2}QV/d$. Denote this by F, and we get

$$F=\frac{V^2S}{8\pi d^2}, \quad \text{or} \quad V=d\sqrt{\frac{8\pi F}{S}}.^* \qquad \text{.......................(1)}$$

In the mode of using the instrument just described, let V' be the difference of potentials between the plates when the movable plate is connected to the body to be tested, V that between the earth and the guard-ring when the other reading of position is taken, and d', d be the two readings; then we have by the result just obtained,

$$V'-V=(d'-d)\sqrt{\frac{8\pi F}{S}}. \qquad \text{..................................(2)}$$

This is the difference of potentials between the body tested and the earth, and is obtained in absolute c.g.s. units of potential, if $d'-d$ be taken in cm, and F in dynes. The result thus depends only on a determination of the difference of the distances of the plates apart in the two positions, and not on the determination of the plates apart in any positions, which it would be relatively difficult to carry out with accuracy.

The electrification independent of that to be tested, which is maintained in the plate B, is in general produced by keeping B in contact with the inner coating of a Leyden jar, the electrification of which can be tested by a proper gauge in the manner described below, and by means of a proper electrifying device brought to the required value, if it should vary from that value. Hence the mode of using the electrometer in which this electrification is employed has been called *heterostatic.* If the electrification to be tested is alone made use of, the instrument is said to be used *idiostatically*.

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* It will be remarked here that the force exerted on one plate by the other is equal to the charge on the former multiplied by the field intensity due to the latter. Thus we have in the present case the whole charge on the attracted disk multiplied by the field intensity due to A, that is by $\frac{1}{2}V/d$.

4. Lord Kelvin's absolute electrometer. The Kelvin absolute electrometer acts according to the principles which have just been explained

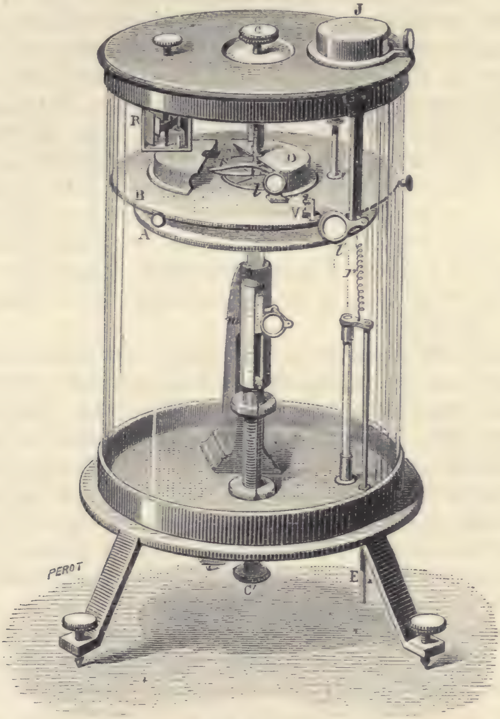

FIG. 217.

It is shown in Fig. 217. The attracted disk and plates are contained within a cylindrical case of white glass, carefully selected for insulation, which is fixed by a brass mounting round its lower end to a horizontal sole-plate of iron, supported on three feet with levelling screws, and is closed above by a stout brass plate screwed to a brass ring cemented round the upper end. The height from sole-plate to cover is 50 centimetres, and the diameter is 30 centimetres. The sides of the case, with the exception of apertures to permit observations of the interior points to be made, are coated inside and outside with tin-foil nearly as high as the plates, which are in the upper part of the jar.

The case thus forms a Leyden jar, the coatings of which can be brought to any necessary difference of potentials. The guard-ring *B* is connected with the interior coating by its supports, which are metal pieces cemented to the inner side of the jar and covered with tin-foil. Within the jar, on the sole-plate, is placed a leaden tray containing pumice moistened with sulphuric acid, which maintains a dry atmosphere within the jar.

The attracting plate *A* is of stout brass, with pieces cut out of it to allow it to pass the supports of the guard-ring, and is supported by an insulating stem of white glass cemented to a vertical brass pillar, which is moved up and down in V guides by the micrometer screw *C′*, working in a fixed nut in the sole-plate. The amount of vertical motion is observed by means of the scale *m*, and a circular plate graduated on its edge, which projects from the screw and turns in front of a fine vertical wire. The former shows the number of complete turns made by the screw, the latter allows any fraction of a turn to be estimated to a degree of accuracy depending on the fineness of the graduation, and the precision with which the position of the wire on the circle can be read. The actual distance traversed is got from this result by multiplying by the step of the screw, which, in the first instrument made, was $\frac{1}{50}$ of an inch.

The attracted disc is made for lightness of thin aluminium strengthened by a thick rim and radial ribs on its upper side; and is made as nearly as possible perfectly plane on its under side. Instead of being hung from one arm of a balance like the disk shown in Fig. 216, it is supported by three delicate springs, similar in shape to coach-springs, of which one only is shown in Fig. 217, projecting from underneath the cover *D*. These springs are placed symmetrically round the disk and meet at their points of crossing above and below. The disk is attached to the lower point of crossing, and the upper point of crossing is attached to the lower end of an insulating stem carried at its upper end by a brass tube which slides in V guides, and can be moved up and down by the head *C* of a micrometer screw similar to that already described as moving the attracting plate *A*. Underneath this screw-head and fast to it is a micrometer circle, which serves to determine fractions of a turn, while complete turns are shown by the divisions on a vertical scale. The actual distance through which the disk is moved in any given case need not be known; all the upper micrometer screw gives is merely a comparison of distances.

Two small uprights stand on the centre of the disk, and between these is stretched a fine black hair, of which an image is formed in the conjugate focus by the achromatic lens *l*. The lens is so adjusted that this focus is between two screw points *V*, which are so placed as to touch the image above and below when the disk is in the sighted position. The image is observed through an eye-lens *l′* attached outside the jar to the brass mounting, and then, since the points and the image of the hair are in focus in the same position of this lens, there is no error due to parallax.

The attracted disk and springs are inclosed within the metallic box *D* (of which one-half is shown displaced) to prevent disturbance by external electrification. The hair is seen through a hole cut in the box opposite the lens.

5. Gauge for testing the electrification of the jar. The difference of potentials between the inner and outer coatings of the jar is tested by an auxiliary attracted disk electrometer used idiostatically. This electrometer, which is called the *gauge*, is contained within the cylindrical box *J* on the cover of the jar. The arrangement is shown in detail in Fig. 218. The disk is a square piece of aluminium forming a continuation of a lever *h* of the same metal. This lever is forked and the prongs joined by a black opaque hair which moves in front of an enamelled plate on which are two black dots as already described. The position of the hair is seen through the plano-convex lens *l*, which is carried by a sliding platform attached to the guard-ring *G*. Instead of the counterpoise shown in Fig. 216, torsion given to the platinum wire *f*, to which the lever is attached in the manner shown in Fig. 218, and round which the lever turns as a fulcrum, forces the disk upwards. This upward force is balanced when the hair is in the sighted position by electric attraction between the disk and a parallel plate below it, which is in contact with the interior coating of the jar while the guard-ring and disk are in contact with the exterior coating. The attracting plate below is mounted on a fine screw, by which its distance from the disk and therefore the sensibility of the gauge can be varied at pleasure within certain limits. The sensibility of the gauge varies with any alteration in the elasticity of the torsional spring *f*. This however is of little consequence as the variations are not sudden, and it is never necessary to know the actual potential of the jar.

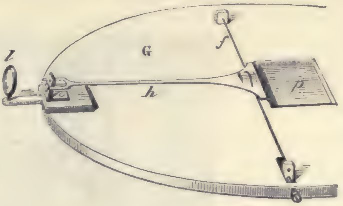

FIG. 218.

Between each end of the wire *f* and the supporting block is interposed a U shaped spring (not shown in Fig. 218) made of light brass. The end of the wire is attached to the extremity of one limb of the U, a pin passing through the supporting block to the extremity of the other limb. The two pins, the extremities of the springs, and the wire are in line. The springs can be turned round the pins as axes, so as to give any initial torsional couple to the wire which may be required, and by their spring cause the wire to be stretched with a nearly constant force.

The mode of attachment of the wire to the lever *h*, deserves notice. The wire is threaded through two holes in the broader part of the lever, near the square disk, so that the part between the holes is above the lever. Halfway between the holes it passes over a small ridge piece of aluminium, which prevents the lever from turning round without twisting the wire.

The plate A when the instrument is used is connected with the body to be tested by the electrode E, which passes through a plug of clean paraffin fixed in an aperture in the sole-plate. The wire r completes the connection between E and A.

6. The replenisher. The difference of potentials between the coatings is kept nearly constant by means of a small induction machine R, called by Sir William Thomson the *Replenisher*. The construction and action of this part will be easily understood from Figs. 219 and 220.

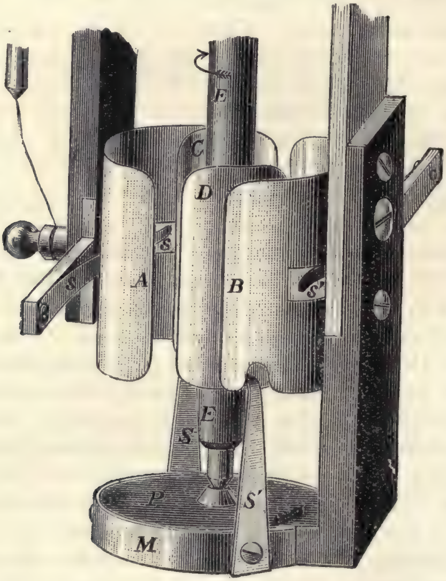

FIG. 219.

Fig. 219 shows the mechanism full-size in perspective elevation ; Fig. 220 the same in plan.

Two similar metal carriers, C, D, each part of a cylindrical surface, are fixed on a cross-bar of paraffined ebonite so as to be slightly non-coaxial but inclined at the same angle to the cross-bar. Through the cross-bar and rigidly fixed to it, passes a cylinder of ebonite having at its ends metal pieces which form the extremities of an axle. The carriers turn round this axle within the circular cylinder marked out by the cylindrical metallic pieces A, B, which are insulated from one another and act as inductors. A receiving spring s or s' projects obliquely inwards through a hole in each conductor, with which it is also connected at the back, and is bent so that the carriers touch the springs on their convex sides, and pass on but little impeded by the friction. Two contact springs, S, S', connected by a metallic arc project slightly inwards beyond the inductors so that the carriers, while opposite the inductors, come into contact with these two springs at the same time, and are therefore put into conducting contact. One of the inductors, A, is connected to the inner coating of the jar, the other, B, is attached by the supporting plate of brass to the cover of the instrument and therefore to the outer coating. A milled head attached to E projects above the cover and forms a handle by which the carriers are twirled round.

The electrical action is on the whole that of an electric machine which multiplies by induction small initial charges. It is easily made out and understood. An initial charge has been given to the jar, so that a difference of potentials exists between the coatings, the interior for example being positive. When the carriers come into contact with the springs S, S', they are brought to the same potential, and, since they are under the influence of the inductors, one carrier becomes charged positively, the other negatively. Then, turning in the direction of the arrow, they come into contact with the receiving springs, and being each (electrically) well under cover of the corresponding inductor, they give up the greater part of their charges, thus increasing the difference of potentials between the inductors.

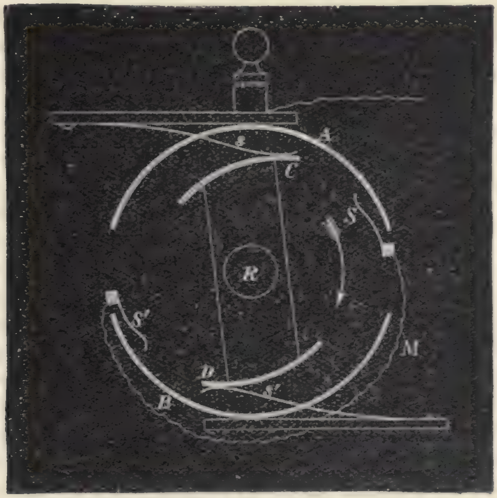

FIG. 220.

If the carriers are turned in the opposite direction the action is of course reversed, and the difference of potentials is diminished. When the replenisher is not in action the carriers are not in contact with any of the springs.

7. Method of using the absolute electrometer. The method of using the absolute electrometer is practically the same as that described for the more rudimentary instrument of Fig. 216. The force required to depress the disk against the action of the springs without overstraining is, however, not determined by torsion, but by weighing. The top cover of the jar and the cover of the balance are removed and the disk is loaded as symmetrically as possible with weights, while all electrical force is avoided by putting the electrode of the plate *A* in contact with the guard-plate *B*. The micrometer-screw *C* is then turned until the disk comes again into the sighted position, and the distance through which the plate was depressed is obtained from the initial and final micrometer reading in terms of divisions of the scale. (It was found in the original instrument made for Sir William Thomson that $\frac{6}{10}$ of a gramme depressed the disk through two divisions of the vertical scale and a fraction of one division on the graduated head.) Several determinations of this distance are made at different temperatures to obtain data for the elimination of the effects of temperature on the springs. The weights are now removed, the covers replaced, and the instrument is ready for use in absloute measurements.

When the electrometer is to be thus used the guard-ring and attracting plate are put into conducting contact by connecting the electrode of the latter with the charging rod let down through the aperture provided for it in the cover, and the disk is put accurately into the sighted position. It is then raised by the micrometer-screw through a distance for which the force *F* has been determined. To bring it back to the sighted position will require the application of that force. The jar is next charged to the degree determined by the sensitiveness of the gauge, and the potential kept constant by using the replenisher as described. The attracting plate is now connected by means of its electrode with the exterior coating of the jar, and the micrometer moved up or down until the disk is brought into the sighted position, when the micrometer reading is taken. This is called the *earth-reading*. The electrode of the attracting plate is now brought into contact with the conductor whose potential is to be tested, and the plate again moved by the micrometer until the disk is once more in the sighted position and the reading once more taken. The difference between the two readings gives $d' - d$ of (2), p. 692 above, which, since F has been determined, and S is supposed known, gives in absolute units the difference of potentials $V' - V$ between the conductor tested and the outer coating of the electrometer jar.

Sir William Thomson also constructed a small attracted disk electrometer capable of being easily carried about from place to place, and therefore adapted for observations of atmospheric electricity at different places at rapid succession by the same observer, for use by explorers, or for any purpose for which smallness of size and portability are necessary. This was called the Portable Electrometer. A description cannot be given here, but the reader may refer to Lord Kelvin's *Reprint of Papers on Electrostatics and Magnetism*, or to the first edition of this book. An account of the Long Range Electrometer, for high potentials, must also be omitted.

8. Symmetrical electrometers : the quadrant electrometer. The carefully constructed form of symmetrical electrometer which we have in Thomson's quadrant electrometer had its beginning in the divided-ring instrument illustrated in Fig. 221. A vertical wire carrying on one side a light horizontal needle is suspended from a fixed point. The wire passes through the centre of two flat semi-circular pieces of metal, which lie in a horizontal plane so as to form a metallic circle complete with the exception of a small space at each extremity of a diameter. These spaces insulate one semicircle from the other. Supposing the needle charged with positive electricity and made to rest in equilibrium above one of the spaces when the two semicircles are put in conducting contact, the arrangement is symmetrical about the needle. If one semicircle be then charged with positive, the other with negative electricity, the needle will be repelled from the positive and attracted toward the negative semicircle. If then the wire be brought back and maintained in the symmetrical position by an applied couple, this couple gives a measure of that due to electric forces tending to deflect the needle, and if the potential of the needle remains constant, differences of potential established between the semicircles can be compared.

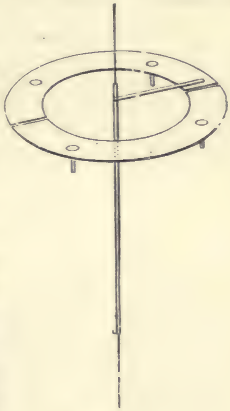

FIG. 221.

It was an obvious but important step to convert the two semicircles into four quadrants by a pair of openings along a diameter at right-angles to the other pair, to put each pair of opposite quadrants into conducting contact, and to make the needle symmetrical about the suspension wire. Thus supposing one pair of quadrants to be charged positively and the other pair negatively, one end of the needle is attracted

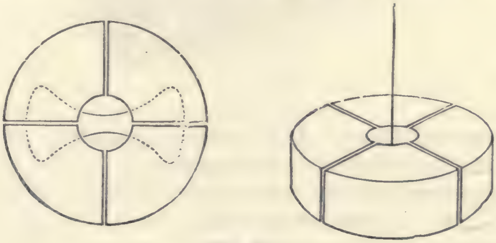

FIG. 222.

by one pair of quadrants, and repelled by the adjacent quadrant of the other pair. The other end of the needle is attracted by the remaining quadrant of the first pair, and repelled by the remaining quadrant of the other pair, which is adjacent. These actions conspire to give a couple turning the needle about the suspension wire.

In the final form of the quadrant electrometer, which is represented in Fig. 223, the four quadrants of the flat-ring are replaced by four quadrants of a flat cylindrical box made of brass. These are shown separately in Fig. 222. Each quadrant is supported on a glass stem projecting downwards from a brass plate which forms the cover of a Leyden jar, within which the quadrants and needles are enclosed. For three of the quadrants the stem passes through a slot in the cover and is attached to a brass piece which closes the slot from above. Thus each of the quadrants can be moved out or in through a small space. The stem of the fourth quadrant is attached to a piece above the cover which rests on three feet. Two of these feet are kept by a spring in a V groove, parallel to which the piece carrying the quadrant with it can be moved by a micrometer screw turning in a nut fixed to

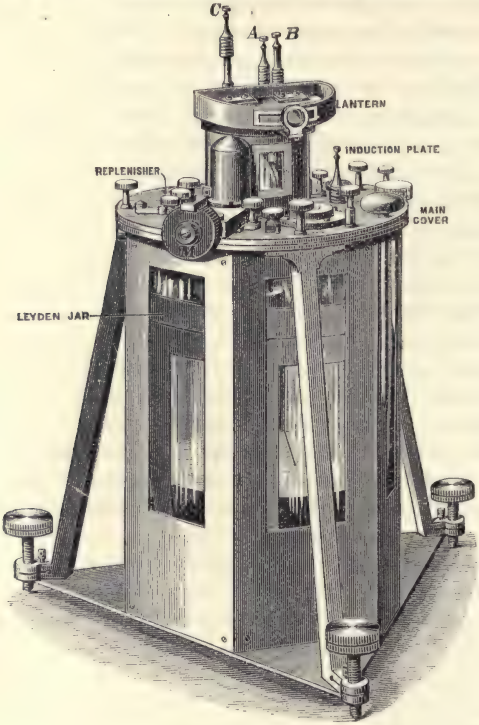

FIG. 223.

the movable piece. The spring which keeps the feet of the movable piece in their groove presses outwards as well as downwards, and so keeps the same sides of the nut and screw threads in contact, to the prevention of "lost time." The details of the instrument will be easily made out by means of Figs. 224 and 225. The former shows a vertical section of the instrument, the latter the suspension-piece and mirror.

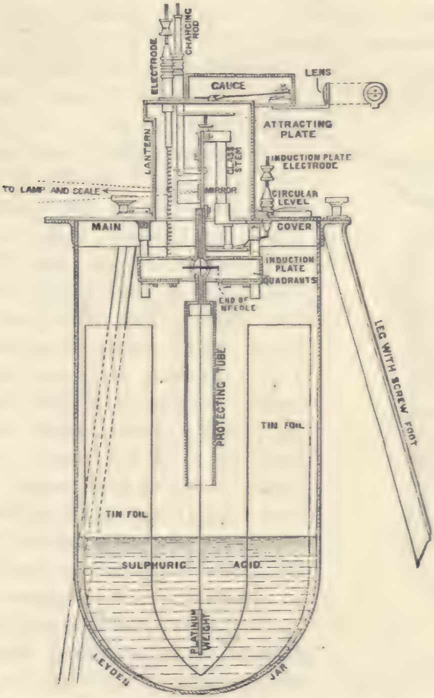

FIG. 224.

A plate rather less in area than the upper surface of a quadrant, but of nearly the same shape, is supported by a glass stem from the cover above a quadrant adjacent to that attached to the micrometer, and is furnished with an insulated electrode passing through the cover. Sufficient length is given to the insulating stem by attaching it to the roof of a cylinder, closed at the top, erected over an opening in the cover. This plate is called the induction plate of the instrument.

9. The needle and its suspension. Within the box formed by the quadrants and about midway between the top and bottom, a needle of sheet aluminium of the form shown by the line drawn, partly full, partly dotted, across the plan of the quadrants on the left in Fig. 222

is suspended horizontally from two pins, *c*, *d* (Fig. 225), carried by a fixed vertical brass plate supported on a glass stem projecting above the cover of the jar. The needle is attached rigidly at its centre to the lower end of a stiff vertical wire of aluminium, which passes down through an opening in the middle of the cover.

To the extremities of a small cross-bar at the top of the aluminium wire are attached the lower threads of a bifilar made of two single fibres, generally of silk. The upper ends of these fibres are wound in opposite directions round the pins *c*, *d*, each of which has, in its outer end, a square hole to receive a small key, by which it can be turned round in its socket so as to wind up or let down the fibre. By this means the fibres can be adjusted so as to be as nearly as may be of the same length; and as the whole supported mass of needle, etc., is then symmetrical about the line midway between the fibres, each bears half the whole weight. The pins *c*, *d*, are carried by the upper ends *e*, *f*, of two spring pieces which form the continuations of a lower plate screwed firmly to the supporting piece. Through *e*, *f*, and working in them, pass two screws *a* and *b*, the points of which bear on the brass supporting plate behind. By the screw *a* the end *e* of the plate *ef* can be moved forward or back through a certain range, and thus the pin *c* carried forward or back relatively to *d*; similarly *d* can be moved by the screw *b*. Thus the position of the needle in azimuth can be adjusted. The distance of the fibres apart can be changed by screwing out, or in, a conical plug shown between the springs *e*, *f*.

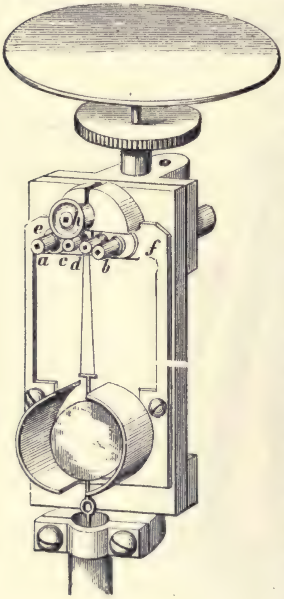

FIG. 225.

The aluminium wire carries between its upper end and the needle a small concave mirror of silvered glass, to be used with a lamp and scale to show the position of the needle. The mirror is guarded against external electric influence by two projecting brass pieces, which form nearly a complete cylinder round it. The part of the wire just above the needle is protected by the tube shown at the bottom of Fig. 225. This tube extends down below the needle a little distance, and is cut away at each side to allow the needle free play to turn round.

The interior coating of the Leyden jar is formed by a quantity of sulphuric acid which it contains, and which also serves to preserve a dry atmosphere within the jar, the exterior coating by strips of tinfoil pasted on its outer surface. The acid has been boiled with sulphate of ammonia to free it from volatile impurities which might attack the metal parts of the instrument. The jar itself is enclosed within a strong metal case of octagonal form, supported on three feet, with levelling screws. The line joining two of these feet (which are in front) is, when level, parallel to the axis of the needle if the latter is properly adjusted.

The needle is connected with the inner coating of the jar by a thin platinum wire kept stretched by a platinum weight at its lower end, which hangs in the acid. The wire is protected from electrical influence by a guard-tube forming a continuation of the narrower guard-tube, partly shown in Fig. 225, and therefore extending from below the quadrants to a short distance above the acid, and connected also by a platinum wire with the acid. [But see below for a modification of this arrangement.]

10. The subsidiary gauge-electrometer. The supporting plate in Fig. 224 carries the disk of an idiostatic gauge of the kind described in 5 above. The height of the disk is adjustable by means of a fine screw and jam-nut below it. The supporting plate, with the suspension and disk of the gauge, etc., is enclosed within an upper brass case, called the *lantern*, which closes tightly the central opening of the cover. The top of the lantern is the guard-plate of the gauge, and carries the aluminium trap-door and lever with sighting plate and lens as already described.

A glass window in the lantern allows light to pass to the mirror, and the suspension to be seen. A small opening in the glass, closed when not in use by a screw-plug of vulcanite, enables the operator to adjust the suspension without removing the lantern.

11. Electrodes, etc. The principal electrodes of the quadrants are brass rods cased in vulcanite, and are arranged so as to be movable vertically. Each is terminated above in a small brass binding screw, and is connected below by a light spiral spring of platinum with a platinized contact piece, which rests by its own weight on a part of the upper surface of the quadrant, also platinized to ensure good contact. They are placed one on each side and in front of the mirror. One is in contact with the quadrant connected below to the micrometer quadrant, the other to the quadrant connected to that below the induction plate.

An insulated charging-rod descends through the lantern, and carries at its lower end a projecting spring of brass. When the rod is not in use the spring is not in contact with anything; but when the jar is to be charged the rod is turned round until the spring is brought into contact with the supporting-plate, which, as stated above, is in contact with the acid of the jar.

The potential of the jar is maintained constant by a replenisher in the manner already described for the absolute electrometer. A spring catch keeps the knob of the replenisher, which is on the upper side of the cover, in such a position when not in use that the carriers are not in contact with any of the springs.

On the upper side of the cover are screws, three in number, by which the cover is secured to a tightly fitting flat ring collar below it, to which the jar is cemented, and to which the case is screwed, two screws, one on each side, which fix the lantern in its place, a cap covering an orifice communicating with the interior of the jar, two binding screws by which wires can be connected to the case, and a knob similar to that of the replenisher, which, when turned against a stop marked "contact," connects by an interior spring the quadrant below the induction plate with the case, and when turned in the opposite direction to an adjoining stop marked "no contact," insulates that quadrant from the case. Two keys, for turning the pins, *a*, *b*, *c*, etc., are kept let down outside the case through holes in the projecting edge of the cover. The cover also carries a small circular level, set so as to have its bubble at the centre when the cover is levelled by an ordinary level. When this has been done the accuracy of construction of the quadrants ensures that they are also level. The level has a slightly convex bottom, and is screwed down with three screws, so that when the instrument is set up for use, a final adjustment, to show horizontality of the quadrants, can easily be made by turning the screws.

12. Adjustments of the quadrant electrometer. Full instructions for setting up and adjusting the quadrant electrometer are sent out with each instrument by the maker, and are therefore available, if kept, as they ought to be, beside it in the case. We shall suppose therefore that the detached parts have been put into their places, the acid poured into the jar, and the instrument set up and levelled; but as a quadrant electrometer is generally part of the equipment of a physical laboratory, and is used over a wide range of electrical work, we describe here the principal adjustments.

The two front quadrants are pulled out as far as possible, to allow the operator to observe the position of the needle, which should rest with its plane horizontal and midway between the upper and under surfaces of the quadrants. If it requires to be raised or lowered, the operator winds or unwinds the fibres by turning the pins *c*, *d*, to which they are attached. The suspension wire of the needle should pass through the centre of the circular orifice formed in the upper surface of the quadrants, when these are symmetrically arranged. If the wire is not in this position the pins *a*, *b*, are turned so as to carry the point of suspension forward or back until the wire is adjusted, and then one pin is carried forward and the other back, without altering the position of the wire, until the black line along the needle is parallel to the transverse slit separating the quadrants.

The scale is placed at the proper distance to give a distinct image of the wire across the line of divisions in front of the lamp flame, then levelled and adjusted so that, when the image is at rest in the centre, the extremities of the scale are at equal distances from the needle.

When the best relative positions of the instrument and the stand for the lamp and scale have been ascertained, these are fixed by the "hole, slot, and plane" arrangement, which enables any instrument supported on three feet or levelling screws to be removed at pleasure, and replaced without readjustment in its original position. A conical hollow, or better, a hole shaped like an inverted triangular pyramid, is cut in the table so as to receive the point (which should be well rounded) of one of the levelling screws, without allowing it to touch the bottom. A V groove, with its axis in line with the hollow, is cut for the rounded point of another levelling screw, and the third rests on the plane surface of the table. If it is desired to insulate the electrometer case it is supported on three blocks of vulcanite cemented to the table; and in one of these the hollow is cut, in another the V groove.

13. Method of charging the electrometer-jar. When the jar is being charged, the main electrodes, the induction plate electrode, and one of the binding screws on the cover, are kept connected by a piece of fine brass or copper wire. The charging electrode is turned round so as to bring the spring at its lower end into contact with the supporting brass piece, and a positive charge is given to the jar by means of the small electrophorus which accompanies the instrument. The cover of the jar is tapped during the process to release the balance lever from the stop, to which it may be adhering. When the lever rises the charging rod is turned so as to disconnect the spring, and the charge is then adjusted to the normal amount (determined by the distance of the attracting disk from the trap door) by the replenisher.

The spot of light may in the process of charging have moved from its position for no electrification, and must be brought back by moving out or in the quadrant carried by the micrometer-screw.

In ordinary circumstances the leakage of the jar will cause the hair to fall down in twenty-four hours about half the breadth of the lower black spot. This loss of charge from the jar is made good by the replenisher; but if the leakage is considerably greater, the main stem should be washed by means of a piece of hard silk ribbon (to avoid shreds) with soap and water, then with clean water, and finally carefully dried. Shreds and dust on the needle and quadrants may tend to discharge the jar, and anything of this kind should be removed by carefully and lightly dusting the needle and quadrants with a clean camel's hair brush. The jar is selected for its high insulating power, but if the acid has in careless handling of the instrument been splashed over the interior surface there may be considerable leakage over the surface of the jar to the case. This can be remedied by removing the acid and carefully washing the jar. The replenisher may also cause leakage of the jar through a deterioration of insulating power of the vulcanite sole-plate which connects the inductors. Such a deterioration with lapse of time is not uncommon in ebonite, and is a consequence of slow chemical action at the surface. A nearly complete cure can be effected by removing the piece and washing it carefully by prolonged immersion in boiling water, and then re-covering its surface with a film of paraffin.

14. Method of testing insulation of quadrants. The insulation of the quadrants is now tested. One pair of quadrants is connected to the case and a charge producing a difference of potentials exceeding the greatest to be used in the experiments is given to the insulated pair by means of a battery, one electrode of which is connected to the electrometer case, while the other is connected for an instant to the electrode of the insulated quadrants; and the deflection of the spot of light is read off. The percentage fall of potentials produced in thirty minutes or an hour is obtained merely by taking the ratio of the diminution of deflection which has taken place in the interval to the original deflection. If this is inappreciable the quadrants insulate satisfactorily. In any case, for satisfactory working the rate of loss of potential shown by the instrument should not be greater than that of the body tested.

If the insulation is imperfect the glass stems supporting the quadrants should be washed by passing a piece of hard silk ribbon well moistened and soaped, then with clean water to remove the soap, and dried by the same piece of ribbon well dried and warmed. If this does not succeed, the fault probably lies in the vulcanite insulators of the electrodes, which should be well steeped in boiling water, then re-covered with clean paraffin and replaced. Care must be taken if this is done not to bend the electrodes.

The final adjustment of the tension of the threads to equality is now made. One pair of quadrants is connected to the case, and the other pair insulated. The poles of a single Daniell's cell are then connected to the electrodes, and the extreme range of deflection produced by reversing the battery, either by hand or by a convenient reversing key, is observed. One side of the instrument is then raised by screwing up that side by one or two turns of one of the front pair of levelling screws, and the range of deflection again noted. If the range is greater the fibre on that side is too short, if the range is smaller the fibre is too long; and the length must be corrected by turning one or other of the pins to which the fibres are suspended. The pins can be reached by the aperture in the window of the lantern ordinarily closed by the vulcanite plug; and to prevent discharge of the jar the key with vulcanite handle should be used to turn them. The black line on the needle will require readjustment by the screws after each alteration of the suspension.

15. Heterostatic use of the quadrant electrometer: theory. The ordinary method of using the quadrant electrometer is heterostatic, since the jar is kept at a potential different from and generally much higher than any potential which the instrument is used to measure. The shape of the needle is such that for most practical purposes equation (3) which follows may be regarded as giving accurately the couple deflecting the needle, when the quadrants are symmetrical about the needle and close. For small deflections we have, for the deflection D, in terms of the potentials V, V_1, V_2 of the needle and the two pairs of quadrants respectively, the equation

$$D = c\,(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right), \quad \text{.........................(3)}$$

where c is a constant depending on the instrument and the mode of reckoning of D. If V be, as it usually is, great in comparison with V_1 or V_2, then

$$V_1 - V_2 = C'D, \quad \text{...(4)}$$

where C' is the now practically constant value of $c\{V - (V_1 + V_2)/2\}$.

To prove this we observe that a symmetrical electrometer may be regarded as consisting of three conductors maintained at different potentials, and fulfilling the following conditions: One (A), the needle, is symmetrically placed with reference to the other two (B and C, Fig. 226), and so formed that one of its two ends, or bounding edges,

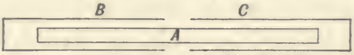

Fig. 226.

is well under cover of B, and the other end or edge under cover of C, so that the electric distribution near each end or edge is uninfluenced by the nearer conductor. Let V, V_1, V_2 be the potentials of A, B, C, respectively, and let A be slightly disturbed from B toward C. This displacement, B say, may be angular or linear, according to the arrangement; in the quadrant electrometer it is the angle through which the needle is turned. Let k be the electrostatic capacity of A per unit of θ at places not near the ends or edges of A, and well under cover of B and C. The quantity of electricity lost by A, because of its displacement relatively to B, is $k\vartheta(V - V_1)$ and that lost by B is $k\theta(V_1 - V)$. Similarly, the quantities gained by A and C in consequence of the motion of A towards C are $k\vartheta(V - V_2)$ and $k\vartheta(V_2 - V)$. Multiplying the first and second of these by V and V_1 respectively, and the third and fourth similarly by V and V_2, subtracting the sum of the first two products from the sum of the second two, and dividing by 2, we get

$$W = k\vartheta(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right), \quad \text{.........................(5)}$$

the work W done by electrical forces in the displacement. This must be equal to the average couple, or average force, multiplied into the displacement, according as the latter is angular or linear. Denoting the force or couple by F, we have

$$F = k(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right). \quad \text{.........................(6)}$$

In an arrangement of this kind, when the displacement is small, the couple or force on A is nearly the same over the whole displacement, and is thus equal to the equilibrating force or couple due to the torsion wire, or bifilar, or other arrangement finally producing equilibrium. For small displacements, this force or couple will generally be proportional to the displacement, and therefore also to the deflection D on the scale of the instrument, and thus

$$D = m\theta = c(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right), \quad \text{..........................(7)}$$

where m and c are constants.

When V is great in comparison with $\frac{1}{2}(V_1 + V_2)$ this equation reduces to $\theta = c(V_1 - V_2)$.

If the angle of deflection θ of the ray of light is not a very small angle, the couple given by the bifilar, it is to be remembered, is proportional to $\sin \frac{1}{4}\theta$. Hence if D be the distance in divisions on the scale (supposed straight and at right angles to the zero direction of the ray) through which the spot of light is deflected, and R the horizontal distance of the scale from the mirror in the same divisions, we have $\tan \theta = D/R$, from which θ can be found and hence $\frac{1}{4}\theta$. We have then

$$K \sin \tfrac{1}{4}\theta = (V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right), \quad \text{..........................(8)}$$

where K is a constant.

Equation (8) would be more nearly satisfied if the central portions of the needle to well within the quadrants were as much as possible cut away, leaving only a framework opposite the orifice at the centre of the quadrants to support the needle.

16. Energetics of the action of a quadrant electrometer. It may be noticed that we have here a system of conductors kept at constant potentials during an alteration of configuration of the system. A very general theorem of energy holds in such a case. In order that the potentials may remain constant, the conductors must be connected with sources of energy. The theorem is that the source or sources of energy supply twice as much energy as is involved in the increase of energy of the system.

To prove this, let Q_1, V_1, Q_2, V_2, ... be the charges and potentials, and E be the energy, before the change of configuration. Then we have

$$E = \tfrac{1}{2}\Sigma(QV).$$

After the displacement the energy has become $E + e$, and the charges $Q_1 + q_1$, $Q_2 + q_2$, ..., while the potentials are unaltered. We have then

$$E + e = \tfrac{1}{2}\Sigma\{(Q + q)V\},$$

that is,

$$e = \tfrac{1}{2}\Sigma(qV).$$

But since the charges have been altered at unvarying potentials the sources must have furnished energy

$$2e = \Sigma(qV).$$

We see thus that $2e$ is the energy furnished by the sources of energy which maintain the potentials. Half of this is stored in the displaced system, the other half has done work, which, when the parts of the system are left at rest, is represented by the potential energy of the twisted suspension fibre, or other directive system controlling the needle. Here we have an example of the very general theorem already illustrated in VIII. 7 above.

As has been mentioned in X. 12 above, the cutting away of the guard-tube for the needle leaves metal cheeks, which, when the needle is deflected to an unsymmetrical position, renders the formula (8) above, obtained on the assumption of the existence of symmetry seriously inaccurate for the measurement of large differences of potentials. In quadrant electrometers made during the last twenty-eight years the guard-tube has been dispensed with.

17. Grades of sensitiveness. The quadrant electrometer described above, when used heterostatically, admits of a number of different grades of sensibility. These are shown in the two following tables, where L denotes the electrode of the pair of quadrants, one of which is below the induction plate, R the electrode of the other pair of quadrants, I the electrode of the induction-plate, O an electrode of the case of the instrument, and C the electrode of the conductor to be tested. LC denotes that L is connected to C, RO that R is connected to O, RLC that RL and C are connected together, and so on, (L) that the quadrants connected with L are insulated by raising L, (R) that the quadrants connected with R are similarly insulated, (RL) that both L and R are raised. The disinsulator mentioned (p. 704 above) is used to free the quadrants connected with L from the induced charge which they generally receive when L is raised.

GRADES OF SENSITIVENESS.

A.

Inductor connected with quadrant beneath it.

FULL POWER.

$$\begin{bmatrix} LC \\ RO \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} RC \\ LO \end{bmatrix}$$

DIMINISHED POWER.

$$(L)\begin{bmatrix} RC \\ O \end{bmatrix} \quad \text{or} \quad (R)\begin{bmatrix} LC \\ O \end{bmatrix}$$

B.

Inductor connected as indicated below.

FULL POWER.

Inductor Insulated.

$$\begin{bmatrix} LC \\ RO \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} RC \\ LO \end{bmatrix}$$

GRADES OF DIMINISHED POWER.

$$(L)\left[\begin{array}{c} \begin{bmatrix} RC \\ IO \end{bmatrix} \\ \begin{bmatrix} RIC \\ O \end{bmatrix} \\ \begin{bmatrix} IC \\ RO \end{bmatrix} \end{array}\right. \quad \text{or} \quad (R)\left[\begin{array}{c} \begin{bmatrix} LC \\ IO \end{bmatrix} \\ \begin{bmatrix} LIC \\ O \end{bmatrix} \\ \begin{bmatrix} IC \\ LO \end{bmatrix} \end{array}\right.$$

$$(RL)\begin{bmatrix} IC \\ O \end{bmatrix}$$

Either of these grades of sensibility may of course also be varied by increasing the distance of the fibres apart.

The quadrant electrometer can be made to give results in absolute measure by determining the constant C' of equation (4), by which the deflection must be multiplied to give the difference $V_1 - V_2$. This can be done by observing the deflection produced by a battery of electromotive force of convenient amount, determined by direct measurement with an absolute electrometer or otherwise. Different such electromotive forces may be employed to give deflections of different amounts and thus give a kind of calibration of the scale to avoid error from non-fulfilment of condition of proportionality of deflection to difference of potentials.

18. Idiostatic use of quadrant electrometer. The quadrant electrometer may also be used idiostatically for the measurement of differences of potential of not less than about 30 volts. The volt is the practical unit of electromotive force, and is about 1·07 times the electromotive force of a Daniell's cell. In this mode of using the instrument the jar is left uncharged, the charging-rod is brought into contact with the inner coating of the jar, and joined by a wire with one of the main electrodes, so as to connect the needle to one pair of quadrants. The other pair of quadrants is either insulated or connected to the case of the instrument. The instrument thus becomes a condenser, one plate of which is movable, and by its change of position alters the electrostatic capacity of the condenser. The two main electrodes are connected with the conductors, the difference of potentials between which it is desired to measure.

A lower grade of sensibility can be obtained by connecting the needle through the charging-rod to the electrode R, and using the induction-plate instead of the pair of quadrants connected with L, which are insulated by raising their electrode.

When the instrument is thus used idiostatically V in equation (7) above becomes equal to V_1, and instead of (7) we have

$$D = \frac{C}{2}(V_1 - V_2)^2, \qquad (9)$$

that is, the deflection is proportional to the square of the difference of potentials, and therefore independent of the sign of that difference. It is to the left or right according to the electrode connected to the needle. This independence of sign in the deflection renders the instrument thus used applicable to the determination of mean squares of differences of potentials in the circuits of alternating dynamo- or magneto-electric generators.

The quadrant electrometer has been modified by different makers. In the form made in Paris for M. Mascart, the needle is kept at a constant potential by being connected to the positive pole of a dry pile, the negative pole of which is connected to the case, and the replenisher is dispensed with.

19. Dolezalek quadrant electrometer. For work on radio-activity and kindred subjects the Kelvin quadrant electrometer is not sufficiently sensitive, and instead of it a modification of the arrangement due to Dolezalek (*Zeitschr. f. Instrumentenk.* 17, 1897) is now very generally used. Smaller quadrants are employed, and have high insulation ensured by being mounted on pillars of amber. The needle is made of silvered or gilded paper, and is therefore extremely light. The suspension is a single fibre of quartz or a very fine drawn metal wire. There is no Leyden jar, and the needle is kept electrified by a battery (as in the arrangement referred to in 17), giving a moderate difference of potentials, of the order of 100 volts. The indicator is a reflected beam of light as in the older form of quadrant electrometer, and the instrument is enormously more sensitive.

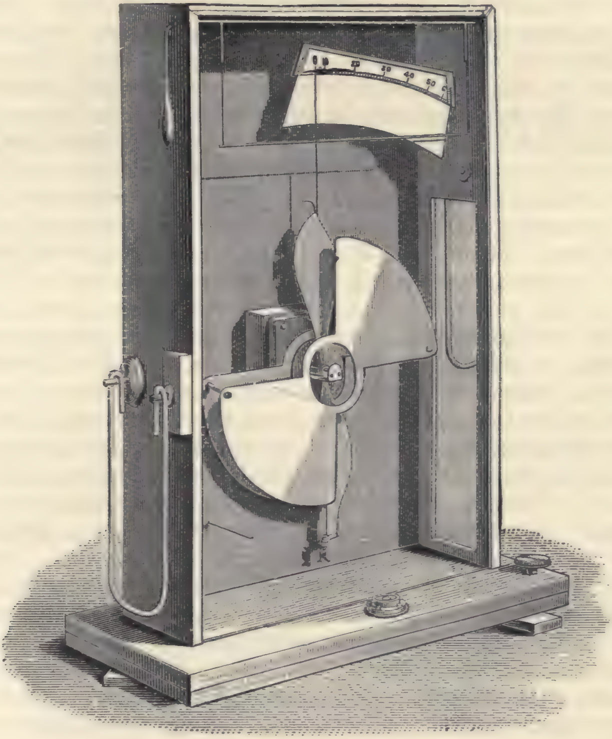

Fig. 227.

20. Electrostatic voltmeters. Electrometers are used idiostatically in practical work, and are called electrostatic voltmeters. One made by Lord Kelvin is represented in Fig. 227, and may be described as an air condenser, one plate of which, corresponding to the needle of the quadrant electrometer, is pivoted on a horizontal knife-edge working on rounded V-grooves cut in the supporting pieces. This plate by its motion alters the electrostatic capacity of the condenser. The fixed plate consists of two brass plates in metallic connection, each of which is, except for the ring-formed middle part, a double sector of a circular plate, and which are placed accurately parallel to one another, with the movable plate between them as shown in the diagram. The upper end of the movable plate is prolonged by a fine pointer which moves along a circular scale, the centre of which is on the axis. The fixed plates are insulated from the case of the instrument; the needle is uninsulated.

Contact is made with the plates by insulated terminals fixed outside the case. The two shown on the left-hand side of the picture belong to the fixed plate, and a similar pair on the right-hand side are in connection with the movable plate through the supporting V-groove and knife-edge. The terminals of each pair are connected by a "safety valve" consistng of a length of fine copper wire contained within a U-shaped glass tube suspended from the terminals, and the terminals in front in the diagram which are separated from the plates by the arc of wire are the working terminals, that is, they alone are used for connecting the instrument to the conductors, or to the two points of an electric circuit, the difference of potentials between which is to be measured.

When a difference of potentials is established between the fixed and movable plates these plates move so as to increase the electrostatic capacity of the condenser, and the couple acting on the movable plate in any given position is, as in the quadrant electrometer when used idiostatically, proportional to the square of the difference of potentials. The couple is balanced by that due to a small weight hung on the knife-edge at the lower end of the movable plate.

The scale is graduated from 0° to 60°, so that the successive division spaces represent equal differences of potential. Three different weights, 32·5, 97·5, 390 milligrammes respectively, are provided to give three different grades of sensibility. Thus the sensibility with the smallest weight on the knife-edge is one division for 50 volts, with the two smaller weights together, that is, with 4 times the smallest weight, one division per 100 volts, with all three weights, or 16 times the smallest weight, one division per 200 volts.

21. Graduation of an electrostatic voltmeter. An electrostatic voltmeter of large range may be graduated as follows. A known difference of potentials is obtained by means of a battery of from 50 to 100 cells with a high standard resistance in its circuit. An absolute galvanometer or current balance measures the current in the circuit, and the product of the numerics of the current and the resistance is the numeric of the difference of potentials between the terminals of the latter. These terminals are connected to the working terminals of the voltmeter, and the deflections with the smaller weights on the knife-edge noted.

For the higher differences of potentials a number of condensers of good insulation are joined in series and charged by an application of the wires from the terminals of the resistance coil to each condenser in succession, and in the same direction, from one end of the series to the other. This is done so as to charge each condenser in the series in the same direction, and as the same difference of potential, V say, is produced between the plates of each condenser, the total difference between the extreme plates is nV, if there be n condensers. A convenient large difference of potentials can thus be obtained with sufficient accuracy, and being applied to the working terminals of the voltmeter is made to give divisions for a series of different weights hung on the knife-edge. These divisions correspond of course to deflections for known differences of potentials with *one* of the weights on the knife-edge.

The divisions thus obtained are then checked by using three instruments which have been dealt with in this way. They are joined in series and a difference of potentials is established between the extreme terminals, which is observed also by the third joined across the other two. Thus by a process of successive halving and doubling the scale is filled up.

22. Standard condensers. Comparison of capacities. One of the chief electrostatic measurements is that of specific inductive capacity. This quantity has been defined in I. 24. In this book even the standard medium is supposed to have an inductivity which may or may not be taken as unity, and the specific inductive capacity of any medium is the ratio of the inductivity of that medium to that of the standard medium, and is therefore essentially numerical.

It is determined by a comparison of the capacity of a condenser with the medium as dielectric with that of a geometrically identical condenser, in which the standard medium, for example, air (see again I. 24) is the dielectric. We shall discuss first methods of comparison of capacities. The methods which depend more or less on electromagnetic principles have already been fully discussed.

The experimental determination of the electrostatic capacity of a condenser is effected by a process in which its charge at a given potential is compared with that required to charge a standard condenser to the same potential. The standard condenser is generally one of which the capacity can be found by calculation from the dimensions and arrangement of the instrument, or which has been itself compared with such a condenser.

There are three forms of standard condenser, the capacity of which can be determined with accuracy by calculation from the geometrical arrangement. These are:

1. Spherical Condensers.
2. Guard-ring Condensers.
3. Cylindrical Condensers.

The simplest form of spherical condenser consists of two spherical conducting surfaces concentric with one another and separated by a dielectric. Such a condenser was used by Faraday in his experiments on Specific Inductive Capacity, and is shown in Fig. 228. An outer brass shell B is supported on a base-piece as shown in the figure, and is fitted above with a tubulure r, filled by a long plug of shellac l. The internal brass ball A is supported in a position concentric with the outer shell by a thin stem passing up through the shellac plug and terminating in a knob a. The support below is perforated so as to form a tube by which the space between the spheres can be filled with dry air or any gas. A stopcock R enables this passage to be closed.

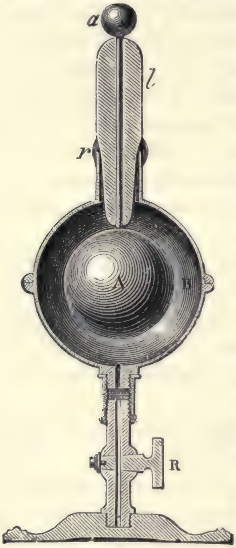

FIG. 228.

This condenser was not used by Faraday for the measurement of capacities in absolute measure, but two of them were employed in the manner described in 29 below for the determination of specific inductive capacities. An absolute condenser on this principle was however constructed by Sir William Thomson, and is shown in section in Fig. 229. The radius of the internal sphere was 4·511 cm, of the inner surface of the external shell 5·857 cm. The inner shell was supported in its place by three pieces of vulcanite, of which one is shown in the figure, and communication was made with the interior conductor by a wire passing through the centre of a circular orifice cut in the outer shell. Calculating the capacity of this condenser by the formula $rr'/(r'-r)$ above, we get 19·628 cm. It was found however that ·225 centimetre had to be added to this number to correct for the effect of the support and the conducting wire. It is difficult to make the surfaces of such a condenser truly spherical, and to fix them so accurately in their places as to enable the capacity to be calculated with sufficient exactness, and comparisons of this condenser with others showed that this value of the capacity was probably too low. This difficulty has however been got over very completely in the spherical condenser used by Mr. E. B. Rosa at the Bureau of Standards, Washington, and described above. [See XVI. 6, and Fig. 205.] When made and used with extreme care this seems the most accurate form of standard condenser.

23. Guard-ring condenser. The guard-ring form of the parallel plate condenser is more easily made and is capable of great accuracy in ordinary use. This is shown diagrammatically in section in Fig. 230. (An actual instrument constructed by Dr. J. Hopkinson is shown in Figs. 248, 249 below in connection with an account of his researches.) The guard-ring R forms as it were part of a cylindrical metal box nearly closed by the disk D which the ring surrounds. This box and disk are supported on a glass stem well covered with clean shellac, and a separate glass stem within the box insulates the disk D from the ring. A wire passing through a hole in the cylindrical wall of the box makes contact with the electrode of the disk. The other plate of the condenser is formed by the large disk P above. This plate is carried by a glass stem mounted in a socket at the extremity of a fine screw working in a fixed nut above. By turning the micrometer head of this screw, the distance of P from the opposite disk can be altered by any required amount. The condenser and its supporting framework are mounted on an iron sole-plate, round which is cut a circular groove to receive a protecting glass cover; and to enable a dry atmosphere to be maintained about the insulating stems, fragments of pumice moistened with strong sulphuric acid are contained in a lead tray placed on the sole-plate. [Rosa and Dorsey's plate condenser is shown in Fig. 209.]

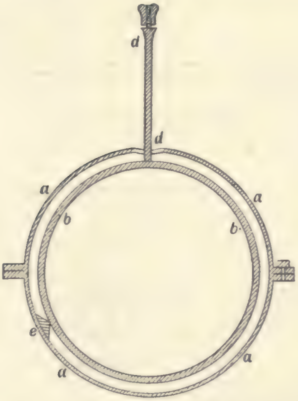

FIG. 229.

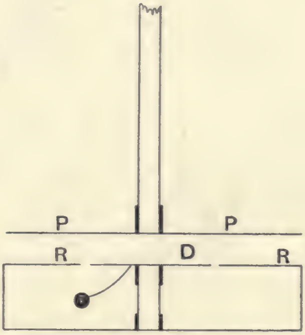

FIG. 230.

The manner of using the condenser is as follows: The guard-ring and disk are connected together and charged to the potential required, while the opposite plate is kept at zero potential. The disk is next disconnected from the guard-ring, which is then brought also to zero potential. The charge which was formerly on the disk remains upon it, and since the distribution was very nearly uniform the capacity can be calculated, and therefore the charge on the disk, from the previously existing potential. The effective area of the disk may be taken as the arithmetic mean of the actual area of the disk and that of the opening in the guard-ring. If S be this mean area we have

$$C = \frac{S}{4\pi d},$$

and therefore for the charge Q upon the disk when the condenser is charged to potential V,

$$Q = \frac{VS}{4\pi d}. \qquad \text{.......................................(10)}$$

24. Cylindrical condenser. A cylindrical condenser of variable capacity was invented by Sir William Thomson, and used by Messrs. Gibson and Barclay in their determinations of the specific inductive capacity of paraffin referred to below. The instrument is represented in longitudinal section in Fig. 231, and in cross-section through C and A in Figs. 232 and 233. The essential parts are two circular cylinders of brass *aa*, *bb* of the same diameter, supported, with their axes in line and a gap between their adjacent ends, on vulcanite pieces, *cc*, *dd*, attached to a sole-plate *h*. The lengths of these cylinders were 26·58 cm and 35·3 cm respectively, and their common diameter 4·9674 cm. These dimensions were determined by a measurement of the volume of water contained by the tubes and an accurate determination of their lengths. A third brass cylinder *e* was supported coaxially within the other two, on four vulcanite feet, near one end, resting on the inner surface of the outer cylinder. The length of this cylinder was 36·6 cm, and its diameter (found by winding fine wire round the cylinder, measuring the length of a certain number of turns, and allowing for the thickness of wire and the spiral arrangement) was 2·303 cm. This last cylinder is loaded so as to rest stably on its supports, and can be slided backwards or forwards in the direction of its length so as to alter the relative lengths of it enclosed within the two tubes *aa*, *bb*. A vertical arm projects upwards through a slot cut in the tube *bb*, and carries an index which moves along a graduated scale *kk*. This scale was graduated into 360 divisions, each 1/40 inch or ·0635 cm nearly.

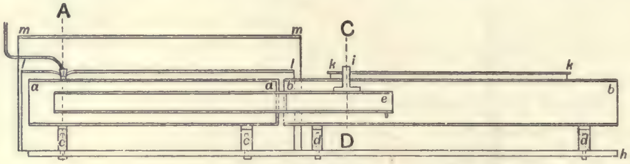

FIG. 231.

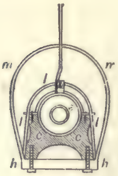

FIG. 232.

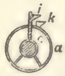

FIG. 233.

A cylinder of metal *ll* fastened to the base of the instrument surrounds the other tube *aa*, to protect it from external influence, and the whole is enclosed within an outer case *mm*.

In the use of the instrument the tube *bb*, the internal cylinder *ee*, and the outer cylinders *ll*, *mm* were connected to earth, while *aa* was insulated and charged. The theory of the instrument is given in XI. 44 above. According as the capacity of the condenser was to be increased or diminished, *ee* was slided towards the left or right, and the amount of change of capacity was given by using the displacement *l*, measured on the scale *kk*, in the formula

$$C=\frac{1}{2}\frac{l}{\log\frac{r'}{r}}, \quad \text{.......................................(11)}$$

where $r'=2{\cdot}4837$, $r=1{\cdot}1515$. The capacity when $l=$ one scale division $={\cdot}0635$ cm, was therefore ·0413 cm.

This instrument has been modified so as to give it greater range by the adoption of the arrangement shown in Fig. 234. Here both *ee* and *ll* (*b* and *c* of the figure) are movable, so as to alter the capacity of *aa*.

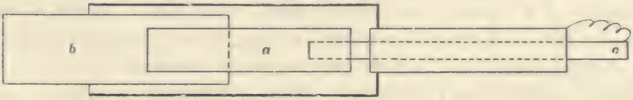

FIG. 234.

25. Electric absorption. Except when the dielectric is a gas, the phenomena of charge and discharge are complicated, and the results of experimental comparisons of the capacities of condensers more or less affected, by what is generally called *electric absorption* and sometimes *electrification*. If a condenser having a solid or liquid dielectric be charged by applying a battery for a time sufficient to give a uniform potential *V* throughout the charged plate of the condenser and then be left to itself, its potential will be found after the lapse of a short time to have considerably diminished. This diminution of potential is only partly due to conduction through the dielectric or to want of proper insulation. Part of it is due to a change produced in the dielectric medium when the condenser is charged, which requires time to bring it about, and is called electric absorption from the original idea that it was caused by the penetration of part of the electric charge into the substance of the dielectric. A further charge is necessary to restore the former potential, and if this be given by a second short application of the original charging battery, a second fall of potential not so great as the first will be produced from this cause, and so on for a third, fourth, fifth, etc., short application. Thus, if the condenser be charged by a long-continued application of the battery, it will take a considerably greater charge than if the same potential had been produced by an instantaneous or short-continued application. Similar results are obtained when a condenser is discharged. If it has been charged by a long contact with the charging battery, or has been left to itself for some time after charge by a short contact, and is then discharged by a short contact, it will be found immediately after to be at zero potential, but after some little time it will be found again to have acquired a potential of the same sign as before, and can be again discharged. In this way three or four or more discharges can be obtained before its plates are permanently reduced to zero potential. These discharges after the first constitute what is called the residual charge of the condenser.

The phenomena of residual charge have been a good deal investigated of late years. Kohlrausch first pointed out the close connection between the phenomena of residual charge and the slow working out of subpermanent strain shown by many elastic substances, and called by German physicists *Elastische Nachwirkung.** He showed that the instantaneous discharge is independent of the residual charge, and that for a given jar left to itself for a given time after charging, the residual charge is proportional to the initial potential. It was found by Dr. Hopkinson that if a Leyden jar be charged positively by an application of a battery continued for a long time, say a week, then negatively for a shorter time, say a day, then positively for a very much shorter time, say a few minutes, the residual discharge will be alternately positive and negative. This behaviour is closely analogous to that of a wire which has been held twisted for different intervals in successively opposite directions. Dr. Hopkinson has also found that mechanical agitation of the dielectric such as that produced by tapping the jar has a marked effect in accelerating the residual discharge.

Attempts have been made with fair success, notably by Clerk Maxwell, to account for electric absorption by imagining the dielectric to be heterogeneous, in the sense of being made up of different imperfectly insulating substances, such that the ratio of the specific inductive capacity to the specific conductivity is not the same for the different media. It is explained in Maxwell's treatise how this supposition accounts for absorption.

It might appear from what precedes that owing to the existence of electric absorption the capacity of a condenser is an indefinite quantity, depending on the time of charge or discharge. This is not the case, however, as it has been found by several experimenters that for ordinary condensers, provided the time of charge or discharge do not exceed an interval of a quarter or half a second, the charge required to produce a potential V, or which is withdrawn in annulling a potential V, are sensibly the same and independent of the duration of the contact. This is called the instantaneous charge of the condenser, and the capacity of a condenser is defined as the amount of the instantaneous charge required to produce unit potential at its insulated coating, while the other is at zero. The methods of comparing capacities described below will not therefore (except in the case of cables which require a sensible time to acquire throughout the same potential) involve any ambiguity.

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* *Pogg. Ann.* 91, 1854. See also on this subject *Encyc. Brit.*, Art. "Electricity," by Prof. Chrystal: Ayrton and Perry, "Viscosity of Dielectrics," *Proc. R.S.* 1878.

26. The platymeter. In the investigation of the specific inductive capacity of paraffin referred to above, the capacities of two condensers were compared by an instrument invented by Sir William Thomson, and called by him a platymeter. This instrument is represented in Fig. 235. A brass cylinder *cc*, 22·94 cm long, and 5·1 cm in diameter, is supported by vulcanite pieces *dd*, and coaxial with it are placed in symmetrical positions, and insulated by the vulcanite supports *ee*, two equal shorter cylinders of thin brass, each 7·68 cm in length and 8·6 cm in diameter. *p*, *p′* thus form corresponding plates of two nearly equal cylindrical condensers, of which the opposite plates are furnished by the cylinder *cc*. The whole is enclosed within a metal case *mm*, through which pass insulated by plugs of paraffin the electrodes *qq* of *p*, *p′*, and the electrode *n* of *cc*.

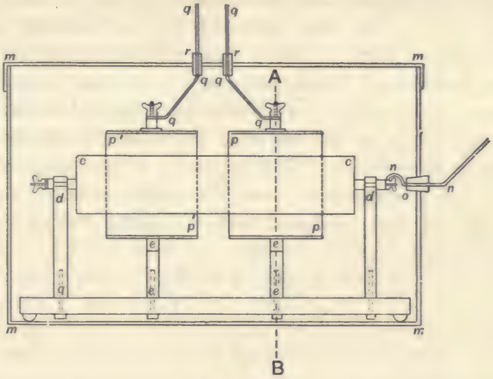

FIG. 235.

The platymeter was used with the sliding condenser in the following manner for the determination of the capacities of other condensers. The cylinder *aa* of the sliding condenser (Fig. 231) was connected to *p′*, the insulated plate of the condenser to be measured to *p*, and the other plate and cylinders *bb*, *ee* to the case of a quadrant electrometer arranged for heterostatic use. The inner cylinder *cc* of the platymeter was connected to the electrode of the insulated pair of quadrants. We shall denote the condenser to be measured and the sliding condenser by *A* and *B*, their respective capacities by *C*, *C′*, and the nearly equal capacities of *p*, *p′* respectively by *c*, *c′*. Now suppose a positive charge given to *A*, and the electrodes of the electrometer connected for an instant to reduce the potential of the cylinder *cc* to zero, and *p* and *p′* then connected so as to share the charge on *A* and *p* with *B* and *p′*. Assuming the action between *p* and *cc* to be equal to that between *p′* and *cc*, that is, the two sides of the platymeter to be precisely equal, it is plain that the resulting potential of *cc* must be positive, zero, or negative according as the capacity $C+c$ is greater than, equal to, or less than $C'+c'$. It is plain also that, under the same conditions, the potential of *cc* must be negative, zero, or positive when B is the positively charged conductor, or positive, zero, or negative, if B be negatively charged. In Gibson and Barclay's experiments one conductor was positively, the other negatively charged, as this gave more marked effects without increased risk of breaking down of insulation.

The capacity of the sliding condenser was adjusted so that when A was connected to p' no alteration in the potential of *cc* was produced by putting pp' in contact after charging. On the assumption that $c=c'$, this gave $C=C'$.

It was found however that when A and B were interchanged without alteration of their capacities the connection of p with p' disturbed the potential of *cc*. The two sides of the platymeter were therefore not exactly equal. But in order that the potential of *cc* should be unaltered after the two condensers are put into contact, it is only necessary that their capacities should be adjusted so as to be in the ratio of the capacities of the sides of the platymeter with which they are respectively in contact. The capacity of the sliding condenser in the interchanged arrangement was therefore altered until the effect of making contact was rendered zero. Calling the new capacity C'_1, we have the two equations

$$\frac{c}{c'}=\frac{C}{C'},\quad \frac{c}{c'}=\frac{C'_1}{C},$$

and therefore

$$C=\sqrt{C'C'_1} \qquad\qquad (12)$$

27. Measurement of the capacity of cylindrical condenser. As an example we may take the measurement of the capacity of the sliding condenser when the index was at a given position of the scale. This was done by comparing it with the spherical condenser already described. The sliding condenser was adjusted so that when connected to the side p of the platymeter, and the spherical condenser to p', the potential of *cc* remained unchanged when after the system was charged as described, p and p' were put into contact. The reading on the scale of the sliding condenser was then 211. The condensers were then interchanged and the same operations repeated, and the reading 183 was obtained on the sliding condenser. A second pair of experiments gave 211 and 186 as the readings.

Now the capacity of the sliding condenser per scale division was found to be ·0413 cm. Hence taking the value 63·519 cm. for the capacity of the spherical condenser, its capacity in terms of that corresponding to a scale division of the sliding condenser taken as unit was 1538. Calling the capacity of the sliding condenser when the slide was at zero, A, we have for the total capacities of the sliding condenser in the first pair of experiments $A+211$ and $A+183$, and in the second pair $A+211$ and $A+186$. Hence taking the arithmetic mean instead of the geometric, we have approximately

$$A=1538-198=1340,$$

and for the capacity C in c.g.s. units

$$C=1340\times{\cdot}0413=14{\cdot}04.$$

28. Comparison of two guard-ring condensers. The following method given by Maxwell for the comparison of the capacities of two guard-ring condensers, is a modification of a method used by Cavendish for the approximate comparison of two parallel plate condensers of the simpler form. The reader can easily make a diagram for himself by drawing diagrammatically two guard-ring condensers side by side. Let A, B, C denote respectively the small disk, guard-ring with metal backing, and large disk of one condenser, A', B', C' the corresponding parts of the other condenser. The following operations are performed while B is kept connected to C', and B' to C, all connections being made with wires of negligible capacity.

1. A is connected to B and C', and with the electrode J of a Leyden jar or a large battery, and A' is connected to B' and C, and with the earth.
2. A, B, C' are insulated from J.
3. A is insulated from B and C', and A' from B' and C.
4. B and C' are connected with B' and C and with the earth.
5. A is connected with A'.
6. A and A' are connected with the electrode of the insulated quadrants of an electrometer or with a sensitive electroscope.

By this process A and A' are charged to equal and opposite potentials, and if their capacities are equal the resulting potential after operation 5 is performed will be zero, and the electroscope will show no deflection. By adjusting therefore one of the condensers until this result is obtained the capacity of the other condenser can be found in terms of that of the first. Thus the effect of putting a slab of some insulating substance between the plates of one of the condensers can be determined by performing this process before and after the introduction of the slab. All the operations here described can be performed in rapid succession by a properly arranged and well insulated key.

If the condensers be not guard-ring condensers this method can yet be applied with accuracy in any case in which A and A' may be regarded as surrounded by the other plates C and C'. For example A may be the insulated cylinder *aa* of a sliding condenser, and A' the internal surface of a spherical condenser, or with sufficient accuracy the interior coating of a Leyden jar. It is only necessary in the above operations to regard B as coincident with C', and B' with C.

29. Faraday's method of comparison of capacities. The following method is practically that used by Faraday in his determination of specific inductive capacity. Two condensers have their plates, which are usually uninsulated, connected to earth, and one of the other plates is charged to a potential which is observed by means of an electrometer. The insulated plate of the other condenser is then brought into contact with the charged plate by means of a fine wire, and the diminished potential is observed by the electrometer. If one of the condensers is an air condenser, that should be the condenser first charged, and the contact with the insulated plate of the other should be made only for an instant and then broken. This avoids the phenomenon referred to above as electric "absorption" which takes place in solid dielectrics. Calling C_1, C_2 the capacities of the condensers, c that of the part of the electrometer charged by being put in contact with the condenser, V the potential before and V' that after the sharing of the charge, then since the charge remains constant we have

$$V(C_1+c)=V'(C_1+C_2+c). \qquad (13)$$

If c is negligible, as it generally is, this gives

$$\frac{C_2}{C_1}=\frac{V}{V'}-1. \qquad (14)$$

Faraday compared the potentials V, V' by bringing a carrier ball into contact with the knob of the condenser before and after the discharge, and comparing by the torsion balance the charges carried off in the two cases.

If the capacity c of the electrometer is not negligible, then if it be supposed independent of the deflection, another equation may be found with which to eliminate it, by first charging the electrometer to some potential V, and then sharing the charge with the condenser of capacity C_1 so as to give a potential V'. This gives

$$vc=v'(C_1+c).$$

Hence substituting in (13) above we get

$$\frac{C_2}{C_1}=\frac{V-V'}{V'}\cdot\frac{v}{v-v'}. \qquad (15)$$

30. Cable testing : determination of capacity. We shall now describe some methods of comparing capacities which are useful in cable testing, and in the determination of the capacities of condensers in cable work generally. The first two methods were given by Sir William Thomson in the early days of cable laying and testing.

The first of these methods requires three condensers of known, one of them of variable, capacity, besides the condenser the capacity of which is to be measured. Let the four condensers be called A, B, C, D, their capacities be denoted by C_1, C'_1, C_2, C'_2, and let C be the variable condenser and D that of which the capacity C'_2 is to be found. (A figure may be made by the reader.) The insulated plates of A, C are first connected together and brought to some convenient potential by giving them a charge from a Leyden jar, or by applying one terminal of a battery the other terminal of which is connected to the earth. They are then disconnected, the charged plate of A put in contact with the insulated plate of B, and that of C with the insulated plate of D. An electrometer of which both pairs of quadrants are insulated, has one electrode connected to A and B, and the other to C and D, and C is varied in capacity, if need be, until both pairs of condensers are brought to the same potential, which will of course be the case when the deflection of the electrometer has been reduced to zero. We have, if V be the potential of A and C before contact with B and D, and V' the common potential after the adjustment has been made,

$$V' = \frac{VC_1}{C_1 + C'_1} = \frac{VC_2}{C_2 + C'_2},$$

or

$$C'_2 = \frac{C'_1}{C_1} C_2. \qquad (16)$$

A well insulated and sensitive galvanometer with insulated key may be arranged instead of an electrometer between the pairs of charged plates, and the criterion of equality of potentials will then be zero deflection of the galvanometer needle when the key, previously kept raised, is tapped down after the operation described above. The use of a galvanometer has however the disadvantage that the whole series of operations must be gone through at each discharge. This is not necessary when an electrometer is used, as then only potentials are compared without discharge.

If D be a condenser of great capacity, such as a long cable with the further end insulated in air, time must be given for the condenser to become charged throughout the same potential, and a corresponding time for the equalization of the potential of D with that of C when these condensers are put in contact. The time generally allowed for a long cable is twenty to thirty seconds and about the same for equalization.

In order to ensure accuracy the condensers C_1, C_2, C'_1, C'_2 should be all, if possible, nearly equal. In any case C_1 should not be small in comparison with C'_1, nor C_1 in comparison with C_2.

31. Second method of finding the capacity of a cable. The next method is much used in cable testing. The arrangement of apparatus is shown in Fig. 236.

A battery of, say, twenty Daniell's cells, insulated by having for the outer containing vessel dry vulcanite or earthenware pots supported on a dry table or board, has its terminals connected through the reversing key K, to the extremities of the series of resistances a, b. These resistances are connected at equal intervals as shown diagrammatically with pieces of metal, which form a set of contact pieces, along which a slider carrying a binding-screw can be moved as in the resistance slide described in XI. 7 above, and so the resistance between the slider and the extremities of a, b, varied. A wire attached to the slider is connected to earth, to which are also connected the uninsulated coatings of the condensers C and L to be compared. C is here supposed to be the standard or known condenser, L a cable with its remote end free in air. The terminal a of the resistance slide is connected with the insulated coating of the condenser L, the terminal b with the insulated coating of C through the insulated key K. This key besides being capable of giving these connections, can also be made to disconnect the resistance slide from the condensers, and to put the insulated coating of the condensers into contact. By being brought into contact with a and b the respective condensers are charged to the potentials of those points. Now, since the slider is at zero potential, if V_1 be the potential of a, R_1 the resistance between a and the slider, and R_2 the resistance between the slider and b, the potential at b will be $-V_2$, where

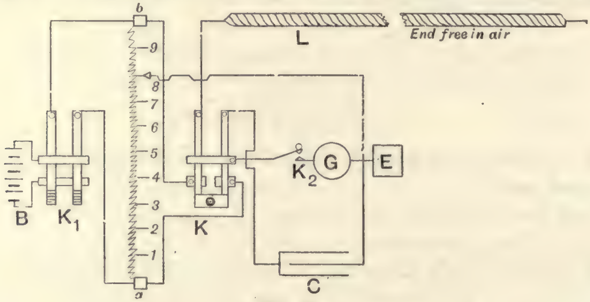

FIG. 236.

$$-\frac{V_1}{V_2}=\frac{R_1}{R_2}. \qquad (17)$$

Hence the potential of the condenser L is $-V_2$ and that of C is V_1, and these potentials are proportional to the respective resistances R_1, R_2. By means of the key K the condensers are brought to one potential, and this is zero if $V_1C_1=-V_2C_2$. To test whether the potential is zero, the key K_2 is depressed and connects the insulated coatings of the condensers to earth through a sensitive galvanometer G. Any difference of potentials between the coatings and the earth is thus annulled and gives rise to a current through the galvanometer. The slider is adjusted until no current is thus produced through the galvanometer. We have then

$$-\frac{V_2}{V_1}=\frac{R_2}{R_1}=\frac{C_2}{C_1},$$

or

$$C_2=\frac{R_2}{R_1}C_1. \qquad (18)$$

For accuracy R_2 and R_1 should be somewhat high resistances so as to ensure an exact knowledge of their ratio, and C_1 should be as nearly as possible equal to C_2.

When a cable is tested sufficient time must be given in charging to enable it to acquire the same potential throughout, and for the discharge of one condenser into the other; and the tests are repeated with the battery reversed on the slide to eliminate the effect of any existing charge in the cable. It is usual also to make a number of tests and take the mean result.

Instead of a more or less elaborate key K arranged to perform all the operations quickly and conveniently, a system of two pairs of cups 1, 2, 3, 4 arranged in the square order

$$\begin{array}{cc} 1 & 2 \\ \\ 3 & 4 \end{array}$$

may be cut in a slab of paraffin and filled with mercury. The terminals of a, b are connected to 1, 2, the insulated plate of the condenser to 4, and that of C to 3. By a connecting bridge of wire held by an insulating handle, 1 and 3 are connected, and in the same way 2 and 4, so as to charge the condensers. These connections are then removed, and 3 and 4 connected so as to discharge one condenser into the other. Then by means of the key K_2, or by another mercury cup, connected by a wire bridge with 3 or 4, the condenser coatings are connected with earth through the galvanometer.

Plainly in this case also an electrometer may be used instead of the galvanometer. One pair of quadrants is connected to earth, the other pair through the key K_2 to the condensers.

32. De Sauty's method of comparing capacities. The following method of comparing capacities is convenient for the comparison of the capacities of condensers in which electric absorption does not come into play. The arrangement of the apparatus is shown in the diagram, Fig. 237. K is a key which when depressed puts into contact with the point of junction of two variable resistances, R_1, R_2, one terminal a of a battery, the other terminal b of which is connected to the earth. The other extremities C, D, of these resistances are connected to the insulated coatings of the condensers C_1, C_2, which are to be compared. The other coatings of these condensers are connected to earth. C and D are connected likewise through a sensitive galvanometer G. When the key K is not depressed it joins A directly through a wire to the earth. R_1, R_2 are adjusted so that neither in charging the condensers by applying the battery to A, nor in discharging by allowing the key to connect A directly to earth, does any current pass through the galvanometer. (If any influence of electric absorption is sensible, the ratio of resistances which gives zero galvanometer current when charging will not generally be the same for charge as for discharge.) When no deflection of the galvanometer needle takes place, the potential at C and D must throughout the discharge have been the same at each instant, for the condensers could not discharge in such a way as to give a current, first in one direction, then in the other, through the galvanometer, and so keep the needle at rest. But if γ_1 be the current through R_1 and γ_2 the current through R_2, V the common potential of C and D, C_1, C_2 the capacities of the condensers connected with R_1, R_2 respectively, we have

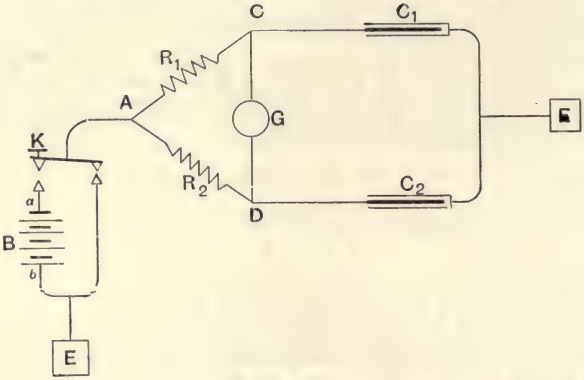

FIG. 237.

$$\gamma_1 = \frac{V}{R_1} = \pm\frac{d(VC_1)}{dt}, \quad \gamma_2 = \frac{V}{R_2} = \pm\frac{d(VC_2)}{dt},$$

and therefore

$$R_1\frac{d(VC_1)}{dt} = R_2\frac{d(VC_2)}{dt},$$

that is the products of R_1, R_2 into the time rates of variation of the charges of the corresponding condensers are equal at each instant. This can only be the case if

$$R_1C_1 = R_2C_2,$$

or

$$C_2 = \frac{R_1}{R_2}C_1. \quad \text{......................................(19)}$$

This result may be seen more easily as follows. Let n equal condensers have their insulated coatings joined to A by wires of equal resistance in the manner shown for two condensers in Fig. 237. Then plainly the charging or discharging current in each wire will be the same at each instant, and the insulated plates will always be at one potential. No change will be caused by joining the insulated coatings in two groups by wires of zero capacity, so as to make the groups virtually two condensers, of capacities equal in each case to the sum of the capacities of the separate condensers of the group, and connected to A by wires of resistances inversely as the capacities. By making n sufficiently large, and the capacity of each condenser sufficiently small, the capacities of the groups may be made of any required value and nearly enough in any ratio commensurable or incommensurable.

33. Direct deflection method of measuring a capacity. Another method, which we shall again refer to later as a method of obtaining the capacity of a condenser in absolute units, is frequently employed to obtain rapidly a comparison of the capacities of two condensers. It is called the direct deflection method. One of the condensers is charged to a measured potential and then discharged by connecting it to earth through a "ballistic" galvanometer, that is a galvanometer the needle system of which has a considerable moment of inertia. Fig. 238 shows the arrangement of apparatus with a form of charge and discharge key, the contact pieces of which are mounted on ebonite pillars to ensure high insulation. The spring lever L is provided with two platinum contacts opposite to the platinum pieces S_1, S_2. When depressed it makes contact for charge, when released it connects the plate of the condenser through the galvanometer. If the duration of discharge is, as it generally is, short, and means are taken, for example, by depressing the key immediately after the discharge contact, to disconnect the galvanometer immediately after the first discharge so as to avoid any effect of residual discharge due to electric absorption, the discharge may be regarded as having wholly taken place before the galvanometer needle has moved from zero. The total deflection of the needle from zero is observed. By placing the galvanometer between the battery and S_1, the deflection produced by charging can be observed. If there is leakage this latter deflection will obviously be greater than the former; and if the leakage be not too great the mean of the two deflections with the same battery may be taken as giving the capacity of the condenser. The other condenser is now charged to a potential V' and discharged in the same manner through the galvanometer and the deflection again observed. V and V' should if possible be chosen so as to make the two deflections nearly equal, in order to eliminate the damping effect which the needle experiences to different degrees in deflections of different amounts. If an instru-

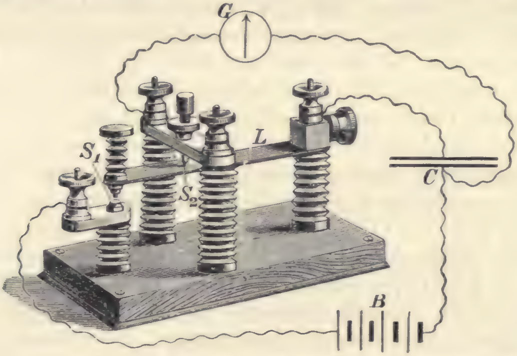

FIG. 238.

ment for comparing the potentials V, V' is not available, they may be produced by applying to the condensers one terminal of a well-insulated battery, the other terminal of which is connected to the earth, and varying the number of cells until equality of deflections is nearly obtained. If the battery be composed of similar cells in good order, the potential may be taken as proportional to the number of cells applied to produce them. For a rough determination it is convenient of course to charge both condensers by the same battery, and thus to the same potential, and to take the capacities as proportional to the galvanometer deflections produced.

34. Comparison of a large with a small capacity. The capacity of a large condenser, such as a long submarine cable with its conductor insulated, may be compared with that of a relatively small condenser by the following method, which is due to the late Sir W. Siemens. Let the large condenser be charged to any convenient potential V by means of a battery. If the capacity be C the charge is VC. Now let the large condenser be connected to the insulated coating of the small condenser, the capacity of which we shall suppose to be c. The common potential of the two condensers will now be $VC/(C+c)$. Now disconnect the small condenser and discharge it, and again connect it to the large condenser, disconnect and discharge as before. The potential will now be $VC^2/(C+c)^2$. Thus after n applications in this manner of the small condenser to the large, the potential of the large condenser will be $VC^n/(C+c)^n$. The deflection on a ballistic galvanometer produced by the nth discharge of the small condenser is now noted. The small condenser is then charged, by the same battery as that used to charge the large condenser, and therefore to the same potential V, discharged, and the deflection noted. If D_n, D be these deflections we have

$$\frac{D}{D_n} = \frac{(C+c)^n}{C^n},$$

and therefore

$$C = \frac{c\sqrt[n]{D_n}}{\sqrt[n]{D} - \sqrt[n]{D_n}}. \qquad \text{.................................}(20)$$

The comparison by this method must be made as rapidly as possible in order that the effect of any leakage of the large condenser may be made as small as possible. On the other hand the theory of the method proceeds on the assumption that the potential of the condenser at each discharge is brought throughout to the same value, and this cannot be done in a long cable unless a sufficient time of contact is given at each discharge. There is further the difficulty of correcting the deflections for air damping, etc. The method therefore cannot be regarded as an accurate one for the cable application.

It is easy, when the ratio C/c is approximately known, to investigate the best value of n to use to give results as little as possible affected by errors in the observation of D, D_n, but on account of the inaccuracies inherent in the method for most practical purposes, it is of little importance to use that value.

35. Leakage method of comparing a capacity with a resistance. The arrangement described above in XI. 44 for the determination of a high resistance gives also a means of determining the capacity of a condenser. For let the coatings of the condenser be connected by a very high known resistance R as described, and let a difference of potential V between the coatings be produced by applying a battery. Let V be observed by means of an electrometer, the insulated quadrants of which are kept connected to the insulated coating of the condenser. As the charge diminishes by conduction through the resistance, the electrometer shows a diminishing deflection which is observed at accurately noted instants of time. If V_0, V be the potentials at the beginning and end of an interval of t seconds, C the capacity of the condenser, and R the resistance connecting the coatings, we have

$$C = \frac{t}{R} \frac{1}{\log \frac{V_0}{V}}. \qquad \text{......(21)}$$

Values of V_0, V, for different values of t are given by the observations, and enable a mean value of C to be obtained free to some extent from errors of observation.

The resistance R must of course be very great in order that the whole charge may not be so quickly lost as to prevent the potentials from being observed before and after a sufficiently long interval of time. If the condenser be not a perfectly insulated air condenser, the actual resistance of the dielectric layer between its coatings may be taken advantage of, and will in general be convenient for the purpose. To determine it we use an auxiliary condenser of known capacity C', and resistance R', which has been determined by some method, for example, the method of p. 370 above. The insulated coating of this condenser is joined to that of the condenser to be measured, so that the capacity of the joint condenser becomes the sum of their separate capacities, and the resistance between their coatings $RR'/(R+R')$. The condenser thus formed is charged and the potential at different instants of time observed as before. Thus if V_0', V' be the potentials before and after an interval of t' seconds, we have

$$C + C' = \frac{t'(R+R')}{RR'} \frac{1}{\log V_0/V'}. \qquad \text{......(22)}$$

This equation with (21) suffices to determine C and R.

36. Werner Siemens' method of determining capacities. We give lastly here a method of measuring capacities, which is of importance in the determination of Specific Inductive Capacities. Fig. 239 shows the arrangement of the apparatus. B is a battery of a number of well insulated constant cells, of which one terminal is connected to earth, and the other terminal to the contact piece *a*. A sensitive galvanometer *G* has one terminal connected to the contact piece *b*, and the other connected to earth. The pieces *a* and *b* are so arranged that a commutator, represented diagrammatically by *K*, connected permanently with the insulated coating of the condenser, makes contact alternately with *a* and *b*. The condenser is charged when *a* is in contact, discharged when *b* is in contact. This is easily managed by means of a rotating cylinder carrying contact pieces which are pressed on by springs represented by *a*, *b*, or by some other suitable mechanical arrangement. The commutator is made to give a constant and large number *n* of discharges, say from 40 or 50 per second upwards. Thus if the battery remains constant, a constant mean current is produced through the galvanometer. Let *E* be the electromotive force of the battery, and *C* the capacity of the condenser, then, on the supposition that we may suppose the condenser completely charged or discharged at each contact, we have for the mean current nEC. If n be sufficiently great this will give the same deflection as a continuous current of the same amount. After this deflection has been observed, the circuit of the battery is completed through the galvanometer, and a resistance R, just of sufficient amount to give a second good measurable deflection of the galvanometer needle. If α, β be these deflections corrected so as to be proportional to the mean current (generally the actually observed deflections may be taken if they are small), we have

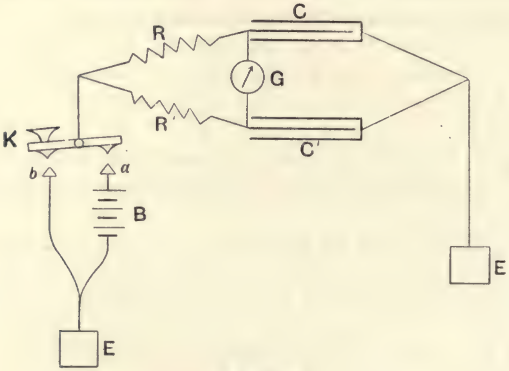

FIG. 239.

$$nEC = m\alpha,$$

$$\frac{E}{R} = m\beta,$$

where m is a constant.

Hence we have
$$C = \frac{\alpha}{\beta}\frac{1}{nR}. \quad\text{...}(23)$$

The commutator may be easily arranged so as to charge the condenser alternately positively and negatively. If C be the mean of the two capacities which the condenser has, according as one or the other coating is made the uninsulated coating, we have, putting n for the *number of reversals* per second, $2nEC$ for the whole quantity of electricity which flows through the galvanometer in a second, that is, the mean current. Hence if α and β have the same meanings as before, we have

$$C=\frac{\alpha}{\beta}\frac{1}{2nR}. \qquad\qquad (24)$$

These values of the current, it is to be remarked, are obtained on the assumption that the times of charge are sufficiently long to allow the condenser to be fully charged to potential E, and the time of discharge also long enough to allow the condenser to be completely discharged. The results of experiments made with different time-intervals have justified this assumption for small condensers even for time-intervals so small as $\frac{1}{20000}$ of a second.

CHAPTER XVIII.

I. EFFECT OF INDUCTIVITY OF THE MEDIUM ON ELECTRIC PHENOMENA.

1. Stress in the dielectric medium. Before we proceed to discuss determinations of specific inductive capacity and other properties of dielectrics, it will be convenient to consider the dependence on inductivity of various electric quantities characteristic of the dielectric medium. For example, according to the theory given by Faraday and Maxwell, there is at every point of the dielectric a tension along the lines of force, and an equal pressure at right angles to that direction, and that the amount of each of these stresses is $\kappa F^2/8\pi$, where F is the resultant field-intensity at the point considered, and κ is the inductivity of the medium.

It is a result of experiment that a soap-bubble is enlarged by electrification, so that the external dielectric medium evidently pulls every part of the bubble surface outward, though this outward action is sometimes attributed to repulsion due to the electric charges elsewhere. There is, however, no doubt that this pull is exerted in liquid and gaseous dielectrics, for there it can be and has been observed.

2. Energy per unit volume equal to pull on unit area of electrified surface. Consider a tube of electric induction drawn in the dielectric and starting from an area dS of an electrified surface which is at potential V; the electric charge on this area is $\sigma\, dS$. Let the tube terminate on an element dS' of a surface at potential V'. The energy in the portion of the tube between the two surfaces is $\frac{1}{2}\sigma\, dS(V-V')$.

Let the first surface be one face of a plane plate and be opposed at a short distance d by a second plane surface, parallel to the former, and at one potential V'. Then for a tube between the surfaces, and not near the edge of either plate, the energy is

$$\tfrac{1}{2}\sigma\, dS(V-V') = \tfrac{1}{2}\sigma\, dS\,.\,4\pi\sigma d/\kappa,$$

where κ is the inductivity of the medium. Hence for this tube

$$\text{Energy per unit volume} = 2\pi\frac{\sigma^2}{\kappa}. \quad \text{.........................(1)}$$

If the inductivity of the standard medium be taken as unity this becomes $2\pi\sigma^2/K$, where K is the specific inductive capacity. For the expression on the right can be written $2\pi\sigma^2/\kappa_0(\kappa/\kappa_0)$, and $\kappa/\kappa_0 = K$ by definition.

This energy per unit volume, it is to be observed, is the pull per unit area on the opposed surfaces. This may be expressed in terms of the field-intensity close to the surface. If E be this intensity, say near the first surface, we have

$$4\pi\sigma = \kappa E$$

or KE when the standard medium has $\kappa_0 = 1$. Thus

$$\text{Energy per unit volume} = \frac{\kappa}{8\pi} E^2, \quad \text{.........................} (1')$$

or $KE^2/8\pi$ when $\kappa_0 = 1$. [In what follows we shall usually take $\kappa_0 = 1$.]

3. Analogy between electrostatic action and heat conduction. The whole matter of the action of the medium may be shortly discussed by means of the analogy of conduction of heat. The mathematical theories are the same, when electric potential is taken as the analogue of temperature, and, with a certain specification as to units, quantity of heat transmitted from a source per unit time is taken as the analogue of quantity of electricity. As an illustration consider a single point-source of heat within a uniform medium infinitely extended in all directions (that is a point at which heat enters the medium from elsewhere, no matter how, for example, it may be supposed *generated* at the point; and let the total heat generated at the source in unit of time be Q. Now if, as we suppose, the distribution of temperature round the source remain unaltered in time, it is clear that whatever heat crosses any closed surface round the source in any time must also cross every other such surface; otherwise heat would be gained by part of the medium and the temperature would be changed. Now if v denote temperature at distance r from the source, symmetry shows that v is constant over every spherical surface of which the source is the centre. The gradient of temperature in any direction at any point is what governs the flow of heat there, and so for the total rate of flow across the spherical surface of radius r, we have

$$-4\pi r^2 k \frac{dv}{dr} = Q,$$

and therefore

$$-k \frac{dv}{dr} = \frac{Q}{4\pi} \frac{1}{r^2}. \quad \text{.......................................} (2)$$

The quantity on the left-hand side of the last equation is (when v is replaced by electric potential, V, and k by electric inductivity, κ) called the " electric displacement " and may be taken to represent the electric strain in the medium, produced by the point-charge $q = Q/4\pi$.

The last equation gives by integration

$$v = \frac{Q}{4\pi k} \frac{1}{r} + C,$$

where C is a constant. To determine C we have the condition $v=0$ when $r=\infty$, so that $C=0$. Thus

$$v=\frac{Q}{4\pi k}\frac{1}{r}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

that is the temperature (potential) is inversely as the distance from the point-source (point-charge) and inversely as the conductivity (inductivity) k of the medium (dielectric).

4. Field containing different media. The thermal analogy shows very clearly how the results for a uniform medium (dielectric) of unit inductivity are modified for any other medium. The total flow of heat across a closed surface in a medium of conductivity k is

$$-\int k\frac{dv}{d\nu}\,dS,$$

where dS is an element of surface, and $d\nu$ a small step along the outward drawn normal to dS, and the integral is taken over the surface. Thus if κ denote inductivity and V electric potential, the analogy gives for the whole quantity Q of electricity within any closed surface in the electric field the equation

$$Q=-\frac{1}{4\pi}\int\kappa\frac{dV}{d\nu}\,dS. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

There $-\kappa dV/d\nu\,.\,dS$ is the integral of electric induction, $-\kappa dV/d\nu$, across the surface element dS, while $-dV/d\nu$ is the component of electric force or electric field-intensity outward at right angles to dS. When the medium is isotropic electric induction and electric field-intensity have the same direction.

The field-intensity at a point B at distance r from a point-charge q situated at a point B is, as we have seen above, $q/\kappa r^2$. Hence if at B another point-charge, of q' units, is situated, the force on this second point-charge is $qq'/\kappa r^2$, and acts outwards along AB, if the charges at A and B have the same sign.

Just as in the thermal analogy, the direction of the flow of heat in an æolotropic body is not in general at right angles to the isothermal surfaces, so in an æolotropic medium the direction of the resultant electric induction at any point is not in general the same as the direction of the resultant electric field-intensity at the same point. Here also the two theories are parallel, but we cannot enter further into the subject here.

5. Conditions which hold at surfaces of separation between media. We also get at once the modified characteristic differential equation for a medium of inductivity κ, varying from point to point, but the same in all directions at any one point,

$$\frac{d}{dx}\left(\kappa\frac{dV}{dx}\right)+\frac{d}{dy}\left(\kappa\frac{dV}{dy}\right)+\frac{d}{dz}\left(\kappa\frac{dV}{dz}\right)+4\pi\rho=0, \quad \ldots\ldots\ldots\ldots(5)$$

and at any electrified surface in the medium,

$$\kappa\left(\frac{dV_1}{d\nu_1}+\frac{dV_2}{d\nu_1}\right)+4\pi\sigma=0. \quad\text{..............................(6)}$$

If the electrified surface be a surface of separation between two media of specific inductive capacities κ_1, κ_2 the surface equation is, by (5) above, modified for the special case of such a surface,

$$\kappa_1\frac{dV_1}{d\nu_1}+\kappa_2\frac{dV_2}{d\nu_2}+4\pi\sigma=0. \quad\text{..............................(7)}$$

In the case of a field occupied in different regions by media of different specific inductive capacities, the characteristic equation is to be applied with the corresponding value of κ for each region, and the surface equation at each separating surface.

It is to be observed that the electric densities ρ and σ are the true electric densities which exist in the form of an electric charge conveyed to the medium or placed on the surface, and do not include the electrification of the medium in consequence of induction.

We may put the theorem into words as follows :

If N_1, N_2 be the normal forces at infinitely near points on opposite sides of the surface of separation between two isotropic media, each force being reckoned in the direction from the surface, K_1, K_2, the specific inductive capacities of the respective media, that is, κ_1/κ_0, κ_2/κ_0, and if there is no electric charge on the surface except that due to induction, then

$$K_1N_1+K_2N_2=0. \quad\text{....................................(8)}$$

This equation may be written in the form

$$N_1+N_2-4\pi\sigma'=0, \quad\text{....................................(9)}$$

where

$$\sigma'=\frac{1}{4\pi}\,\frac{K_2-K_1}{K_2}\,N_1=\frac{1}{4\pi}\,\frac{K_1-K_2}{K_1}\,N_2. \quad\text{....................(10)}$$

6. Apparent electrification on the surface of a dielectric. This value of σ' is the electric surface density which would exist on the separating surface of the media if each had unit specific inductive capacity and N_1, N_2 their actual values, and was called by Maxwell* the *apparent* electric density on the surface. If a distribution of this density be made over the surface of the space occupied by K_2, and the specific inductive capacities K_1, K_2 be made each unity, the same electric force will be produced at all points internal or external. For the distribution if made gives the actual values of N at the surface, and equation (5) will plainly be satisfied ; and we have seen that under these conditions there can be only one value of the potential at any point.

If this apparent electrification be removed during the action of the inducing force by bringing every part of the surface to zero potential, say, by passing a flame over it, and the inducing force be then removed, there will appear a true electrification equal and opposite to σ'. This fact was used by Sir William Thomson to explain the phenomena of pyro-electricity shown by certain crystals.*

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* *El. and Mag.* vol. i. 3rd ed. p. 100.

7. Refraction of lines of force at common boundary of dielectrics. If the surface of separation is not at right angles to the lines of force, then resolving the forces at two infinitely near points on opposite sides of the surface along and at right angles to the normal, we have by (7), if the surface is not electrified,

$$K_1\frac{dV_1}{d\nu_1}+K_2\frac{dV_2}{d\nu_2}=0, \quad\text{..............................(11)}$$

and since $V_1=V_2$, at every point of the surface,

$$\frac{dV_1}{d\omega}=\frac{dV_2}{d\omega}, \quad\text{.....................................(12)}$$

where $dV/d\omega$ denotes rate of variation of potential in a direction parallel to the surface of separation, and in the plane of the line of force and the normal. Hence if θ_1, θ_2 be the angles which the line of force makes with the normal in the first and in the second medium respectively, we have

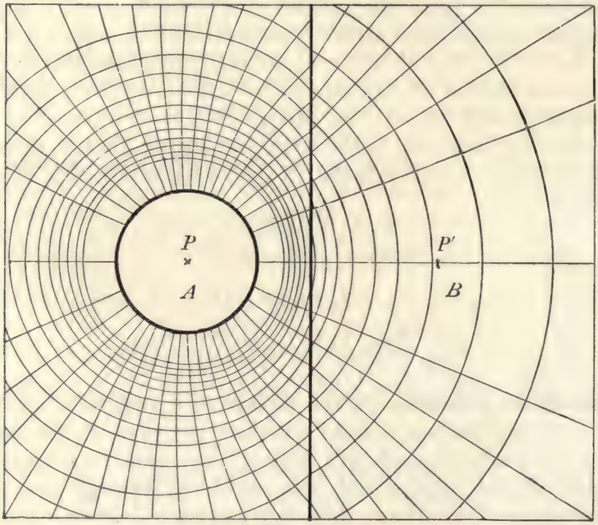

FIG. 240.

$$\tan\theta_1=\frac{dV_1}{d\omega}\Big/\left(-\frac{dV_1}{d\nu_1}\right), \quad \tan\theta_2=\frac{dV_2}{d\omega}\Big/\frac{dV_2}{d\nu_2},$$

and therefore

$$\tan\theta_1=\frac{K_1}{K_2}\tan\theta_2. \quad\text{..................................(13)}$$

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* *Ibid.* p. 61.

The line of force thus undergoes a species of refraction in which the tangents of the angles of incidence and refraction are related as are the sines of the corresponding angles in the refraction of light. It is to be observed that according to the law of refraction of lines of force they can show nothing corresponding to the optical phenomenon of total reflection. This refraction is illustrated in Fig. 240, which represents a section of the field due to a point-charge q at P in a medium A of inductivity κ, in contact over a plane surface with a medium B of inductivity κ_2. The change of direction ("refraction") of the lines of induction will be observed.

8. Spherical portion of a dielectric imbedded in another medium.—Undisturbed external field uniform. The following case is of great importance in the theory of magnetism and of practical interest in the experimental determination of specific inductive capacities. A spherical portion of an isotropic dielectric medium in which the electric force has everywhere the same magnitude and direction, that is, in which there is a uniform field of force, is replaced by an equal spherical portion of another isotropic dielectric. It is required to find the apparent electrification, and thence the force at any point without or within the sphere.

Let K_1, K_2 be the specific inductive capacities of the surrounding medium and the sphere respectively, F the uniform electric force in the first medium produced independently of the apparent electrification, N_1, N_2 the external and internal normal component forces at any point due to the apparent electrification, σ' the surface density of the apparent electrification at that point of the separating surface, and θ the angle which a radius drawn to the point makes with the positive direction of F. Taking N_1, N_2 in the direction from the surface on both sides we get by (8) and (9),

$$F\cos\theta + N_1 + \frac{K_2}{K_1}(-F\cos\theta + N_2) = 0;$$

or by (10)

$$\sigma' = -\frac{1}{4\pi}\,\frac{K_1 - K_2}{K_2}(F\cos\theta + N_1)$$

$$= -\frac{1}{4\pi}\,\frac{K_2 - K_1}{K_1}(-F\cos\theta + N_2). \quad\text{.................(14)}$$

This is the surface characteristic equation.

9. Density of apparent electrification given by "couches de glissement." The distribution supposed formed in the following manner satisfies this equation at the surface, and Laplace's equation at every internal and external point, and gives therefore the apparent surface density for the case. Two equal spherical volume distributions of electricity of uniform density ρ, one positive, the other negative, and of the same radius as the sphere, are placed in coincidence; then, according as

K_2 is greater or less than K_1, the positive or the negative distribution is displaced (Fig. 241) in the direction of F through a finite distance a less than the sum of the radii. A positive volume distribution of meniscus shape is thus formed on one side, and a similar negative distribution on the other, and in the space occupied by the coincident parts of the distributions there is zero electric density. Now let the distance a be diminished indefinitely and the density ρ of the volume distribution increased so that ρa is not altered in value. Drawing then any radius making an angle θ with the direction of F, we have for the thickness of the stratum in the direction of the radius the value $a \cos\theta$. Hence if $\sigma'_0 = a\rho$, the surface density at the extremity of the radius is $\sigma' = \sigma'_0 \cos\theta$. Its value is σ'_0 or $-\sigma'_0$ according as $\theta = 0$ or $= 180°$.

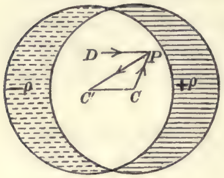

FIG. 241.

10. Field within sphere. The force at any internal point P due to the distribution is plainly the resultant of the two forces due to the two spherical portions of the volume distributions which have C, C' as centres and P a common point on their surfaces. These forces are in magnitude respectively $4\pi\rho . CP/3$, $4\pi\rho . C'P/3$, and act in the directions shown in Fig. 241, and therefore their resultant acts in the direction CC'. Putting R for this resultant, taken positive in the direction of F, we have

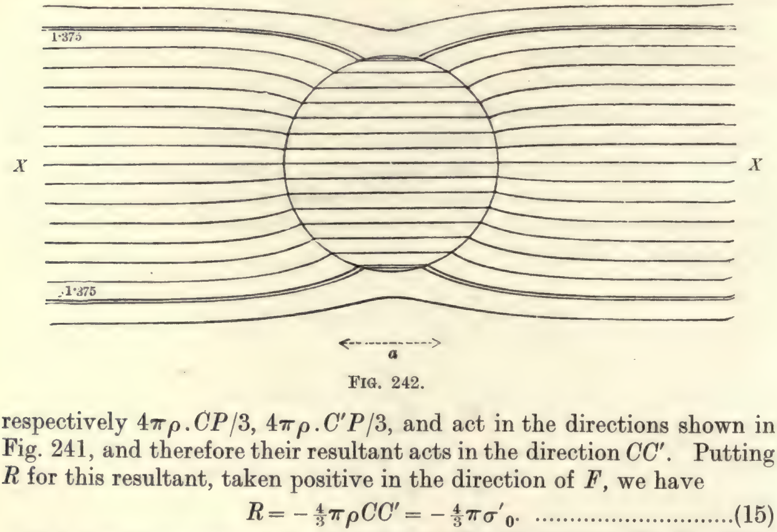

FIG. 242.

$$R = -\tfrac{4}{3}\pi\rho CC' = -\tfrac{4}{3}\pi\sigma'_0. \qquad \text{.............................(15)}$$

It is therefore constant in magnitude. The total force, $F + R$, within the sphere is therefore also constant in magnitude and direction.

By
$$N_2 = \frac{4}{3}\pi\sigma'_0 \cos\theta = \frac{4}{3}\pi\sigma',$$

which gives by substitution in (14)

$$\sigma' = \frac{3}{4\pi}\left\{\frac{K_2 - K_1}{2K_1 + K_2}\right\} F \cos\theta = \sigma'_0 \cos\theta. \qquad \text{...................(16)}$$

Therefore
$$R = -\frac{K_2 - K_1}{2K_1 + K_2}\,F, \qquad\qquad (17)$$

and
$$F + R = \frac{3K_1}{2K_1 + K_2}\,F. \qquad\qquad (18)$$

Hence according as K_2 is greater or less than K_1 the force within the sphere is less or greater than the force F without.

The directions of the lines of force outside and inside the sphere are shown in Fig. 242 for the case of $K_2 = 2{\cdot}8K_1$, and radius of sphere $= 1{\cdot}1a$; in Fig. 243 for $K_2 = {\cdot}48K_1$ and radius of sphere $= 1{\cdot}34a$.

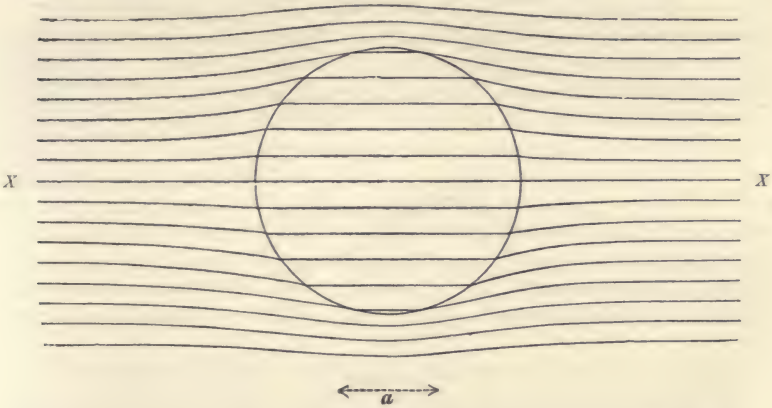

FIG. 243.

11. Case of conducting sphere situated in impressed uniform field. If the sphere is of conducting material, $K_2 = \infty$, and $F + R = 0$, as it ought to be. In this case also we have

$$\sigma' = \frac{3}{4\pi}\,F\cos\theta. \qquad\qquad (19)$$

The directions of the lines of force for the case of the conducting sphere are shown in Fig. 244. The radius of the sphere is $a/\sqrt[3]{2} = {\cdot}794a$.

The equation of the curves external to the circle in Figs. 242...244 is

$$x = \left\{\left(\frac{a^2y^2}{b^2 - y^2}\right)^{\frac{2}{3}} - y^2\right\}^{\frac{1}{2}}.$$

The centre of the circle is the origin, and the curve XX, which in each case is a straight line, is the axis of x. In Figs. 242 and 243, y^2 is everywhere less than b^2; in Fig. 244, y^2 is everywhere greater than b^2. Each set of curves is drawn for a constant value of a which is indicated below the diagram, and values of b equal to 0, $\cdot 2a$, $\cdot 4a$, $\cdot 6a$, ... $1{\cdot}6a$. In Figs. 242 and 244, the curve for $b = \sqrt{3}/\sqrt[3]{2}\,.\,a = 1{\cdot}375a$ is drawn. This curve has a pair of double points through which the circle in

Fig. 244 passes: in Fig. 242 these points fall within the circle and are not shown. In Fig. 244 the circle has radius $=a/\sqrt[3]{2}=\cdot794a$ and cuts orthogonally all the curves except that on which are the double points: in Figs. 242 and 243 the radii of the circles are $1\cdot1a$ and $1\cdot34a$ respectively. (See Sir W. Thomson's *Reprint of Papers on Electrostatics and Magnetism*, p. 492, from which these figures are taken.)

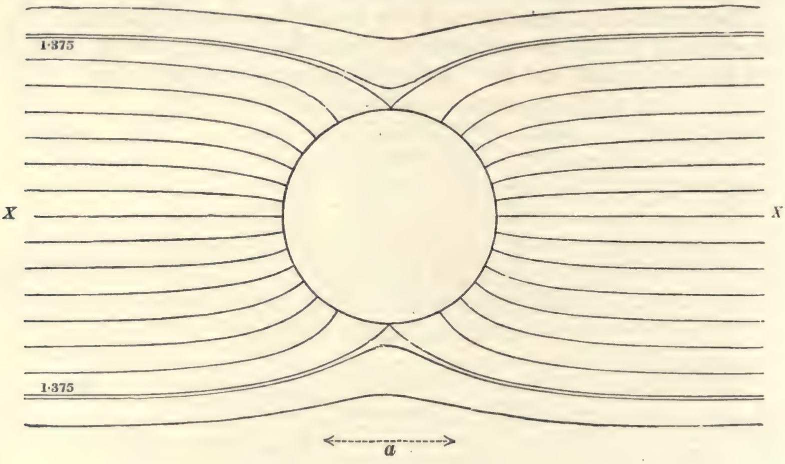

FIG. 244.

12. Energy of a dielectric sphere in a uniform field. Force on imbedded sphere. The potential energy of the dielectric sphere in the uniform field is found simply by calculating the work done by electric forces in the relative displacement of the imaginary volume distributions. If r be the radius of the sphere, the total quantity of electricity in the positive volume distribution is $4/3\,.\,\pi\rho r^3$. The work done by electric forces in displacing this through a distance a is $4/3\,.\,\pi\rho r^3\,.\,Fa$. Hence, if E be the energy of the sphere *in the field*,

$$E=\frac{4}{3}\pi r^3F\,.\,\rho a=\frac{4}{3}\pi r^3F\sigma'_0$$

$$=-r^3\frac{K_2-K_1}{2K_1+K_2}\cdot F^2. \quad \text{.................................(20)}$$

This expression has been obtained for a uniform field, but it will also hold for a variable field if r be so small that the value of F is sensibly constant in magnitude and direction at every point of the sphere.

On this supposition, the rate of diminution of E in any direction ν in a variable field is given by the equation

$$-\frac{dE}{d\nu}=r^3\frac{K_2-K_1}{2K_1+K_2}\frac{d(F^2)}{d\nu}, \quad \text{...........................(21)}$$

and this must be the total electric force on the sphere.

Writing x, y, z respectively for ν in this formula we get X, Y, Z the component forces in the direction of these variables. The direction of the resultant force on the sphere is that for which $d(F^2)/d\nu$ is a maximum, and in which F^2 increases. The direction therefore in which the sphere tends to move is towards a place of maximum value of F^2; that is, in which the value of F is numerically greatest without distinction of sign.

For a conducting sphere (21) becomes

$$-\frac{dE}{d\nu}=r^3\frac{d(F^2)}{d\nu}, \quad \text{.....................................(22)}$$

and the sphere tends to move in the same direction as the dielectric sphere.

Since, as we have seen, there is no place of maximum or minimum potential in space not occupied by any part of the electrification, a point-charge, or small sphere supposed uniformly electrified, would nowhere be in stable equilibrium except in contact with some part of the electrification; and the proposition may be extended to any electrified body. Hence in the cases here considered the spheres move along the line of greatest variation of force towards a place where the force is numerically greatest. Generally, this is the direction in which all bodies of small dimensions, placed in the electric field without charge, tend to move.

By (21) and (22) $(K_2-K_1)/(2K_1+K_2)$ is the ratio of the force on a dielectric sphere of specific inductive capacity K_2 to the force on a conducting sphere of the same radius placed at exactly the same place in the field of specific inductive capacity K_1.

This relation has been used by Boltzmann for the determination of specific inductive capacities (see 19, *et seq.*).

13. Consideration of particular cases of different media in contact. We shall now apply the results stated above to one or two important cases:

(1) An electric field consists of two regions, one bounded by equipotential surfaces, and filled with a dielectric of specific inductive capacity K the same in all directions, and the other, the remainder of the space within the zero equipotential surface, occupied by a dielectric of unit specific inductive capacity. It is instructive to refer this example directly to the thermal analogy. The analogue of the electrified system is a geometrically corresponding system of heat-sources and isothermal surfaces in a medium of conductivity everywhere unity, except in a region bounded by isothermal surfaces, where the conductivity is k. Suppose the whole medium at first of unit conductivity, and that then a medium of conductivity k is substituted for the former medium in the space referred to, while everything else remains unaltered. The effect of introducing the medium of (say) higher conductivity is to diminish the difference of temperature between the inner and outer

surfaces of the new medium in the ratio of 1 to k, since everywhere in that medium the flux along a line of flow becomes $-kdv/dr$, which as the generation of heat is unchanged, must be equal to the former value of $-dv/dr$. Hence also the flux at every point which is not in the new medium is unchanged, and we have therefore at every such point the same gradient of temperature as before, and therefore also the same difference of temperature as before, between any point of the system of sources and the inner surface of the new medium, and between any point in the outer surface of the new medium and the surface of zero temperature. If then the temperature of the inner surface was formerly v, and that of the outer surface v', the temperature of any point of the source has been lowered by the introduction of the medium of conductivity k by an amount $(v-v')(k-1)/k$.

In precisely the same way in the electrical problem, if the electric charges are kept the same, the electric force at every point inside or outside the new medium is unaltered, and, at every point within the substance of the medium itself is changed from its former value F to F/K, and the potential of any part of the electrified system is lowered by the amount $(V-V')(K-1)/K$, where V and V' are the respective potentials of the inner and outer separating surfaces of the media.

If the new medium fill the whole space between the electrified system and the surface of zero potential $V'=0$, the potential V of any part of the system has been diminished in the ratio of 1 to K, and the charge of the whole system necessary to produce a given potential at any part of it has therefore been increased in the ratio of K to 1; that is, the electrostatic capacity of the system has been increased in this ratio.

The same results would be obtained by imagining the medium of inductive capacity K replaced by a medium of unit inductive capacity, and the internal and external surfaces of the region electrified so that the surface density at any point of the inner surface is $\{(K-1)dV/d\nu\}/4\pi$, and at any point of the outer surface $-\{(K-1)dV/d\nu\}/4\pi$, where $dV/d\nu$ is the rate of variation outwards along a line of force passing through the point taken in the first case just inside, in the second case just outside, the region in question. These being equilibrium distributions would not alter the actual distribution, and the force inside and outside the region at any point would be the same as before, while within the region it would be diminished at any point in the ratio of 1 to K.

We see in the same way that if the specific inductive capacities, instead of being 1 and K, were respectively K_1 and K_2, the difference of potential between the two sides of the layer K_2 would be less than its value for the same space occupied by the medium K_1 in the ratio of K_1 to K_2, and the density of the imaginary distribution described in the last paragraph would be

$$+\frac{K_2-K_1}{4\pi}\frac{dV}{d\nu}$$

for the inner surface, and for the outer surface

$$-\frac{K_2-K_1}{4\pi}\frac{dV}{d\nu}.$$

(2) The same method applies to the case of a field composed of dielectrics of inductive capacities K_1, K_2, K_3, etc., each bounded by equipotential surfaces and arranged in this order outwards from the electrified system, which we suppose in the medium K_1. Let V be the potential of any part of the electrified system, V_1 the potential of the outer surface of K_1 and the inner surface of K_2, V_2 the potential of the outer surface of K_2 and the inner surface of K_3, and so on. Then if K_1 alone were replaced by vacuum, $V-V_1$ would become $K_1(V-V_1)$, the other differences of potential remaining the same as before; if K_2 were then replaced by vacuum, V_1-V_2 would become $K_2(V_1-V_2)$, and so on. Hence, if all the media were replaced by vacuum, the potential of any part of the electrified system would be changed from V to

$$K_1(V-V_1)+K_2(V_1-V_2)+\text{etc.}$$

Hence, if C be the new value of the elctrostatic capacity of the system and C' its former value, we have

$$\frac{C}{C'}=\frac{V}{K_1(V-V_1)+K_2(V_1-V_2)+\text{etc.}}\cdot \quad\text{.....................(23)}$$

14. Maxwell's conception of the system of stress in a dielectric. Electric displacement. Maxwell * has considered a dielectric medium surrounding an electrified system as in a state of strain under stresses consisting of a tension (as in a stretched wire or cord) acting at each point along the direction of the electric force, and an equal pressure at the same point in all directions at right angles to that of the electric force. The amount of the tension and pressure (each taken in units of force per unit of area) at any point at which the electric force is F in a medium of specific inductive capacity K is $KF^2/8\pi$; that is, equals the electric energy of the medium per unit of volume at that point.

Further, he has regarded the electric charge of the system as the surface manifestation of a change which took place in the medium when the electrification was set up. This change he has called *Electric Displacement,* and consists in a passage, across every surface drawn in the medium so as to enclose the electrified system, of a quantity of electricity equal to the charge on the system, so that the introduction of a charged system within a closed space does not produce any change in the total quantity of electricity within the space. Thus when one coating of a condenser is charged positively an equal quantity of positive electricity passes towards the other coating across every intermediate surface, and the charges on the coatings are to be regarded as the charges of the surfaces of the separating dielectric. If any change take place in the charge, a corresponding change takes place in the displacement. Hence when a quantity of electricity is transferred from one coating, A, to the other, B, as when charge or discharge takes place along a wire connecting them, an equal quantity of electricity crosses every section of the dielectric from B towards A. If therefore we regard the process of displacement as an electric current, the dielectric and the wire constitute a closed circuit round which a current passes so long as any change in the electric state of the system is taking place.

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* *El. and Mag.* vol. i. 2nd. ed. pp. 59-67 and 153-156.

The magnitude of the electric displacement is $KF/4\pi$, and the displacement across any element δs of a surface drawn everywhere at right angles to the lines of induction is $KF\delta s/4\pi$. The integral of this expression taken over the surface is the whole quantity of electricity in the form of a charge within the surface.

The ratio $4\pi/K$ of the electric force to the electric displacement Maxwell has called by analogy the *Co-efficient of Electric Elasticity* of the medium. In virtue of the electric elasticity a force opposing the displacement is set up which restores the medium to its former state when the electric force is removed. In a conducting wire this elastic force is continually giving way, and being restored by the displacement continually going on, which therefore constitutes an electric current.

If K vary with the temperature heat must be supplied to prevent the temperature of the dielectric from undergoing change. For if the condenser, charged to surface density σ, have the distance of its plates apart increased by $d\psi$, at constant temperature θ, the work done on the system will be $-(2\pi\sigma^2/K)\,d\psi$. Let heat dH be absorbed, and let another change be performed in the opposite direction, at temperature $\theta - d\theta$, with corresponding evolution of heat. The work done in the double operation will be $\{2\pi\sigma^2 d\psi\,(dK/d\theta)d\theta\}/K^2$. With two opposite adiabatic changes this gives a cycle of changes for which the work has the value just written. But this has also the value $d\psi\, dH\, d\theta/\theta$. Thus we obtain the result

$$dH = \frac{2\pi\sigma^2}{K}\,\frac{\theta}{K}\,\frac{dK}{d\theta}\,.$$

From experiments made by W. Cassie (J. J. Thomson's *Applications of Dynamics to Physics and Chemistry*) it appears that the heat supplied to keep the temperature constant, at about 30° C., was for glass 0·6, for mica 0·12, and for ebonite 0·21, of the work done in the expansion.

II. MEASUREMENTS OF SPECIFIC INDUCTIVE CAPACITY.

15. Relation between specific inductive capacity and index of refraction. All measurements of Specific Inductive Capacity involve in practice a comparison of the capacity of a condenser with air as the dielectric with that of the same condenser with the whole or part of the space between the plates occupied by the substance of which the specific inductive capacity is to be found. For practical purposes the specific inductive capacity of air (which is nearly the same at all ordinarily attainable temperatures and pressures) at 0° and under standard atmospheric pressure (760 mm. of mercury) is usually taken as unity, and it will be convenient at present to follow this custom.

According to the electromagnetic theory of light, the specific inductive capacity of a dielectric should be equal to the square of the index of refraction μ_∞ of the medium for light waves of infinite length. Strictly, $\mu^2_\infty = \kappa \times$ magnetic permeability, or magnetic inductive capacity, of the medium. But there is no transparent dielectric for which the magnetic permeability differs much from that for air, which is here taken as unity. This index is usually calculated from the measured values of the index for known wave lengths by the formula $\mu = A + B/\lambda^2$, where λ is the wave length. It is however to be noted that this is a formula of extrapolation, and that the value which it gives may very frequently be seriously in error. The values of μ_∞ thus calculated are given below in some cases for comparison; in others the value of μ for the line D is given. [See for results 26, 27, 30, 31, 32 below.]

16. Determinations of specific inductive capacity. The first measurements of this kind were made by Cavendish,* by a method the same in principle as that described above, p. 722. He found for glass a mean value of about 8·22, for shellac 4·47, and for wax 4·04. These values later experiments have shown to be too great, no doubt in great measure from the effects of electric absorption.

Faraday's experiments were made by the method and apparatus sketched at pp. 714 and 715 above. Two condensers of the form shown, and as nearly equal as possible, were constructed. The inner surface of each had a diameter of 2·33 inches, and the outer shell of each an internal diameter of 3·57 inches. To test the equality of the condensers the following process was employed. The condensers were set at some little distance apart, so that the inductive influence of one on the other might be neglected, and in positions such that they were as nearly as possible similarly placed with respect to all external conductors, including the observer. The external coatings were then connected once for all to the earth. The interior coating A of one condenser was then charged, while that of the other, B, remained uncharged. The potential of A was then tested by bringing a small carrier ball into contact with the knob and observing the force produced at a given distance on the suspended ball of a torsion balance. To observe the rate of loss of charge the observations were repeated after a short interval, and the result showed only a slight dissipation. The charge of A was then shared with B by bringing A and B symmetrically into contact by their knobs. The potentials of *B* and *A* thus produced were then tested by the carrier ball as before, the charge from *B* being taken by the ball at the instant of contact with *A*. The following are two sets of results. The numbers are degrees of torsion of the glass thread of the balance and may be taken as proportional to the charges.

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* *Elect. Res.* p. 144, *et seq.*

I.		II.	
Centres of Balls in Balance 160° apart.		Centres of Balls in Balance 150° apart.	
A	*B*	*A*	*B*
	0		152
254			148
250			
Charge divided.		Charge divided.	
	122	70	
124			78
Both discharged.		Both discharged	
1			5
	2	0	

Thus, taking the experiment I., the charge divisible between *A* and *B* may be taken as 249. As *B* was found immediately after discharge with 122 it may be taken as having received that amount at least. The other may be taken as having retained 124. These numbers do not differ much from 124·5, the half of the disposable charge. Again, taking experiment II., the disposable charge on *B* may be taken as 143, and the amount of this given to *A* is 70, and the amount retained 73. These numbers are again nearly equal to half the disposable charge 71·5, and the discrepancy is in the opposite direction. Hence the capacities of *A* and *B* may be regarded as very nearly equal.

To make sure that the instrument would plainly show changes of capacity, Faraday put a metallic lining into the lower hemisphere of one of the instruments so as to bring down the distance between the internal ball and the outer coating to ·435 inch. A comparison of the capacities of the condensers made by the same process as before gave 1·08/1 as the ratio in which the capacity of the condenser had been increased. The true ratio was more nearly 1·2/1. But the result showed that a real alteration of capacity of the condenser could be unmistakably recognized in spite of the unavoidable errors of experiment.

Having thus satisfied himself of the sensibility of his apparatus, Faraday introduced a thick hemispherical cup of shellac into the lower hemisphere of one of the equal condensers, and compared the capacities in the manner described above, by first charging one and then sharing the charge with the other and observing the reduced potential immediately after. Each of the apparatus was made in turn the condenser to be first charged. The following are the results of such an experiment:

I.		II.	
A (Shellac).	B (Air).	A (Shellac).	B (Air).
0			0
	304	215	
	297	204	
Charge divided.		Charge divided.	
113			118
	121	118	
Both discharged.		Both discharged.	
0			0
	7	0	

Calling C' the capacity of the shellac condenser, C that of the air condenser, V the potential before and V' the potential after the sharing of the charge, we have by (4) above

$$C' = \frac{V - V'}{V'} C.$$

Hence from experiment I. we get

$$C' = \frac{290 - 113\cdot5}{113\cdot5} C = 1\cdot55C \text{ nearly},$$

and from experiment II.

$$C' = \frac{118}{204 - 118} C = 1\cdot37C \text{ nearly}.$$

The much smaller result in the second case is due to dissipation and absorption in the shellac condensers between the instant at which the reading 204 was obtained and that of the division of the charges. Faraday estimated the corrected result as nearly $1\cdot47C$.

From four experiments made by this method Faraday obtained a mean result of $1\cdot5C$ for the capacity of the shellac condenser. Now plainly, if we regard the direction of the lines of force in the space between the coatings as everywhere radial, that is, neglect the curving down towards the shellac of lines starting from the lower part of the upper hemisphere of the inner ball, we have, denoting by K the specific inductive capacity of shellac relatively to air,

$$\frac{1+K}{1+1} = \frac{C'}{C} = 1\cdot5,$$

or

$$K = 2.$$

In the same way Faraday found for flint glass $K = 1\cdot76$, for sulphur $K = 2\cdot24$, and for spermaceti that K was between $1\cdot3$ and $1\cdot6$. For oil of turpentine and naphtha he obtained results which indicated a higher specific inductive capacity than that of air, though here the results were rendered uncertain by the influence of conduction.

A long series of experiments was also made by Faraday on different gases, and it was found that so far as the means of measurement went all had the same specific inductive capacity, and that this was independent of temperature and pressure.

For further information as to Faraday's experiments the reader is referred to the original memoirs.*

17. Specific inductive capacity of paraffin wax. The specific inductive capacity of paraffin was determined by Messrs. Gibson and Barclay† in 1870, using the platymeter and sliding condenser described above. The paraffin condenser compared is shown in Fig. 245. *aa* is a cylindrical brass vessel 15·5 centimetres deep and 8·61 centimetres in diameter. At the bottom of this cylinder is a layer of paraffin 1 centimetre thick. On this layer rests coaxial with the outer cylinder, a brass tube *bb*, 4·3 centimetres long, 7·2 centimetres in internal diameter, and ·115 centimetre thick. Inside *bb* and coaxial with it is a cylinder *cc*, 13·1 centimetres long and 6·1 centimetres in external diameter. The space between *aa* and *cc* was filled up with paraffin, and from the imbedded tube *bb* an electrode *d* of fine wire was led to the outside.

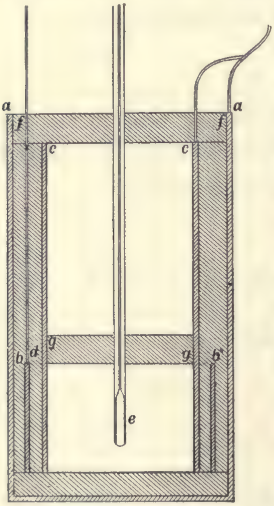

FIG. 245.

The condenser thus formed was placed in an outer vessel containing water of which the temperature was given by a thermometer. A second thermometer fixed in a paraffin plug *ff*, resting on *cc*, gave the temperature of the interior. The paraffin plug *gg* inserted at the level of the top of *bb*, together with *ff*, prevented the passage of heat between the interior of *bb* and the air above the condenser.

The outer vessel *aa*, and the tube *cc* were connected with the earth, and the inner tube *bb* to one side of the platymeter, and balance obtained against the sliding condenser as described above, p. 719. Taking the capacity of the sliding condenser as 1384 times that for each scale division, which it now was in consequence of a small addition which had been made to its value at zero, the mean of a large number of experiments gave for the value of that of the paraffin condenser 1684 times the same unit, or an absolute capacity of 69·552 c.g.s. electrostatic units. These experiments, which were made at different temperatures, showed no alteration of specific inductive capacity with change of temperature. The capacity of the same condenser with the paraffin between the cylindrical plates removed was found in the same way to be 35·394 c.g.s. units, but this was subject to a correction for the cake of paraffin which was left at the bottom to support *bb* and *cc*. The final result was that for paraffin $K=1{\cdot}977$.

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* *Exp. Res.* Series XI. p. 371, *et seq.*

† *Phil. Trans.* 1871, p. 573.

18. Boltzmann's determinations. Some very important determinations of specific inductive capacities have been made by Boltzmann.* In his first series of experiments he determined the value of K for ebonite, paraffin, sulphur, and rosin. The method was a modification of that of Cavendish referred to above. A parallel-plate air condenser, the plates of which were supported on insulated stems carried by sliding pieces movable along a graduated horizontal bar, and so could be placed at different measurable distances apart, had one plate connected to earth while the other plate was charged by means of a battery. Different battery-powers of from 6 to 18 Daniell's cells were used in the experiments. After the condenser had been thus charged, the charge was shared with the insulated quadrants (formerly at potential zero) of a Thomson's electrometer, the capacity of which had been increased by means of a small air condenser.

The potential after the charge was thus shared, and while the condenser was still connected, was observed. A direct application of the battery to the electrometer gave in the same way the previous potential of the condenser.

The addition of the small condenser to the electrometer rendered the united capacities of the electrometer and small condenser nearly the same for all deflections, leaving only an increase of capacity of about 1/5 per cent. for each 100 divisions of deflection from zero. This was to some extent eliminated by a double set of observations, first as just described, then by connecting the condenser for the sharing of the charge, and the battery when applied direct, for so short a time that the charging was over before the needle had appreciably moved. As however the error from this source could hardly be greater than the inevitable inaccuracies in a determination of this kind, we shall here neglect it.

If c be the capacity, assumed constant, of the electrometer and added condenser, C_1 that of the sliding condenser, V_1 the potential before, and V'_1 the potential after the charge was shared, d_1 the distance between the plates, supposed so close that the effect of the edges may be neglected, we have by (4) above

$$C_1=c\,\frac{V_1-V'_1}{V'_1}=\frac{m}{d_1}, \quad\text{.................................}(24)$$

where m is a constant.

In order to make the results depend not on the absolute distance between the plates, but on the much more accurately measurable difference of two distances, a similar set of observations was made, still with air only between the coatings, but with another distance d_2. Calling the capacity C_2, the potentials V_2, V'_2 in this case, we have

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* *Wien. Ber.* 66, 67 (1872, 3).

$$C_2 = c\,\frac{V_2 - V'_2}{V'_2} = \frac{m}{d_2}. \quad\text{.................................}(25)$$

A disk of the substance, the value of K for which was to be found, somewhat larger than the plates of the condenser, was placed in a parallel position between them, so that the induction between the plates took place everywhere across the disk. The same process was followed, and gave potentials V_3, V'_3 for a distance d_3, and a thickness of disk e. Hence if C_3 be the capacity of the condenser,

$$C_3 = c\,\frac{V_3 - V'_3}{V_3} = \frac{m}{d_3 - e + \dfrac{e}{K}}. \quad\text{..........................}(26)$$

Putting $C_1 = 1/\lambda_1$, $C_2 = 1/\lambda_2$, $C_3 = 1/\lambda_3$, we get from equations (24) and (25) $m = (d_1 - d_2)/(\lambda_1 - \lambda_2)$, and hence from (26)

$$\lambda_3 = (\lambda_1 - \lambda_2)(d_3 - e + e/K)/(d_1 - d_2).$$

Hence remembering that $\lambda_1(d_1 - d_2)/(\lambda_1 - \lambda_2) = m\lambda_1 = d_1$, we have finally

$$K = \frac{e}{\dfrac{\lambda_3 - \lambda_1}{\lambda_1 - \lambda_2}(d_1 - d_2) - d_3 + d_1 + e}, \quad\text{.......................}(27)$$

which involves besides e only differences of distances, and the ratio $(\lambda_3 - \lambda_1)/(\lambda_1 - \lambda_2)$, which can be calculated without any knowledge of C from the observation of potential, and for these of course the properly corrected deflections may be taken.

Boltzmann found that no sensible difference in the values of K for ebonite, paraffin, sulphur, and rosin, was produced in the values of K by varying the time of charging or the amount of the charging battery. He also in one set of experiments tried the effect of excluding air from between the disks and the coatings of the condenser, by laying the disks on a mercury surface, and pouring a thin coating of mercury on a portion of the upper surface surrounded by an edging of paper.

The results are given in the following table, in which the main columns I., II., III. give the results of experiments made with different distances between the plates. The first of the two sub-columns in each case gives the result for air between the disk and armatures, the second the result for mercury armatures.

Substance.	Values of K.						
	I.		II.		III.		Mean.
Ebonite - -	3·17	3·07	3·11	3·10	3·20	3·24	3·15
Paraffin - -	2·28	2·30	2·34	2·33	2·31		2·32
Sulphur - -	3·85		3·83				3·84
Rosin - - -	2·57		2·53				2·55

19. Boltzmann's determinations by the force on a dielectric sphere in a known field. Boltzmann also determined the specific inductive capacities of the same substances by comparing the force on a small ball of the dielectric placed in a field of electric force of known intensity with the force on a conducting ball of equal size placed in the same field. This he did by hanging the ball as shown at *s* in Fig. 246, by a double thread from one end of a light rod, itself hung by a bifilar and forming therefore an arrangement akin to a torsion balance. The other end of the rod carried a mirror *M* by which the deflection of the balance could be obtained by means of a telescope and scale. The field was produced by a larger ball which was kept charged by means of a Leyden jar connected to its supporting rod.

Experiments were made for electrifications of the large ball of different durations,—(*a*) for a constant electrification of considerable duration, (*b*) for a comparatively short electrification, (*c*) for a rapidly alternating positive and negative electrification. The electrification (*b*) was obtained by making the charging and discharging contacts by the pendulum of a metronome, the electrification (*c*) by means of a vibrating tuning fork, one prong of which connected *B* alternately to each of two Leyden jars oppositely charged. By the result of p. 739 above, if we put $K_1=1$, and write K for K_2, and r denote the ratio of the force on the dielectric sphere to that on the conducting sphere, we have

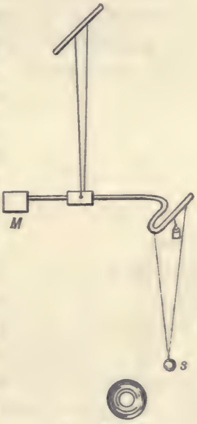

FIG. 246.

$$\frac{K-1}{K+2}=r,$$

or

$$K=\frac{2r+1}{1-r}. \qquad (28)$$

For a sulphur ball, as will be seen from the table below, the force was practically the same for an alternating electrification of about $\frac{1}{200}$ sec. duration as for a long-continued electrification. Hence in this short interval the polarization of the dielectric was fully set up.

The following are some of the results obtained, with the duration of electrification noted. For reference the mean value obtained with the condenser is added.

Substance.	K			Value of K by Condenser.
	$\frac{1}{360}$ sec. to $\frac{1}{64}$ sec.	45 secs.	90 secs.	
Ebonite	3·48	3·74	...	3·15
Paraffin	2·32	8·12	...	2·32
Sulphur	3·90	3·70	...	3·84
Rosin	2·48	5·28	5·61	2·55

The effect of increasing the duration of charge is therefore apparently to increase the specific inductive capacity, but in the cases of sulphur and ebonite to a much smaller extent than for the other two substances.

20. Inductivity in different directions in a crystal. Suspending a ball of crystallized sulphur with different diameters successively in the direction of the force of the field, Boltzmann found that the specific inductive capacity had different values in different directions. For the greatest mean and least axes he found the following values :

	Greatest Axis.	Mean Axis.	Least Axis.
K	4·773	3·970	3·811

Experiments have been made by this method under Boltzmann's direction by Messrs. Romich and Nowak.[*] Results were obtained for (α) permanent electrification, and (β) for electrification reversed 64 times per minute. The values of K are given in the following table :

	K	
	β	α
Glass	7·5	159
Fluorspar	6·7	7·1
Quartz	4·6	>1000
Calc Spar, perp. to axis	7·7	9·9
,, parallel to axis	7·5	8·5
Selenium, freshly melted	10·2	151
Sulphur, mixed with Graphite	4	4·4

The difference between the results for permanent and for short continued electrification seem surprisingly great in some cases.

21. Experiments on mica and on ice. Klemenĉiĉ experimented on the specific inductive capacity of mica, and found it independent of the potential to which the condenser in which the substance formed the dielectric, and practically independent of the duration of charge. K for the specimens used was 6·64.[†] So long as the condenser was kept thoroughly dry, the mica was found to insulate well and give constant results.[‡]

By freezing distilled water in a shallow copper vessel in which was supported on three insulating feet a horizontal plate of copper in contact with the water surface, Professors Ayrton and Perry[§] made a condenser with ice as the dielectric. They then determined the capacity of this condenser and found from its dimensions the specific inductive capacity of ice. At $-13{\cdot}5°$ C. the value of K thus obtained was 22·168. It is of course to be remembered that the insulating power of ice is comparatively slight. Professors Ayrton and Perry found 2240×10^6 ohms for its specific resistance at $-12{\cdot}4°$ C.

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[*] *Wien. Ber.* 70 (1874). See also Wiedemann, *Lehre von der Electricität,* Bd. ii. p. 34.

[†] The value of K for mica is given as 5 in Jenkin's *Electricity and Magnetism*, but it is not stated on what authority.

[‡] *Beiblätter*, vol. xii. No. 1. 1888.

[§] *Phil. Mag.* 1878, p. 43.

22. Five plate balance method. An extended series of experiments on solids was made by Mr. J. E. H. Gordon,* using a form of induction balance, the idea of which is due to Sir William Thomson and Prof. Clerk Maxwell. It is represented diagrammatically in Fig. 247. *A*, *B*, *C*, *D*, *E* are five parallel coaxial disks separated by intervals about an inch wide, of which the three *A*, *C*, *E* are six inches in diameter and the two *B*, *D* four inches in diameter. *A* and *E* are connected by a wire, the middle plate *E* is connected to the needle of a quadrant electrometer, the plates *B*, *D* to the electrodes of the pairs of quadrants. It is evident that, if a difference of potentials between *C* and *A*, *E* be established, it is possible so to place *A*, *B* that the needle will not be affected, and it is also evident that when this position has been attained, the equilibrium will subsist whatever be the difference of potentials. The position of the plate *A* was adjustable by a micrometer screw, and equilibrium was attained by this means. It is to be noted that the effects of the edges of the plates are neglected.

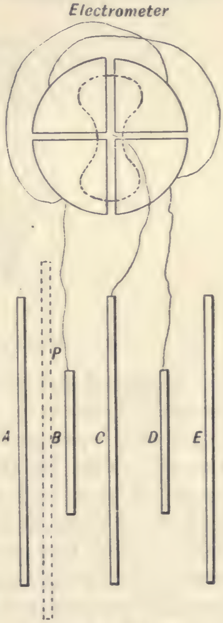

FIG. 247.

The method of proceeding was therefore simply as follows. Having obtained equilibrium with air only between the plates, the experimenter introduced a plate *P* of the dielectric to be experimented on, and measured by means of *S* the distance through which *A* had to be displaced in order to restore equilibrium. This distance gave the thickness of a plate of air, equivalent to the plate *P* of the dielectric. The ratio of this thickness to the thickness of *P* is the specific inductive capacity of the material.

In the experiments the plates *A*, *B*, and *C* were connected to the terminals of an induction coil, the primary circuit of which was broken as many as 12,000 times a second by an interrupter arranged for the purpose. Thus the potential was rendered alternately positive and negative 12,000 times a second, and all effects of absorption were obviated.

We do not give here the results obtained by Gordon, as it became clear later that with the sizes of plates and distances apart used by him, the five-plate balance could not give accurate results. Under proper conditions, however, the method may be useful.

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* *Phil. Trans.* 1879, p. 417.

23. Hopkinson's experiments on glass. The values of K for glass obtained by Mr. Gordon were not in agreement with some previously obtained by Dr. John Hopkinson,* who experimented according to the method of comparison of capacities described above, p. 722. The capacity of a guard-ring condenser was compared with that of a sliding condenser (the identical instrument used in Gibson and Barclay's experiments described above) (1) when air only was the dielectric, (2) when a plate of glass was introduced between the plates. The guard-ring condenser is shown in Fig. 248, half in section, half in elevation. k is the protected disk 15 centimetres in diameter with a gap 1 millimetre in breadth between it and the guard-ring, ee the opposite plate, h the guard-ring bearing a brass cylinder box (not shown in the drawing) which forms a shield for the back of the protected disk. The guard-ring is insulated on a stiff frame of iron formed by two triangular pieces of iron ab, cd connected by three wrought-iron stays. The insulators are three ebonite legs gg, which are screwed to the tops of the stays. The attracting disk is carried on a screwed stem of 1/25 inch step, and can be raised or lowered without rotation by a nut f divided as a micrometer. Fig. 249 is a plan of the instrument with the brass backing removed. It shows the protected disk and its supports, which are two bars ll, ll of vulcanite attached to the back of the disk and resting on the upper surface of the guard-ring.

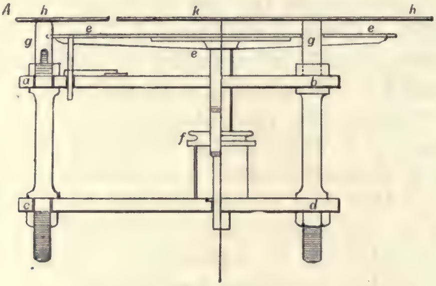

FIG. 248.

NOTE.—The protecting cylindrical box on the guard-ring is here omitted.

This instrument served also to measure the thickness of the glass plates used in the experiments. The screw f was turned until the brass plates were in contact, and the micrometer reading taken; then the glass plate was placed above ee, which was screwed up until the plate came into contact with h, k, h. Slips of tissue paper were interposed between the ebonite legs gg and the plate hh, and the contact was judged by these slips becoming loose. A reading of the screw micrometer was taken for each slip, and the mean of the three taken as the reading of contact. A correction was determined for the effect of bending of the plates and compression of the slips before their release.

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* *Trans. R.S.* 1878, p. 17.

A special switch supported above the guard-ring condenser enabled the connections to be made in the required order. A battery, in some cases of 48, in others of 72 small Daniell's cells, had its middle point connected to earth, one of its poles to *hkh*, and the other to the inner coating of the sliding condenser, while the outer coating and the plate *ee* were connected to the electrometer case. Thus the inner plates of the two condensers were charged to equal and opposite potentials. Then one pair of quadrants of a Thomson's electrometer, both pairs of quadrants of which were connected to earth, were insulated, the guard-ring was connected to earth, and the protected plate and the insulated plate of the sliding condenser connected together and to the insulated quadrants of the electrometer. The direction of the electrometer deflection, if any, at the instant of the combination of the charges, was observed. If no deflection took place the guard-ring condenser and the sliding condenser had equal capacities, and the latter was adjusted until this was the case.

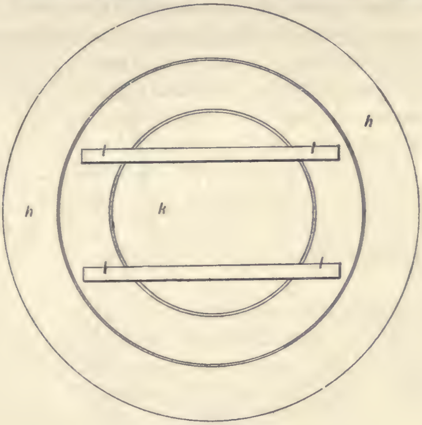

FIG. 249.

The following are mean results of two or more experiments for each substance :

	Density.	*K*
Glass, Light Flint - - -	3·2	6·85
,, Double-extra Dense -	4·5	10·1
,, Dense Flint - - -	3·66	7·4
,, Very Light Flint - -	2·87	6·57

The plates of glass were in most cases in contact with both plates of the condenser.

24. Further results of Hopkinson. Hopkinson continued his investigations, and besides coming to the conclusion stated above as to the five-plate balance, arrived at the important results :

1. That the specific inductive capacity of glass is the same for discharges lasting $\frac{1}{10000}$ second, $\frac{1}{20000}$ second, or $\frac{1}{4}$ second.

2. That it is independent of the potential to which the condenser is charged.

At the same time he extended his former results, and applied his method of experimenting to the investigation of the specific inductive capacity of liquids. A flask of flint glass, with thin walls and a long thick neck, was filled up to the junction of the neck, with strong sulphuric acid. A wire passing down through the neck connected the acid with a metal piece *A* (Fig. 250), supported on an insulating stand of ebonite. On this metal piece rested the horizontal arm of a kind of bell-crank (or *L*-shaped piece of metal pivoted at the angle). The flask was first charged by means of a battery and the potential measured by a quadrant electrometer which was then detached and discharged. Then a previously deflected metallic pendulum, *D*, connected to earth through its supports, was released, and striking the vertical arm of the lever, connected the flask for an instant to earth and discharged it. The electrometer was then applied to detect any residual charge. The leakage method described above [XI. 44] was used to measure the duration of discharge. A paraffin condenser of known capacity had its plates connected for the time of discharge to be measured, first by a resistance of 256 ohms, then by a resistance of 512 ohms, and the remaining potential in each case was observed. These operations obviously gave data for the calculation of the time interval *t*. With

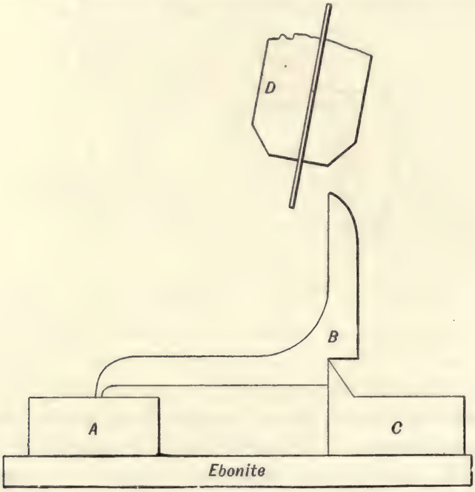

FIG. 250.

a duration of discharge of about $\frac{1}{2000}$ second, less than 3 per cent. of the original charge given by a battery of 20 elements remained. Longer and shorter times of discharge gave similar results. The practical result of all the experiments was that determinations of specific inductive capacity by observations of discharge may be taken as correct for glass if the period of discharge be anything between $\frac{1}{20000}$ sec. and $\frac{1}{2}$ sec.

25. Hopkinson's experiments on glass plates. The method adopted for determining the specific inductive capacity of glass plates was practically the same as that already described at p. 754. The guard-ring and protected disk were first connected to one pole of a well insulated battery of 1,000 chloride of silver cells, the other pole of which was connected to the insulated plate of a cylindrical sliding condenser. Thus the two condensers were charged to equal and opposite potentials. By means of a special commutator changes of connections similar to those described above were made so as to combine the charges of the condensers, with the addition that the electrometer quadrants connected to the condensers after combination were immediately after insulated to avoid effects of residual charge. The capacity of the sliding condenser was adjusted till no electrometer deflection was produced.

The glass plate was then placed between the plates of the guard-ring condenser and the operations repeated until equilibrium was again obtained. The two results gave the ratio of the capacities, and from the distance between the plates of the condenser and the thickness of the glass plate the value of K was found.

The capacity of the glass flask described above was determined in a similar way by aid of the sliding condenser, with a charging battery varying from 10 to 1,800 chloride of silver cells, with only a little over $\frac{1}{4}$ per cent. of alteration.

The values of K are given in the following table with the thicknesses of the plates, and for comparison the earlier results obtained by the same experimenter :

Substance.	Density.	Thickness of Plate in mms.	K	Value of K formerly obtained.
Glass, Double-extra Dense Flint	4·5	4·5	9·896	10·1
,, Dense Flint - - -	3·66	16·57	7·376	7·4
,, Light Flint - - -	3·2	15·04	6·72	6·83
,, ,, ,, - - -	—	10·75	6·89	6·85
,, Very Light Flint - -	2·87	12·70	6·61	6·57
,, Hard Crown - - -	2·485	14·62	6·96	—
,, Plate - - - -	—	6·52	8·45	—
Paraffin - - - - -	—	20·19	2·29	—

26. Hopkinson's experiments on liquids. Hopkinson obtained results also for liquids by the method just described.* The space between two coaxial metal cylinders was filled with the liquid to be experimented on. These two cylinders connected together formed one coating of a condenser of which the liquid formed the dielectric, and the other coating was given by a cylinder suspended from an ebonite plate above, and immersed in the liquid. The latter plate was charged and the other connected to earth, and the capacity compared with that of the oppositely charged sliding condenser. The capacity of the same apparatus with air as the dielectric had previously been obtained in the same way, and the results gave at once the value of K for the liquid. The following table gives some of the results. The column headed μ^2_∞ contains for the purpose of comparison the square of the index of refraction of the liquid for light of infinite wave length. This was calculated from the formula $\mu_\infty = A + B/\lambda^2$ from observations of the index of refraction which were made on each of the substances for the Fraunhofer rays C, D, F, G, of the spectrum.

Name of Liquid.	K	μ^2_∞
Petroleum Spirit - - - - -	1·922	1·92
Petroleum Oil, Field's - - - -	2·07	2·075
,, ,, Common - - - -	2·10	2·078
Ozokerite - - - - - -	2·13	2·086
Turpentine, Commercial - - - -	2·23	2·128
Castor Oil - - - - - -	4·78	2·153
Sperm Oil - - - - - -	3·02	2·135
Olive Oil - - - - - -	3·16	2·131
Neat's Foot Oil - - - - -	3·07	2·125

The closeness of the agreement between the numbers for K and for μ^2_∞ for the mineral oils and for turpentine is very remarkable. The divergence in the other cases is to be expected, as from the composition of the substances it is probable that the results included effects of electrolytic action.

27. Experiments of Silow. Results with which Hopkinson's agree very well had been previously obtained for turpentine, benzene, and petroleum by Silow.† Two series of experiments were made. In the first a very ingenious and simple method was employed. A kind of quadrant electrometer was constructed by pasting on the inside of a cylindrical glass vessel, 10 centimetres deep and 15 centimetres in diameter, four symmetrically placed vertical strips of tinfoil each 10 centimetres broad, and joining the opposite pieces together by strips across the bottom. Within was hung a platinum needle of the shape of an inverted

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* *Phil. Trans. loc. cit.*

† *Pogg. Ann.* 156 (1875), p. 389, and Wiedemann, *Die Lehre von der Elektricität*, Bd. ii. p. 45.

T, in which the vertical pieces at the ends of the horizontal cross-piece were semi-cylinders of platinum. The needle was left uncharged, and one of the pairs of strips was connected to earth and the other charged to a convenient potential. The deflections of the needle for the same difference of potential (1) with the vessel filled with air, (2) with the liquid under experiment, were observed, and it was assumed that the angles of deflection were proportional to the specific inductive capacities in the two cases. This would have been strictly true of the angles through which a torsion head at the top of the suspension thread would have had to be turned if the needle had been brought back in both cases to a position of equilibrium after deflection.

For two kinds of turpentine, I., II., and for petroleum he obtained :

	K	μ^2_∞
Turpentine I., mean of three experiments	2·173	2·129
Turpentine II. - - - - - -	2·221	
Petroleum - - - - - -	2·037	2·148

A second set of experiments was made by Silow by a method similar to that described above, p. 730. A condenser formed of two gilded circular plates kept $1\frac{1}{2}$ mm apart by small pieces of ebonite, and enclosed within a glass vessel covered on its interior surface with tinfoil, had one of its plates alternately connected to earth and to one pole of a water battery of 175 zinc-copper elements. The connections were made by a rotating commutator kept running at a constant speed sufficiently great to give a constant deflection of the needle of a galvanometer placed in the charging or discharging circuit. Three deflections were taken (1) with the vessel filled with air, (2) with the liquid under experiment in the vessel and therefore between the plates, (3) with only the joining wires attached. Denoting by α, β, γ, these deflections corrected so as to be proportional to the currents, we have for the ratio of the capacity of the apparatus with the liquid between the plates, to its capacity with air between the plates $(\beta-\gamma)/(\alpha-\gamma)$, that is for the liquid,

$$K=\frac{\beta-\gamma}{\alpha-\gamma}. \qquad \text{.......................................(29)}$$

Different battery powers applied gave the same values for K. The following are the mean values of K for the substances mentioned, with the values of μ^2_∞ for comparison.

Substance.	K	μ^2_∞
Turpentine - - - - - - -	2·153	2·134
Benzene - - - - - - - -	2·198	2·196
Petroleum, first specimen - - - -	2·071	2·048
Petroleum, second specimen - - -	2·037	2·048

28. Experiments of Quincke. Some interesting experiments on the specific inductive capacity of liquids have also been made by Quincke.* According to the theory of Faraday and Maxwell, referred to at p. 743 above, there is, at every point of the electric medium, a tension along the lines of force, and an equal pressure at right angles to that direction, the amount of which reckoned in units of force per unit of area is $KF^2/8\pi$, where F is the resultant electric force at the point. Quincke's method amounted to measuring not only the tension, but the pressure also, in different liquid dielectrics, and his results, besides giving (1) from the observed tension, (2) from the pressure, values of K which he compared with those obtained by the ordinary condenser method, are interesting in their bearing on electrical theory.

His apparatus for the measurement of the tension consisted of two horizontal circular plates placed a short distance apart in a glass vessel. The upper plate was suspended from one end of the beam of a balance, and was connected to earth. The lower plate was charged by means of a battery of Leyden jars, the outer coatings of which were to earth. The potential was observed in arbitrary units by means of a Thomson's standard electrometer (see p. 693 above). The attraction of the upper plate towards the lower was then measured by weights put on the other scale of the balance. The mean pull per unit of area was therefore obtained.

Now, from what has been proved above (p. 732) it follows that the force f, per unit of area, on any part of the upper plate not near the edge, is $2\pi\sigma^2/K$, and we have $\sigma = -KF/4\pi = -KV/4\pi d$ if V be the difference of potentials, d the distance between the plates. Hence

$$f = \frac{KV^2}{8\pi d^2}. \qquad \text{.......................................(30)}$$

The weighing therefore gave, taking the mean pull as nearly enough equal to f, directly the tension.

By comparison of results for two different media using the same value of V for both cases, the ratio of the values of K could be at once obtained. Thus if f_1, f_2, be the tensions, and the corresponding specific inductive capacities determined in this manner be denoted by K_{f_1}, K_{f_2}, we have

$$\frac{K_{f_1}}{K_{f_2}} = \frac{f_1}{f_2}. \qquad \text{.......................................(31)}$$

The pressure at right angles to the lines of force was found in an ingenious manner. The upper disk of the apparatus just described was removed and replaced by a plate of the same diameter with a short vertical tube at its centre, by means of which communication could be obtained with the space between the plates. Attached to this vertical tube was an india-rubber bag which could be cut off by means of a stopcock. A branch tube communicated with an ordinary open U manometer containing bisulphide of carbon. Enough of air was blown by the rubber bag into the space between the plates to form a flat bubble of from 2 to 5 centimetres in horizontal diameter, bounded by the plates above and below. The stopcock was closed and the pressure was read off on the manometer. The lower plate was now charged to the same potential as before while the upper plate was connected to earth. The increase of pressure was read off from the manometer, and gave the difference of pressures in the air and the liquid due to the electrification.

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* *Wied. Ann.* 19 (1883).

If h be the difference of heights of the liquid produced by the electrification, and ρ the density of the liquid, we have, denoting the value of K determined in this way by K_p, and the acceleration due to gravity by g,

$$gh\rho = \frac{K_p - 1}{8\pi} \frac{V^2}{d^2}, \qquad \text{.............................(32)}$$

if K be taken $= 1$ for air.

Using the value of f given in (30) for the same medium, this gives

$$K_p = \frac{gh\rho}{f} K_f + 1. \qquad \text{.............................(33)}$$

The following are some of the results obtained :

Substance.	Density at Temp. stated.		Temp. of Exp.	Value of K Sp. Ind. Cap.		
				By Condenser K	By Tension K_f	By Pressure K_p
		°C.	°C.			
Sulphuric Ether	·7205	14·9	6·60	3·364	4·851	4·672
Bisulphide of Carbon I.	1·2760	12·20	7·50	2·217	2·669	2·743
„ „ „ II	1·2796	10·20	12·98	1·970	2·692	2·752
Benzene (from Coal Tar)	·8825	15·91	13·20	1·928	2·389	2·370
„ (from Benzoic Acid)	·8822	17·64	14·40	2·050	2·325	2·375
Light Benzene	·7994	17·20	11·60	1·775	2·155	2·172
Colza Oil	·9159	16·40	16·41	2·443	2·385	3·296
Turpentine	·8645	17·10	16·71	1·940	2·259	2·356
Petroleum	·8028	17·00	16·62	1·705	2·138	2·149
Ether 5 vols. + 1 vol. Bisulph. of Carbon	·8134	16·40	8·50	2·871	4·136	4·392
Ether 1 vol. + 1 vol. Bisulph. of Carbon	·9966	16·60	10·50	2·458	3·539	3·392
Ether 1 vol. + 3 vols. Bisulph. of Carbon	1·1360	17·40	5·30	2·396	3·132	3·061

The values of K obtained by tension and pressure here seem uniformly greater than those obtained by the condenser method, which must be regarded of course as the true values. But they agree very well with one another, and go far to prove the equality of the pressure and tension.

29. Correction of Quincke's results for connections. Electric stress and strain. It was pointed out by Hopkinson that* perhaps the capacity of the key and connecting wires might be appreciable, and that if so the values of K given for the condenser method in the above table would be increased by the correction. This was found by Professor Quincke to be the case, and the following corrected results obtained by him are given by him in a note to Dr. Hopkinson's paper :

	Values of Sp. Ind. Cap.	
	By Condenser K.	By Tension K_f.
Sulphuric Ether - - -	4·211	4·394
Bisulphide of Carbon - -	2·508	2·623
,, ,, - - -	2·640	2·541
Benzene - - - - -	2·359	2·360
Petroleum - - - -	2·025	2·073

This shows that for these substances K, K_f, K_p are sensibly equal. Further the experiments seem to confirm fairly well the theoretical values $KF^2/8\pi$ for the pressure within the medium.

The question of stress and strain in the dielectric medium is yet far from having been fully investigated. It is certain that opposite charges of electricity, on the opposite plates of a condenser, for example, are the surface aspects of a state of strain in the medium. But the nature of this strain cannot be said to be yet known. Experiments which have been made by Quincke and others show that in glass, and in most liquids, except certain oils, the electric strain results in a uniform dilatation ; whereas in elastic strain, consisting of elongation in one direction, and an equal shortening in every direction at right angles to that, would result in a negative dilatation, which is contrary to the observed facts. Liquids would not support the shear which in electric strain seems to be operative. Hence the electric strain appears to be distinct from elastic strain.

30. Hopkinson's experiments on liquids. Hopkinson, at a later date,† made experiments on the specific inductive capacity of a number of oils and other liquids. The method adopted was a modification of the five-plate balance method described above. The arrangement of apparatus is shown in Fig. 251. Two air condensers E, F, of determinate and nearly equal capacity, and two adjustable sliding condensers I, J, were joined as shown like the four branches of a Wheatstone bridge. The inner coatings of E, I were joined to one pair of quadrants of an electrometer, and those of F, J to the other pair of quadrants.

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* *Proc. R.S.* vol. xli. 1886.

† *Proc. R.S.* Oct. 1887.

To the inner coating of *J* could be attached the inner plate of a liquid condenser containing the substance to be experimented on. The outer coatings of *E*, *F* were connected to the case of the electrometer and to one terminal of an induction coil; the outer coatings of *I*, *J*

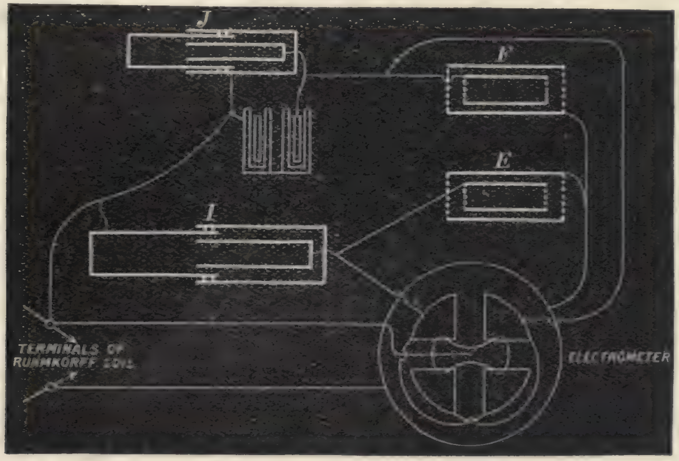

Fig. 251.

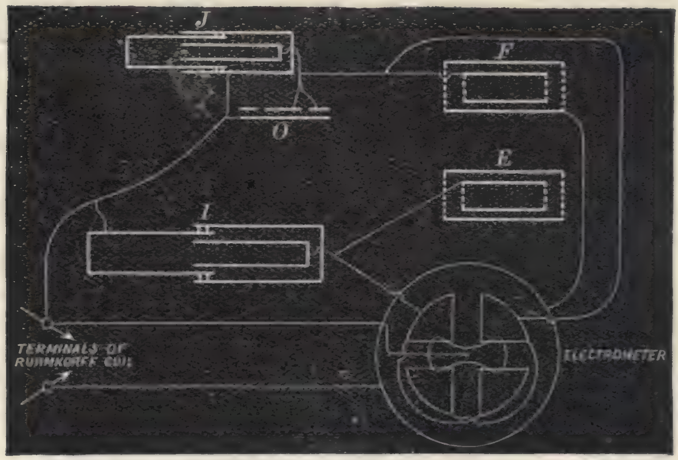

Fig. 252.

were connected to the needle of the electrometer and to the other terminal of the induction coil.

In order that there might be no deflection of the electrometer needle it was necessary that the capacities of *E* and *I* should be in the same ratio as those of F and J respectively. An adjustment of one or both of the sliding condensers was made until this relation was fulfilled in each of four cases, (1) when no fluid condenser was introduced, (2) when the condenser without the interior plate, but fitted with a "dummy" to represent the necessary supports or connections outside the liquid, was connected to J, (3) when the complete condenser charged with air was added to J, (4) when the complete condenser charged with liquid was connected to J. Assuming for simplicity the sliding condenser I to remain unaltered and x, y, z, z_1 to be the respective readings of J in the four cases, we must have

$$\frac{\text{capacity of condenser with liquid}}{\text{capacity of same condenser with air}}=K=\frac{x-z_1-(x-y)}{x-z-(x-y)}$$
$$=\frac{y-z_1}{y-z}. \quad \text{................}(34)$$

The following is an abstract of the results obtained:

	K.	$\mu^2{}_D$ for line D.
Colza Oil, six samples - - - -	3·07 to 3·14	
,, ,, another sample* - - -	3·23	
Arachide - - - - - -	3·17	
Sesame - - - - - - -	3·17	
Linseed Oil, raw - - - - -	3·37	
Castor Oil - - - - - -	4·82	
,, ,, another sample - - -	4·84	
Ether - - - - - - -	4·75	
Carbon Bisulphide - - - -	2·67	
Amylene - - - - - -	2·05	1·9044

It is to be noted with respect to colza oil that, as given by Quincke (p. 761 above), the value of K_p is 3·296 and of K_f 2·385.

31. Hopkinson's experiments on the benzene series. Dr. Hopkinson also experimented with the following liquids of the benzene series, for which also he determined the index of refraction μ_D for the line D of the spectrum.

	K.	$\mu^2{}_D$.
Benzene - - - -	2·38	2·2614
Toluene - - - - -	2·42	2·2470
Xylene - - - - -	2·39	2·2238
Cymene - - - - -	2·25	2·2254

* Doubtful as to purity.

The same method, but with a guard-ring condenser instead of the fluid condenser as shown in Fig. 252, was applied to the measurement of the specific inductive capacity of solids. The connections shown in Fig. 252 were first made, that is the guard-ring and protected disk both connected to the inner coating of J. The arrangement was then adjusted to balance, then the guard-ring remaining connected to J, the protected disk was transferred to I and balance again obtained. The difference of the readings of the sliding condenser gave on an arbitrary scale the capacity of the guard-ring condenser for the given distance of the plates apart. These operations were then repeated with a plate of the substance for which K was to be found placed between the plates of the guard-ring condenser.

Only three substances were experimented on, with the following results. The previously obtained values (p. 757 above) are given for the first two for comparison.

	K.	Previously found value of K.
Flint Glass, double-extra dense - -	9·5	9·896
Paraffin Wax - - - - - -	2·31	2·29

Rock salt was the third substance with a result of 18 for K, but the sample was very rough and too small, and possibly conducted so greatly as to affect the result. In these experiments the effect of the connecting wire of the guard-ring condenser was not allowed for.

32. Experiments on hydrocarbon series. Negreano* has applied the five-plate balance method to the determination of the specific inductive capacity of a number of hydrocarbons of homologous chemical composition. The balance was arranged with its plates horizontal and well insulated on ebonite rods; the diameter of the larger plates was 16 centimetres, of the smaller 12 centimetres, and the distance of adjacent plates apart 1 centimetre. The liquid experimented on was placed on a flat shallow dish attached to the ebonite supports between the uppermost plate and that next to it. Balance was obtained (1) with the instrument used simply as an air condenser, (2) with the empty dish in position, (3) with the liquid in the dish. The corresponding positions of the movable plate were obtained by a micrometer. Another micrometer measured the thickness of the stratum of liquid. The index of refraction μ_D was also determined for the D line in the case of each liquid.

It was found that the value of K increases as the composition of the substance becomes more complicated, and that the value of

$$(K-1)/(K+2)\rho,$$

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* *Comptes rendus*, tome civ. 1887.

where ρ is the density, is approximately constant. The following is a synopsis of the results :

	Temp.	Density.	K.	μ_D.
Benzene, C_6H_6, with thiophene -	26°	·8803	2·3206	1·4974
,, ,, ,, ,, another specimen -	25	·8756	2·2988	1·4978
,, ,, pure - - -	14	·8853	2·2921	1·5062
Toluene, C_7H_8 - - - -	27	·8608	2·242	1·4912
,, ,, - - - -	14	·8711	2·3013	1·4984
Xylene, C_8H_{10} - - - -	27	·8554	2·2679	1·4897
Metaxylene, C_8H_{10} - - -	12	·8072	2·3781	1·4977
Pseudocumene, C_9H_{12} - - -	14	·857	2·4310	1·4837
Cymene, $C_{10}H_{14}$ - - - -	19	·851	2·4706	1·4837
Terebenthene, $C_{10}H_{16}$ - - -	20	·875	2·2618	1·4726

It will be noticed that the value of $\sqrt{K}$ is only a little greater than μ_D in each case, and that $(K-1)/(K+2)\rho$ has the value ·34 approximately in the first six cases and the last, and is slightly greater in the remaining three.

33. Experiments of Cohn and Arons. Discussion by Quincke of results for liquids. Experiments on liquids have also been made by E. Cohn and L. Arons. Two quadrant electrometers were employed, one with air filling the quadrants, the other specially designed to contain the liquid experimented on as in Silow's method described above, p. 758. One pair of quadrants of each electrometer was connected to one terminal of a Helmholtz induction coil, the other pair of quadrants, the needle and the case were connected to earth and to the other terminal of the coil. Denoting by δ_1, δ_2 the (corrected) deflections on the ordinary and special electrometers respectively when both are filled with air, δ'_1, δ'_2 the corresponding deflections when the special electrometer contains the liquid, we get easily by (29) above

$$K=\frac{\delta_2\delta'_1}{\delta_1\delta'_2}. \quad \text{.................................}(34')$$

The following results were obtained :

	K.
Distilled Water - - - -	76
Ethyl Alcohol - - - -	26·5
Amyl Alcohol - - - - -	15
Petroleum - - - - - -	2·04
Xylene, two kinds - - -	2·39 2·36

The numbers here given it will be observed are high in the first three cases. These substances have however considerable conductivity, which would tend of course to give an apparently high specific inductive capacity. The authors believe that the results are correct within 5 per cent.

Prof. Quincke* re-examined the question of the values of K for liquids obtained by the different methods, as described above. All liquids experimented on except colza oil gave practically the same result whatever the method employed. For that substance however the result stated above held, that is the pressure method gave the highest value, the electrical balance the lowest, and the condenser method a mean value; and this anomaly was found to hold good for different kinds of colza. That it could not be due to electrolytic action was clear from the fact that the products of decomposition at the condenser plates could not alter the pressure at the surface of the bubble.

Prof. Quincke† also measured the index of refraction of pure ether for ultra-red rays by passing them through the medium and receiving them upon a thermopile. He found that for pure ether $K = 4{\cdot}3$, and that for ultra-red rays its index of refraction is less than 2. The substance seems therefore not to conform to Maxwell's relation.

34. Experiments on gases. Determinations of the specific inductive capacity of gases were made by Boltzmann‡ and by Professors Ayrton and Perry.§ Boltzmann's method was as follows: A condenser consisting of two horizontal circular plates was supported within a closed metallic vessel, through the walls of which passed wires to make connection with the plates, and which could be connected with an air-pump or a gas generating apparatus. Two metallic plates were placed above and two below the condenser to preserve it at a uniform temperature. The vessel was exhausted, then one plate A of the condenser was charged by being connected to one terminal of a battery of 300 Daniell's cells, while the other plate B and the other terminal of the battery were connected to earth. B was then disconnected from earth and connected to the insulated electrode of an electrometer which had been previously brought to zero potential. The electrometer showed no deflection, proving that there was no leakage. The charge on A therefore remaining constant, it was found in accordance with theory that the admission of air altered only the specific inductive capacity between the plates, and therefore the potential of A, but not the potential of B which remained zero. After the admission of air the potential of A was restored to its original value, and the change of potential of B read off on the electrometer. The number of cells was then increased by one, and the increased potential of B again read off. The ratio of the specific inductive capacities could now be calculated.

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* *Wied. Ann.* 33, 1888.

† *Wied. Ann.* 32. No. 12. 1887.

‡ *Wien. Ber.* 69 (1874); *Pogg. Ann.* 15 (1875).

§ *Trans. Asiatic Society of Japan* (1877).

If V_1, V_2 be the potentials of A before and after the admission of air, and K_1, K_2 the corresponding specific inductive capacities, we have $V_2/V_1 = K_1/K_2$. Hence by the restoration of the potential to V_1 the potential of B was increased by an amount proportional to $V_1 - V_2$, that is by an amount $m(1 - K_1/K_2)$, where m is a constant. By the increase of the number of cells from n to $n+1$ the increase of the potential of B was therefore $mV_1(n+1)/n$. Hence calling these changes as measured by the electrometer δ, δ', we have $\delta/\delta' = n(1 - K_1/K_2)(n+1)$, or

$$K_2 = \frac{n\delta'}{n\delta' - (n+1)\delta} K_1. \quad \text{..............................(35)}$$

35. Effect of pressure on specific inductive capacity of gases. It was found by Boltzmann that the alteration of capacity was very nearly in simple proportion to the alteration of pressure of the air, and he assumed that the effect of alteration of temperature was only that corresponding to the consequent alteration of density. Hence if we denote by K the specific inductive capacity of air under pressure equal to that due to p millimetres of mercury under standard circumstances, suppose that for absolute vacuum to be unity, and assume the proportionality to hold for all pressures, we may write

$$K = 1 + \frac{kp}{760}, \quad \text{.......................................(36)}$$

where $1 + k$ is the specific inductive capacity of air at standard atmospheric pressure.

By (35) and (36), putting p_1, p_2 for the pressures corresponding to K_1, K_2, we get

$$k = 760 \frac{n\delta' + (n+1)\delta}{n\delta'(p_2 - p_1) - (n+1)p_2\delta}. \quad \text{.......................(37)}$$

Boltzmann found similar results to hold for other gases than air, and gave the following values for K at standard atmospheric pressure. The value of $\sqrt{K}$ is given also for comparison with the index of refraction.

Gas.	K.	$\sqrt{K}$.	μ.
Air	1·000590	1·000295	1·000294
Carbonic Acid	1·000946	1·000473	1·000449
Hydrogen	1·000264	1·000132	1·000138
Carbonic Oxide	1·000690	1·000345	1·000340
Nitrous Oxide	1·000994	1·000497	1·000503
Olefiant Gas	1·001312	1·000656	1·000678
Marsh Gas	1·000944	1·000472	1·000443

36. Experiments of Ayrton and Perry. In Ayrton and Perry's method the capacities of two condensers were compared with different gases at different pressures between the plates of one of them, while the other had continually air at ordinary temperature and pressure for its dielectric. The latter condenser consisted of a square horizontal uninsulated plate of tin-foil of 1815 square centimetres area, cemented to the upper surface of a plate of hard wood which rested on the horizontal top of a block of stone, and an insulated upper plate of the same size supported on ebonite levelling screws, the lower ends of which rested on the stone. The other condenser was contained within an air-tight rectangular vessel of sheet brass, and consisted of eleven parallel plane plates, each 324 square centimetres in area, kept at equal distances of three millimetres apart in racks of ebonite. The first, third, etc., and last plates, reckoning from one side, were connected to the case, the other plates were insulated and connected to a platinum wire passing out through a glass tube 35½ centimetres long to the outside of the case. This glass tube, which had been chemically cleaned and covered with paraffin, to prevent leakage over the surface, was very carefully cemented into a brass socket attached to the metallic case, and was nowhere in contact with the platinum wire except at the outer end, where it was drawn to a point and hermetically sealed. Cement contained in a metal cap surrounding the junction of the tube and socket prevented leakage there, and a second cap filled with cement surrounded the point of the tube, and guarded the point from being broken by motion of the wire. By means of another tube the case could be filled with the gas to be experimented on, or connected to a Sprengel or other pump by which the required degree of exhaustion was produced. This tube was made of special form to prevent mercury from the Sprengel pump from passing by any accident into the condenser case.

The method of making a determination was as follows. The insulated plates of the condenser were charged to equal and opposite potentials in the following manner :—The battery of 87 Daniell's cells had its poles joined by a resistance of 10,000 ohms, and by means of a reversing key one terminal a of this coil was connected to the insulated plate of one condenser, while the other terminal b was connected to earth; then b was connected by the reversing key to the insulated plate of the other condenser and a to earth.

The battery was then removed and the charged plates connected together, and with the insulated electrode of a quadrant electrometer of which the other electrode and case were to earth, and the reading taken.

If the potential of each condenser was numerically V, the capacity of the constant air condenser C_1, and the capacity of the other C_2, the charge left after the two condensers were connected was $V(C_1 - C_2)$, supposing the constant condenser to have been positively charged. The corrected deflection a shown by the electrometer was therefore $mV(C_1 - C_2)/(C_1 + C_2)$, where m is a constant.

To eliminate m and V the terminals of the battery were kept joined by the resistance of 10,000 ohms, and one terminal was connected to earth, while a point on the resistance was connected to the insulated quadrants of the electrometer now detached from the condensers. The difference of potentials of the battery between the extremities of the resistance was $2V$, and if the resistance intercepted between the terminals of the electrometer be denoted by R, the difference of potentials shown by the corrected deflection β of the electrometer was $2VR/10000$. We have therefore $\beta = 2mVR/10000$. Hence

$$\frac{\alpha}{\beta} = \frac{10000}{R}\frac{C_1 - C_2}{C_1 + C_2} = \frac{10000}{R}\frac{1 - \frac{C_2}{C_1}}{1 + \frac{C_2}{C_1}}. \qquad (38)$$

This enabled the ratio C_2/C_1 of the capacities to be calculated. Another experiment made with C_2 changed by alteration of the medium, gave at once the ratio of the two values of C_2, that is of the specific inductive capacities in the two cases.

The following table gives the mean results for many experiments in different gases at standard pressure: taking the value of K for air as unity.

Dielectric.	K.
Vacuum - - - -	·9985
Air - - - - -	1·0000
Carbonic Acid - - -	1·0008
Hydrogen - - - -	·9998
Coal Gas - - - -	1·0004
Sulphurous Acid - - -	1·0037

It was observed that when air was allowed to mix with the carbonic acid the value of K more and more nearly approached unity.

Experiments on the specific inductive capacity of a high Sprengel vacuum were undertaken by a Committee of the British Association consisting of Professors Ayrton and Perry, Prof. O. J. Lodge, and Mr. J. E. H. Gordon. A preliminary report was presented * containing a plan of experimenting and some results which seem to show that at a pressure of about $1/10^6$ of an atmosphere the specific inductive capacity is ·6 or ·8 per cent. less than that for ordinary air.

37. Experiments at low temperatures. The effect of very low temperatures on the specific inductive capacity of various substances has been investigated by Dewar and Fleming. The temperature varied from $-200°$ C. upwards through a considerable range. The condenser used consisted of two coaxial cones, with a space of 3 mm thickness between them, which was filled with the dielectric to be experimented on, and the cooling was produced with liquid air. The condenser thus formed was charged to a convenient difference of potentials, and discharged. The galvanometer deflection at charging or discharging was observed. The dielectric was then melted out and the space filled with air (gaseous) at the same temperature as before. The charging and discharging were performed as before and the galvanometer deflections again observed. A comparison of the results gave the specific inductive capacity of the dielectric used.

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* *Brit. Assoc. Rep.* 1880.

An interesting result was that for pure ice, which rose from 2·43 at $-200°$ C. to about 71 at $-7°{\cdot}5$ C. At the higher temperature the ice had considerable conduction which made the specific inductive capacity difficult to estimate exactly. For a good conductor this constant should be practically infinite.

In the case of various other substances (solutions and compounds) it was found that the specific inductive capacity at $-200°$ C. did not greatly exceed μ^2_∞. The agreement was nearly exact for carbon disulphide, olive oil, and castor oil.

38. Experiments with electric waves. A circular exciter of electric waves was properly placed within a concentric circle of wire at one end of a pair of parallel wires at a distance apart of 2 cm, and produced between two wire bridges across the wires, a system of stationary waves which were propagated along the conductors, with successive nodes half a wave length apart. A vacuum tube with such a length of wire joining its terminals that its period coincided with that of the exciter was lighted up when situated at a loop in the waves, and so the distance between two nodes, and therefore the wave length, was determined. Observations were made with the wires prolonged into a trough of liquid, and again with the trough removed and the wires continued in air. The bridge next the exciter being kept fixed in position, the farther bridge was moved out in the trough or in the air until for different positions of this bridge the lamp lit up.

Thus the half lengths of the wave in air and liquid were found, and so also their ratio λ_1/λ_2. Thus $K=\mu^2=(\lambda_1/\lambda_2)^2$ was obtained.

Two wave frequencies, $1{\cdot}5\times10^8$ and 4×10^8, were employed. For water the change of frequency had little effect, but for glycerine the change in μ^2 was considerable,—from $\mu^2=39{\cdot}1$ at the lower frequency to 25·4 for the higher.

For water the formula

$$\mu^2=88{\cdot}23-0{\cdot}4044t+0{\cdot}001035t^2,$$

at temperature $t°$ C., was obtained. In the case of solutions the refractive index found was nearly that for water. Increase of frequency gave a decided diminution of refractive index with increase of conductivity.

In connection with the subject of electric stress and strain the results of Kerr's investigations of double refraction in solids and liquids are of very great interest and importance. We give a very brief account here of his experiments on liquids.

Let the reader imagine two horizontal cylinders of brass, placed with their axes parallel and in a horizontal plane, at a distance of an inch or so apart, in a trough containing the liquid to be experimented on. The trough has glass ends, and a beam of light of definite wave-length is passed along between the cylinders, but before entering the liquid passes through a polarizer, so that the beam is polarized in a plane inclined, let us say, at an angle of 45° to the vertical. After passing between the cylinders the light is received by an analyzer set at right angles to the polarizer. The beam is thus extinguished by the analyzer.

The cylinders are now connected to the poles of an electric machine, and it is then found that when the machine is worked the extinction is no longer perfect, but that a ray passes on from the analyzer to the eye of an observer. One component of the light passing through the liquid is delayed relatively to the other component, and the two unite in a plane polarized beam which is not extinguished by the analyzer.

An electrometer was used to measure the difference of potentials between the cylinders, and therefore also the strength of the electric field there existing. It was found, by bringing the beam again to extinction by a compensator, which measured the difference of phase of the two components, that the effect produced was proportional to the square of the field-intensity.*

Kerr found that some liquids acted on the light as would have done a quarter-wave plate † with the crystal axis along the lines of force, while others had the opposite effect, that of a crystal of Iceland spar, with its axis similarly placed. The former he called positive liquids, the latter negative.‡ Benzene, paraffin oil, and toluene are positive; various oils, such as colza and olive oil, are negative.

If the difference of paths traversed in the same time by the components be δ, λ the wave-length, l the length of path in the field and E the field-intensity, the formula expressing the results was

$$\frac{\delta}{\lambda} = BlE^2,$$

where B was a constant. This constant was determined by Quincke for carbon disulphide to be approximately $3{\cdot}1 \times 10^{-7}$. The difference of refractive indices for the two rays is of course $B\lambda lE^2$, where λ is the wave-length in air.

The value of B depends on the wave-length, and has been found to be considerably increased by rise of temperature.

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* *Phil. Mag.* Ser. 5, ix. (1880).

† A plate which produces a phase-difference of $\frac{1}{4}\lambda$ between the waves propagated through it.

‡ *Phil. Mag.* Ser. 5, xiii.

When a conducting liquid is used in which a field cannot be maintained, it is observed that if a spark-gap exists in one of the wires connecting to the electric machine the field is lighted up, when a spark passes. This effect is no doubt due to rapid electric oscillations set up by the spark.

Of the Kerr and Zeeman magneto-optic effects this is not the place to treat, and unfortunately space is not available to discuss them in the present book.

APPENDIX I.

THEORY OF THE INDUCTION COIL.

By Professor E. Taylor Jones, D.Sc.

An induction coil consists of two coupled circuits, each having inductance, resistance, and capacity, the capacity of the secondary being, in the ordinary use of the instrument, mainly distributed along the coil, but including also the capacity of any bodies (*e.g.* the electrodes of a spark-gap or of an exhausted tube) connected with its terminals. The secondary capacity C_2 may be defined as the charge on one half of the secondary coil (and the bodies connected with its terminal) divided by the difference of potential of the terminals.

When the secondary terminals are connected with bodies of very small capacity between which no discharge is passing, the current induced in the secondary coil when contact is broken at the interrupter is not uniformly distributed along the wire, but is greatest at the central winding and nearly zero at the terminals. In these circumstances the secondary self-inductance L_2 may be defined as the magnetic induction through the secondary coil, due to the current in this coil, divided by the value of the current in the central winding. The value of L_2 is of course smaller when the current is distributed in this way than when it is distributed uniformly, as for instance when the secondary coil is short-circuited, or when its terminals are connected with a condenser of considerable capacity.

For similar reasons the coefficient of induction of the secondary on the primary (*i.e.* the magnetic induction through the primary coil due to the secondary current, divided by the value of the latter in the central winding) is smaller than the coefficient of induction of the primary on the secondary, which is the mutual inductance as usually defined. Employing the notation used by Drude in his theory of the Tesla coil, we shall denote these two coefficients by L_{12} and L_{21}, L_{21} being the coefficient of induction of the primary on the secondary.

The coupling k^2 of the primary and secondary circuits, *i.e.* the square of the coefficient of coupling, is defined by

$$k^2 = L_{12}L_{21}/L_1L_2.$$

In the following sketch of the theory of the secondary potential at break we shall neglect the resistances of the coils and all other causes of dissipation of energy, so that the oscillations are treated as undamped. Denoting the potential difference of the plates of the primary condenser by v_1, that of the secondary terminals by V_2, and the electromotive force of the battery by E, the equations for the currents after contact is broken at the interrupter are

$$L_1 \frac{d\gamma_1}{dt} + L_{12}\frac{d\gamma_2}{dt} + v_1 = E, \quad \text{......(1)}$$

$$L_2 \frac{d\gamma_2}{dt} + L_{21}\frac{d\gamma_1}{dt} + V_2 = 0, \quad \text{......(2)}$$

where γ_2 is the current in the central winding of the secondary coil. Also

$$\gamma_1 = C_1 \frac{dv_1}{dt}, \quad \text{......(3)}$$

$$\gamma_2 = C_2 \frac{dV_2}{dt}. \quad \text{......(4)}$$

Substituting for γ_1, γ_2 from (3) and (4), and writing V_1 for $v_1 - E$, equations (1) and (2) become

$$L_1C_1 \frac{d^2V_1}{dt^2} + L_{12}C_2\frac{d^2V_2}{dt^2} + V_1 = 0, \quad \text{......(5)}$$

$$L_2C_2 \frac{d^2V_2}{dt^2} + L_{21}C_1\frac{d^2V_1}{dt^2} + V_2 = 0. \quad \text{......(6)}$$

The assumed solutions $V_1 = Ae^{ipt}$, $V_2 = Be^{ipt}$, lead, after elimination of the ratio B/A, to the equation for $p\ (=2\pi n)$,

$$p^4L_1C_1L_2C_2(1-k^2) - p^2(L_1C_1 + L_2C_2) + 1 = 0. \quad \text{......(7)}$$

Each circuit has therefore two frequencies of oscillation n_1, n_2 $(n_2 > n_1)$ given by the equation

$$8\pi^2({n_1}^2, {n^2}_2) = \frac{1}{1-k^2}\left[\frac{1}{L_1C_1} + \frac{1}{L_2C_2} \mp \sqrt{\left\{\left(\frac{1}{L_1C_1} - \frac{1}{L_2C_2}\right)^2 + \frac{4k^2}{L_1C_1L_2C_2}\right\}}\right]. \quad \text{......(8)}$$

The conditions at the moment of break $(t=0)$ are

$$v_1 = 0, \quad i.e.\ V_1 = -E,$$

$$V_2 = 0,$$

$$\frac{dV_1}{dt} = \frac{\gamma_0}{C_1},$$

$$\frac{dV_2}{dt} = 0,$$

where γ_0 denotes the primary current just before break.

It can be shown that the solution of (5) and (6) corresponding to these initial conditions is

$$V_2 = \frac{2\pi L_{21}\gamma_0 n_1 n_2}{n_2{}^2 - n_1{}^2}(n_2 \sin 2\pi n_1 t - n_1 \sin 2\pi n_2 t), \quad \text{.............(9)}$$

$$\begin{aligned} V_1 = &-\frac{2\pi\gamma_0 n_1 n_2{}^2}{C_1(n_2{}^2 - n_1{}^2)}\left(\frac{1}{4\pi^2 n_2{}^2} - L_1 C_1\right)\sin 2\pi n_1 t \\ &+\frac{2\pi\gamma_0 n_1{}^2 n_2}{C_1(n_2{}^2 - n_1{}^2)}\left(\frac{1}{4\pi^2 n_1{}^2} - L_1 C_1\right)\sin 2\pi n_2 t. \quad \text{..........(10)} \end{aligned}$$

The wave of potential in the secondary circuit after break therefore consists of two oscillatory components of different frequencies, which begin in opposite phase, and the amplitudes of which are inversely proportional to their frequencies.

In the primary circuit the two components begin in the *same* phase, since the value of L_1C_1 lies between those of $1/4\pi^2 n_1{}^2$ and $1/4\pi^2 n_2{}^2$.

For given values of L_{21}, γ_0, n_1 and n_2 the value of V_2 is stationary at times given by $dV_2/dt = 0$, *i.e.*

$$\cos 2\pi n_1 t - \cos 2\pi n_2 t = 0, \quad \text{..............................(11)}$$

or

$$\sin \pi(n_1 + n_2)t \,.\, \sin \pi(n_2 - n_1)t = 0.$$

The stationary values of V_2 therefore occur at the times

$$\left.\begin{array}{ll} t = 0, & \dfrac{1}{n_1+n_2},\ \dfrac{2}{n_1+n_2},\ \dfrac{3}{n_1+n_2},\ \dots, \\ \text{and} & \dfrac{1}{n_2-n_1},\ \dfrac{2}{n_2-n_1},\ \dfrac{3}{n_2-n_1},\ \dots. \end{array}\right\} \quad \text{....................(12)}$$

At any stationary value we have, by (11),

$$\sin 2\pi n_2 t = \pm \sin 2\pi n_1 t, \quad \text{................................(13)}$$

the upper sign giving the numerical minima of V_2, the lower sign the maxima.

Substituting in (9), we find that the numerical maxima lie on the (t, y) curve

$$y = 2\pi L_{21}\gamma_0 \frac{n_1 n_2}{n_2 - n_1} \sin 2\pi n_1 t, \quad \text{........................(14)}$$

the minima on the curve

$$y = 2\pi L_{21}\gamma_0 \frac{n_1 n_2}{n_2 + n_1} \sin 2\pi n_1 t. \quad \text{........................(15)}$$

In dealing with the maximum secondary potential of an induction coil we are only concerned with the greatest maximum of V_2 in the first half-period of the slower component. Even though there should be a closer coincidence of maxima of the two components in some subsequent half-period, the amplitudes are by this time so much reduced by the damping that the potential seldom, if ever, reaches a value equal to the greatest in the first half-period.

We may distinguish as the *principal maximum* of V_2 that which occurs nearest to the first summit of the curve (14), *i.e.* at the time nearest to $t=1/4n_1$. The first maximum occurs at the time $t=\frac{1}{n_1+n_2}$, and this is the principal maximum if the frequency-ratio n_2/n_1 is between 1 and 5. If $n_2=5n_1$ the first maximum is equal to the second, and they occur at times $\frac{1}{n_1+n_2}$, $\frac{2}{n_1+n_2}$. If n_2/n_1 is between 5 and 9 the second maximum is the principal maximum, occurring at the time $\frac{2}{n_1+n_2}$. If $n_2=9n_1$ the second and third maxima are equal, and if n_2/n_1 is between 9 and 13 the third maximum is the principal maximum, and it occurs at $t=\frac{3}{n_1+n_2}$; and so on.

Consequently the principal maximum secondary potential is given by the equation

$$V_{2m}=2\pi L_{21}\gamma_0\frac{n_1n_2}{n_2-n_1}\sin\phi, \quad\text{......(16)}$$

where

$$\left.\begin{array}{llllll}\phi=\dfrac{2\pi n_1}{n_1+n_2} & \text{if } \dfrac{n_2}{n_1} \text{ is between} & 1 & \text{and} & 5,\\[2ex] \phi=\dfrac{4\pi n_1}{n_1+n_2} & \text{,,} \quad \text{,,} \quad \text{,,} & 5 & \text{,,} & 9,\\[2ex] \phi=\dfrac{6\pi n_1}{n_1+n_2} & \text{,,} \quad \text{,,} \quad \text{,,} & 9 & \text{,,} & 13,\\ \dots\dots\dots\dots & & & & \end{array}\right\} \quad\text{......(17)}$$

If n_2/n_1 has one of the values 3, 7, 11, ... maxima of the two oscillations occur simultaneously, the principal maximum occurring at the time $1/4n_1\left(\phi=\frac{\pi}{2}\right)$, and being equal to the sum of the amplitudes of the components.

The expression $2\pi L_{21}\gamma_0 n_1n_2/(n_2-n_1)$ represents the sum of the amplitudes of the components of the potential wave in the secondary circuit. It is convenient to express this quantity in terms of k^2 and the ratio L_1C_1/L_2C_2. Calling the latter u we find from (8),

$$\frac{n_1n_2}{n_2-n_1}=\frac{1}{2\pi\sqrt{L_2C_2}}\sqrt{\frac{1}{1+u-2\sqrt{u(1-k^2)}}}.$$

The sum of the amplitudes is therefore

$$\frac{L_{21}\gamma_0}{\sqrt{L_2C_2}}\cdot\frac{1}{\sqrt{[1+u-2\sqrt{u(1-k^2)}]}}, \quad\text{......(18)}$$

or

$$\frac{L_{21}\gamma_0}{\sqrt{L_2C_2}}U,$$

where

$$U^2=\frac{1}{1+u-2\sqrt{u(1-k^2)}}. \quad\text{......(19)}$$

The principal maximum secondary potential is therefore

$$V_{2m} = \frac{L_{21}\gamma_0}{\sqrt{L_2 C_2}} U \sin\phi, \quad \text{.........}(20)$$

the angle ϕ being found from (17) and the frequency-ratio n_2/n_1, which is given by

$$\frac{n_2^2}{n_1^2} = \frac{1+u+\sqrt{\{(1-u)^2+4k^2u\}}}{1+u-\sqrt{\{(1-u)^2+4k^2u\}}}. \quad \text{.........}(21)$$

One important problem connected with the induction coil is that of determining the optimum primary capacity, *i.e.* the capacity of the condenser connected across the break which gives the greatest potential at the secondary terminals when the induction coefficients of the circuits, the secondary capacity, and the primary current immediately before the interruption are all given. In these circumstances the only variable in the expression (18) for the sum of the amplitudes is u (which is proportional to the primary capacity), and we find from (19) that U has a maximum value of $1/k$ when $u = 1 - k^2$. Also $\sin\phi$ has its maximum value unity when the frequency-ratio n_2/n_1 is one of the numbers 3, 7, 11, 15, ... Both of these conditions are satisfied if the coupling k^2 has one of a series of values which may be calculated from (21) by putting in this equation $u = 1 - k^2$, and n_2/n_1 successively equal to 3, 7, 11, ... The first four of the series, with the corresponding values of u (*i.e.* $1 - k^2$), are given in Table I.

TABLE I.

n_2/n_1.	k^2.	$u = 1 - k^2$.
3	0·571	0·429
7	0·835	0·165
11	0·902	0·098
15	0·931	0·069

If the coupling has one of the values given in the second column of Table I., the optimum value of u is given by the corresponding number in the third column, and the most effective value of the primary capacity is $(1-k^2)L_2C_2/L_1$.

In any one of these adjustments of the system the principal maximum secondary potential is, by (20),

$$\begin{aligned} V_{2m} &= \frac{L_{21}\gamma_0}{\sqrt{L_2C_2}} \cdot \frac{1}{k} \\ &= \gamma_0 \sqrt{\frac{L_{21}}{L_{12}}} \sqrt{\frac{L_1}{C_2}}, \quad \text{.........}(22) \end{aligned}$$

so that

$$\frac{1}{2} L_1 \gamma_0^2 = \frac{1}{2} C_2 V_{2m}^2 \frac{L_{12}}{L_{21}}. \quad \text{.........}(23)$$

Turning now to the primary circuit equation (10) enables us to calculate the potential difference in the primary condenser at the moment when the secondary potential reaches its greatest value. This occurs at the time $t=1/4n_1$ if n_2/n_1 has one of the values 3, 7, 11, ... , so that $\sin 2\pi n_1 t=1$ and $\sin 2\pi n_2 t=-1$. The value of V_1 therefore becomes at this instant

$$\begin{aligned} V_1 &= \frac{-\gamma_0}{2\pi C_1}\cdot\frac{1}{n_2-n_1}+2\pi\gamma_0 L_1\frac{n_1 n_2}{n_2-n_1} \\ &= \frac{-\gamma_0}{2\pi C_1(n_2-n_1)}(1-4\pi^2 L_1 C_1 n_1 n_2) \\ &= \frac{-\gamma_0}{2\pi C_1(n_2-n_1)}\left(1-\sqrt{\frac{u}{1-k^2}}\right), \text{ by (8).} \end{aligned}$$

The condition $u=1-k^2$ makes this expression vanish, since the denominator cannot be zero in any actual case.

Consequently in any of the adjustments specified in Table I. the primary condenser is uncharged at the moment when the secondary potential reaches its greatest value; the two potential waves in the primary have in fact equal amplitudes, and at the instant in question the potentials in the two components are at their maxima, but in opposite phase. Further, since $dV_1/dt=0$ and $dV_2/dt=0$, there is no current in either circuit, and as we are neglecting all causes of dissipation of energy, including hysteresis and eddy currents in the core, it is clear that at the moment in question the whole of the energy exists as electrostatic energy in the secondary circuit, the value of which must be equal to the electrokinetic energy $\frac{1}{2}L_1\gamma_0^2$ initially supplied to the primary circuit. The expression on the right-hand side of (23) therefore represents the energy of the charge on the secondary circuit when its terminals are at a potential difference V_{2m}.

What may be called the efficiency of conversion of electrokinetic into electrostatic energy is unity in each of the adjustments specified in Table I. if all damping losses are neglected.

If the coupling has not one of the above special values the maximum value of $U\sin\phi$ does not occur when $\phi=\pi/2$. In such cases one must resort to numerical calculation and find by trial the value of u which gives the greatest value of $U\sin\phi$. The value of $U\sin\phi$ for any given values of u and k^2 can be calculated from (21) (which gives the frequency-ratio), (17), and (19). For any given value of k^2 the frequency-ratio n_2/n_1 is smallest when $L_1C_1=L_2C_2$, its value then being equal to $\sqrt{(1+k)/(1-k)}$. Thus if the coupling is 0·64 the smallest ratio is 3, if it is 0·779 the smallest ratio is 4, and the frequency-ratio 5 is the smallest for the coupling 0·852. The ratio 2 does not occur if the coupling is greater than 0·36, a value much lower than those usually found in induction coils.

When the primary capacity is reduced from the value given by the equation $L_1C_1 = L_2C_2$ the frequency-ratio steadily increases. We shall confine our attention to this range, because we find therein the highest values of the secondary potential and the optimum value of the capacity. The diagram in Fig. 253 shows the manner in which, according to the theory with damping neglected, the maximum secondary potential varies with the capacity of the primary condenser. The coupling in this case is 0·571, the first of the values given in Table I. The abscissa u represents the ratio L_1C_1/L_2C_2, which is proportional to C_1, the other factors being constant. The ordinate of the full-line curve represents $U \sin \phi$, which is proportional to the maximum secondary potential produced at the interruption of a given primary current. This curve consists of a series of arches of which the first—counting from the right—is the largest and highest, and the second, third, etc., form a diminishing series. The abscissa of the summit of the highest arch determines the optimum value of the primary capacity.

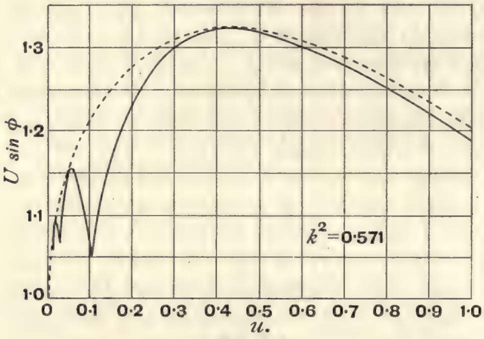

FIG. 253.

The ordinate of the broken-line curve in Fig. 253 represents the function U, which is proportional to the sum of the amplitudes of the components. It therefore represents what the maximum potential would be if the two components in the secondary were always in the same phase at the moment of maximum potential.

The difference of the ordinates of the two curves represents the deficiency in the maximum potential arising from the fact that the two component oscillations are not generally in phase with each other at the instant when the maximum occurs. This deficiency is greatest at the points of intersection of the arches, which must therefore represent those adjustments in which the phase-difference of the components (at the peak of the potential wave) is greatest, *i.e.* in which the frequency-ratio has one of the values 5, 9, 13, … . The broken-line curve touches each of the arches of the maximum potential curve. In the adjustments corresponding to these points of contact the maximum potential is therefore equal to the sum of the amplitudes of the components, so that the frequency-ratios are 3, 7, 11, … .

It will be observed that one of the points of contact in Fig. 253, viz. that at which $n_2/n_1=3$ occurs at the summit of the broken-line curve. In this adjustment, therefore, not only is the maximum potential equal to the sum of the amplitudes, but the sum of the amplitudes has also its maximum value. The conditions are therefore, as already explained, the most favourable possible for the production of high secondary potential and spark-length.

Turning now to the experimental side of the question, one way of changing the coupling of an induction coil is to draw out the primary and core to various distances along the axis of the secondary. In order to determine the effect of varying the coupling it is desirable to connect a condenser to the secondary terminals in parallel with the spark-gap, so as to minimize any effects arising from the variation of the capacity of the secondary coil due to the displacement of the primary. When this experiment was tried with a certain 18-in. coil, the primary capacity being adjusted to the optimum value for each position of the primary, it was found that the length of spark produced at the interruption of a given primary current fell to a minimum in a certain position of the primary, and increased to a maximum when the primary was drawn out beyond this position. In the position of maximum spark-length the primary was about one foot from the central position, *i.e.* from the position of maximum mutual inductance. With the primary in the position of greatest spark-length the coupling was found to be 0·58, which agrees fairly closely with the first of the values indicated by the theory as the most effective when the oscillations are undamped.

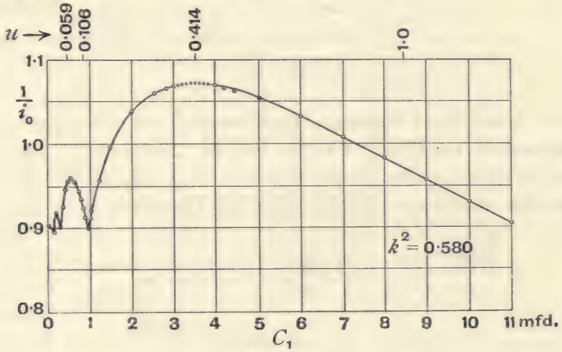

FIG. 254.

The manner in which the maximum secondary potential varies with the primary capacity, for this position of the primary coil, is shown in Fig. 254. In this diagram the abscissa is the primary capacity in microfarads, the ordinate the reciprocal of the smallest primary current required to cause a spark to appear at a constant gap, which is proportional to the maximum secondary potential due to a given primary current. It will be seen that this curve also consists of a series of arches, which have much the same relative proportions as those of Fig. 253. The values of u at the chief maxima and minima of the curve (shown above the diagram), determined from measurements of the inductances and capacities, also show fairly good agreement with the corresponding abscissae of Fig. 253. It is clear that in this case the damping of the oscillations is not sufficient to cause any great difference in the optimum capacity or in the relative proportions of the arches of the "capacity-potential" curve, though the value of the potential is, of course, considerably reduced by the damping.

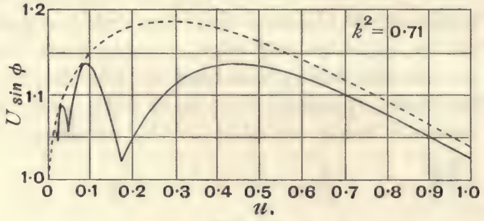

FIG. 255.

When the coupling is increased the first arch of the capacity-potential curve diminishes relatively to the others, becoming equal in height to the second arch at the coupling 0·71. Fig. 255 shows the calculated curves for this coupling. In this case the first arch does not come into contact with the broken curve, since the coupling is greater than 0·64, the limiting value for the frequency-ratio 3. There are two optimum capacities, given by $u = 0·44$ and $0·09$, the abscissae of the summits of the first two arches. These summits are well below the highest point of the broken curve, so that at the coupling 0·71 it is impossible to obtain the highest efficiency of conversion of electrodynamic into electrostatic energy. This coupling represents, in fact, the least efficient value likely to be met with in induction coils.

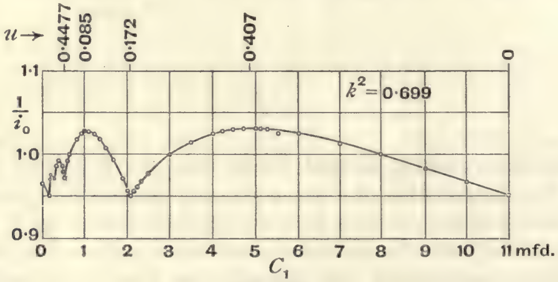

FIG. 256.

In actual coils the least efficient coupling appears to be rather smaller than the corresponding value for a coil devoid of damping. When the primary of the 18-inch coil was placed in the position of minimum spark-length the coupling was found to be 0·699. The experimental curve for this position is shown in Fig. 256. This curve shows the same general characteristics as the theoretical curve of Fig. 255. In particular it shows two optimum capacities having nearly the same ratio as those of the theoretical curve.

If the coupling is further increased the first arch continues to decline, and it finally disappears at the value 0·852. At the coupling 0·87 the second and third arches are equal in height, and there are again two optimum capacities. At 0·9 the third arch touches the broken curve at its summit, this being one of the values which give maximum efficiency of conversion.

When the coupling is varied by drawing out the primary coil the two values 0·571 and 0·835 would, in the absence of damping, be equally effective from the point of view of spark-length for a given primary current, but on the experiment being tried with the 18-inch coil the spark-length was found to be decidedly greater with the smaller of these two couplings than with the other. It is clear therefore, both on theoretical and on experimental grounds, that the spark-length of an induction coil does not necessarily increase with the mutual inductance* nor with the coupling, and it appears that the effect of the damping (chiefly due to core losses) in reducing the secondary potential increases with the coupling.

The form of the capacity-potential curve is characteristic of the coupling and may be used for approximately determining the coupling, since the relative proportions of the principal arches, in a well-constructed coil, are not greatly modified by the damping.

The wave-form of the secondary potential of an induction coil may be observed by means of the electrostatic oscillograph, an instrument which can be connected directly to the secondary terminals of the coil and can be used for potentials up to about 200,000 volts. The essential parts of the instrument include a metallic strip under tension, which carries a small mirror at its middle point and is placed between two metallic plates, to one of which it is connected. The other plate may be enclosed in an ebonite sheath. The lower edge of the mirror can be held fixed by an adjustable insulated platform brought into contact with it. The instrument is placed in an ebonite vessel containing a damping oil and provided with a window. The angular deflection of the mirror is proportional to the square of the difference of potential of the plates.

A narrow pencil of light proceeding from a pinhole falls upon the mirror, a part of the pencil striking another small mirror attached to a tuning-fork mounted in front of the instrument. The two reflected rays fall upon a rotating mirror by which they are focussed upon a photographic plate where two small images of the pinhole are formed.

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* In the position of greatest spark-length the mutual inductance was 61 per cent., in that of least spark-length 81 per cent., of the maximum.

Some examples of the photographs obtained when the instrument was connected with the 18-inch coil (no discharge passing between the terminals) are shown in Fig. 257. The lowest curve was obtained when the coupling and the capacity were those corresponding to the summit of the first arch in Fig. 254, *i.e.* the primary coil was drawn out to the position of maximum spark-length and the primary capacity adjusted to its optimum value. The curve shows the peaked summits and flattened zeroes characteristic of the frequency-ratio 3, which we find, on referring to Fig. 253, should be the ratio for this adjustment.*

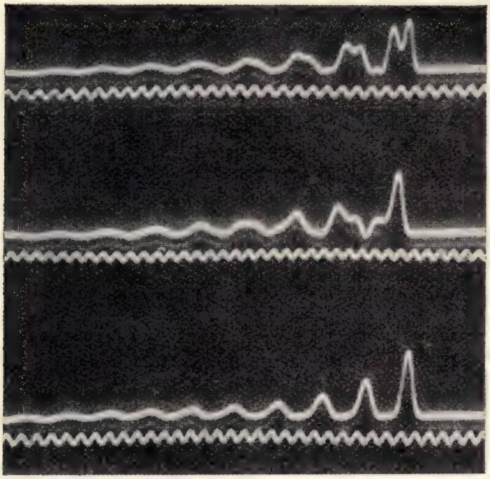

FIG. 257.

The second curve was obtained when the primary was placed in the position of minimum spark-length and the primary capacity adjusted to the value corresponding to the summit of the first arch in Fig. 256. The frequency-ratio in this case is about 3·8.

The uppermost curve of Fig. 257 was obtained when the coil was approximately in the adjustment corresponding to the point of intersection of the first and second arches of Fig. 256. The frequency-ratio is clearly about 5, as it should be according to the theory.

With the primary capacity adjusted to give the summit of the second arch of Fig. 256, the curve was that shown in Fig. 258, the frequency-ratio being about 6·6.

Fig. 259 was obtained when the capacity was that corresponding to the second minimum of Fig. 256, the frequency-ratio here being 9.

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* The period of the time curve in each photograph is 1/768 sec.

The curves of Figs. 257...259 represent cases in which the secondary terminals of the coil are connected with a condenser, but similar curves are found without the condenser. These curves preserve their form, the amplitude only changing, when the primary current is varied over wide limits. The greatest ordinate of the curve in any case is approximately proportional to the square of the primary current at break.

It may be noticed that the theoretical and experimental results here described are not in agreement with those of Lord Rayleigh,* according to which the longest secondary spark should be obtained without any primary condenser, provided the interruption of the primary current takes place with sufficient rapidity, the only use of the condenser with ordinary interrupters being to quicken the break by preventing the formation of an arc between the contact surfaces. Rayleigh obtained experimental evidence on this point by interrupting the current by firing at the primary wire with a rifle, and found that in these circumstances the secondary spark for a given primary current was longer without any condenser than with the condenser attached to the coil. A repetition of the rifle-bullet experiment,† however, has shown that even with this rapid interruption a longer spark is obtained with than without a condenser, provided the capacity of the condenser is the optimum indicated by the theory described above.

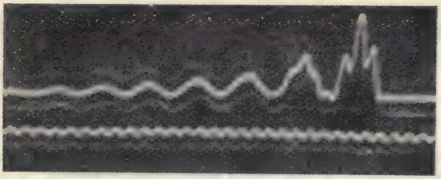

FIG. 258.

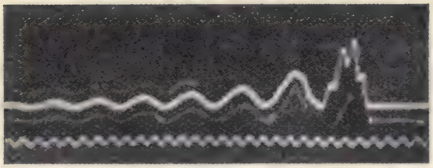

FIG. 259.

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* *Phil. Mag.* ii. p. 593 (1901).

† *Phil. Mag.* April, 1914, p. 583.

APPENDIX II.

ZONAL SPHERICAL HARMONICS.

A SPHERICAL harmonic may be defined as a homogeneous function of x, y, z which satisfies Laplace's equation,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0. \quad \text{.................................(1)}$$

Since it is homogeneous it satisfies also the relation

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=nV \quad \text{................................(2)}$$

if n be the degree of the function.

The fundamental equation may be transformed by the substitution of the variables, r, θ, ϕ, connected with x, y, z by the equations

$$\left.\begin{aligned} x&=r\sin\theta\cos\phi,\\ y&=r\sin\theta\sin\phi,\\ z&=r\cos\theta.\end{aligned}\right\} \quad \text{....................................(3)}$$

Of these θ may be regarded as the co-latitude and ϕ the longitude, or θ and ϕ may be taken as respectively the polar distance and right ascension of the point x, y, z, of which r is in both cases the radius vector from the origin.

When these substitutions are made Laplace's equation becomes

$$r\frac{\partial^2(rV)}{\partial r^2}+\frac{1}{1-\mu^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\}=0 \quad \text{................(4)}$$

if μ denote $\cos\theta$.

Equation (2) becomes plainly

$$\frac{\partial V}{\partial r}=\frac{n}{r}V. \quad \text{..(5)}$$

The last result gives

$$r\frac{\partial^2(rV)}{\partial r^2}=n(n+1)V.$$

Hence (4) takes the form

$$\frac{1}{1-\mu^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\}+n(n+1)V=0. \quad \text{..............(6)}$$

If V denote a spherical harmonic of degree n, we may write it in the form $r^n S_n$. S_n is a function of θ, ϕ, but not of r, and is called a spherical surface harmonic of degree n. It satisfies by (6) the equation

$$\frac{1}{1-\mu^2}\frac{\partial^2 S}{\partial\phi^2}+\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial S}{\partial\mu}\right\}+n(n+1)S=0. \quad \text{................(7)}$$

If $r^n S_n$ denote a spherical harmonic of degree n, $r^{-(n+1)}S_n$ denotes a spherical harmonic of degree $-(n+1)$. To prove this we have only to notice that it clearly satisfies (6), since S_n satisfies (7). Again if we denote it by V, we have

$$\frac{\partial V}{\partial r}=-(n+1)r^{-(n+2)}S_n=-\frac{n+1}{r}V,$$

which is what (5) becomes when n is changed to $-(n+1)$.

If S_n is symmetrical about an axis it is called a zonal surface harmonic (or simply a zonal harmonic) of order n. We may take the axis of symmetry as axis of z, so that the symmetry is expressed by making S_n independent of ϕ. We shall denote a zonal harmonic of order n by Z_n.

The differential equation satisfied by Z_n is, by (7),

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial u}{\partial\mu}\right\}+n(n+1)u=0. \quad \text{........................(8)}$$

The discovery of zonal harmonics resolves itself then into finding particular solutions of this equation. The most important case, and the only one which we here consider, is that in which n is a positive integer.

We assume first that u may be expanded in a series of powers of μ. Thus writing

$$u=A_1\mu^{m_1}+A_2\mu^{m_2}+\dots, \quad \text{............................(9)}$$

substituting in the differential equation (8), and equating coefficients of like powers of μ, we get first from those of μ^{m_1},

$$(m_1-n)(m_1+n+1)A_1=0.$$

Since A_1 is not zero this gives $m_1=n$, or $m_1=-(n+1)$. Thus there are two solutions according as m_1 is taken $=n$, or $=-(n+1)$. m_2 is then found to be m_1-2, $m_3=m_1-4$, etc.

Again the successive coefficients in (9) are found to be connected by the relation

$$A_r=-\frac{(m_1-2r+4)(m_1-2r+3)}{2(r-1)(2m_1-2r+3)}A_{r-1},$$

whichever value is given to m_1.

Hence if we take $m_1 = n$, (9) becomes

$$u = A_1\left\{\mu^n - \frac{n(n-1)}{2\,.\,(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\,.\,4\,.\,(2n-1)(2n-3)}\mu^{n-4} - \dots\right\}. \quad \dots(10)$$

The series within brackets in (10) is finite and has for last term $(-1)^{\frac{1}{2}n} n!n!n!/(\frac{1}{2}n!\frac{1}{2}n!2n!)$ if n be even, and

$$(-1)^{\frac{1}{2}(n-1)}\mu n!n!(n-1)!/\{\tfrac{1}{2}(n-1)!\tfrac{1}{2}(n-1)!(2n-1)!\}$$

if n be odd. The numbers of terms in the two cases are $\frac{1}{2}(n+2)$ and $\frac{1}{2}(n+1)$.

Another series is obtainable by putting $m_1 = -(n+1)$. This and the former multiplied each by an arbitrary constant and added together give the complete solution of (8).*

The series in (10) with $2n!/2^n(n!)^2$ substituted for A_1 is what is called the zonal surface harmonic of order n. Thus

$$Z_n = \frac{2n!}{2^n n!n!}\left\{\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\,.\,4\,.\,(2n-1)(2n-3)}\mu^{n-4} - \dots\right\}. \quad (11)$$

It may be verified by differentiation that

$$Z_n = \frac{1}{2^n n!}\frac{d^n}{d\mu^n}\{(\mu^2-1)^n\}, \quad \dots\dots(12)$$

and by expansion of $(1 - 2\mu h + h^2)^{-\frac{1}{2}}$ in ascending powers of h that Z_n is the coefficient of h^n in the resulting series. It is this latter fact that renders the choice of the value above assigned to A_1 convenient.

By means of (11) we can at once write down the zonal surface harmonic for any assigned value of n. Thus, for values of n from 0 to 7,

$$Z_0 = 1, \quad Z_1 = \mu, \quad Z_2 = \tfrac{3}{2}\mu^2 - \tfrac{1}{2},$$

$$Z_3 = \frac{5}{2}\mu^3 - \frac{3\,.\,1}{2}\mu, \quad Z_4 = \frac{7\,.\,5}{2\,.\,4}\mu^4 - \frac{5\,.\,3}{2\,.\,2}\mu^2 + \frac{3\,.\,1}{2\,.\,4},$$

$$Z_5 = \frac{9\,.\,7}{2\,.\,4}\mu^5 - \frac{7\,.\,5}{2\,.\,2}\mu^3 + \frac{5\,.\,3}{2\,.\,4}\mu,$$

$$Z_6 = \frac{11\,.\,9\,.\,7}{2\,.\,4\,.\,6}\mu^6 - \frac{9\,.\,7\,.\,5}{2\,.\,4\,.\,2}\mu^4 + \frac{7\,.\,5\,.\,3}{2\,.\,4\,.\,2}\mu^2 - \frac{5\,.\,3\,.\,1}{2\,.\,4\,.\,6},$$

$$Z_7 = \frac{13\,.\,11\,.\,9}{2\,.\,4\,.\,6}\mu^7 - \frac{11\,.\,9\,.\,7}{2\,.\,4\,.\,2}\mu^5 + \frac{9\,.\,7\,.\,5}{2\,.\,4\,.\,2}\mu^3 - \frac{7\,.\,5\,.\,3}{2\,.\,4\,.\,6}\mu.$$

A numerical table of the first seven zonal surface harmonics calculated by Professor Perry for values of μ for every degree from 0 to 90° is given at the close of this *note*.

The following method of defining a solid spherical harmonic is due to Clerk Maxwell (*El. and Mag.* vol. i. chap. ix.). Let an electric doublet of moment Φ_1 be placed at the origin with its axis in any direction the cosines of which are l, m, n, then, by (8), p. 787 above, its potential at the point (x, y, z) at distance r from the origin is

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* For a full discussion of the solutions of (8), see Forsyth's *Differential Equations*, §§ 89–99.

$$V_1 = -\Phi_1\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)\frac{1}{r} = \Phi_1\left(l\frac{x}{r} + m\frac{y}{r} + n\frac{z}{r}\right)\frac{1}{r^2}.$$

If then the operation $l\partial/\partial x + m\partial/\partial y + n\partial/\partial z$ be denoted by d/dh_1, where h_1 is a distance along the axis, we may call the operation differentiation with respect to the axis h_1, and we have

$$V_1 = -\Phi_1\frac{d}{dh_1}\left(\frac{1}{r}\right) = \Phi_1\frac{\mu_1}{r^2}, \quad \text{.........................(13)}$$

where μ_1 is the angle between the direction of h_1 and of the line drawn from the origin to (x, y, z).

With respect to this kind of differentiation we may notice that if the suffix j indicate any axis whatever with direction cosines l_j, m_j, n_j, and μ_j denote the cosine of the angle between the axis referred to and the line from the origin to (x, y, z), and λ the cosine of the angle between the axes, we have

$$\frac{dr}{dh_j} = \mu_j. \quad \text{.......................................(14)}$$

Again, if the suffix k indicate another axis,

$$\begin{aligned}\frac{d\mu_j}{dh_k} &= \frac{d}{dh_k}\left(l_j\frac{x}{r} + m_j\frac{y}{r} + n_j\frac{z}{r}\right)\\ &= \frac{1}{r}\left\{(l_jl_k + m_jm_k + n_jn_k) - \left(l_j\frac{x}{r} + m_j\frac{y}{r} + n_j\frac{z}{r}\right)\frac{dr}{dh_k}\right\}\\ &= \frac{1}{r}(\lambda_{jk} - \mu_j\mu_k). \quad \text{...(15)}\end{aligned}$$

Now let two doublets of moments $-\Phi_1$, $+\Phi_1$, with axes parallel to h_1, be placed with their centres on another axis h_2 at distances $-\frac{1}{2}\partial h_2$, $+\frac{1}{2}\partial h_2$ from the origin, the potential at (x, y, z) due to the pair of doublets is

$$V_2 = -V_1 + V_1 - \partial h_2\frac{dV_1}{dh_2} = -\Phi_1\partial h_2\frac{d}{dh_2}\left(\frac{\mu_1}{r^2}\right).$$

If we diminish $\frac{1}{2}\partial h_2$ indefinitely and increase Φ_1 so that $\Phi_1\partial h_2$ remains a finite quantity $\Phi_2/2$, we have

$$V_2 = -\Phi_2\frac{1}{2}\frac{d}{dh_2}\left(\frac{\mu_1}{r^2}\right). \quad \text{...............................(16)}$$

Hence performing the differentiation we get

$$V_2 = -\Phi_2\frac{1}{2}\left(\frac{1}{r^2}\frac{d\mu_1}{dh_2} - \frac{2\mu_1}{r^3}\frac{dr}{dh_2}\right) = \frac{1}{2}\frac{\Phi_2}{r^3}(3\mu_1\mu_2 - \lambda_{12}). \quad \text{........(17)}$$

This is the potential due to what may be called a doublet of the second order placed at the origin. It may be written

$$V_2 = (-1)^2 \Phi_2 \frac{1}{2} \frac{d}{dh_1} \frac{d}{dh_2} \left(\frac{1}{r}\right). \quad \text{.........................(18)}$$

Let now the doublet of the second order we have just supposed built up, be imagined placed with change of direction with its centre on a third axis h_3 at a distance $\frac{1}{2}\partial h_3$ from the origin, and an equal doublet of the second order but of opposite sign placed with its centre on the same axis at the same distance from the origin on the opposite side. Then the potential of this arrangement at (x, y, z) is

$$V_3 = (-1)^3 \Phi_2 \partial h_3 \frac{1}{1 \,.\, 2} \frac{d}{dh_1} \frac{d}{dh_2} \frac{d}{dh_3} \left(\frac{1}{r}\right).$$

If we diminish ∂h_3 and increase Φ_2 so that $\Phi_2 \partial h_3$ remains finite and equal to $\Phi_3/3$, we get a doublet of the third order at the origin with axes h_1, h_2, h_3, which produces a potential at (x, y, z) of amount

$$V_3 = (-1)^3 \Phi_3 \frac{1}{1 \,.\, 2 \,.\, 3} \frac{d}{dh_1} \frac{d}{dh_2} \frac{d}{dh_3} \left(\frac{1}{r}\right). \quad \text{...................(19)}$$

Proceeding in this way we can build up a doublet of any order n with axes h_1, h_2, ... h_n. The potential produced at (x, y, z) by this doublet is

$$V_n = (-1)^n \Phi_n \frac{1}{n!} \frac{d}{dh_1} \frac{d}{dh_2} \cdots \frac{d}{dh_n} \left(\frac{1}{r}\right). \quad \text{.....................(20)}$$

If $\Phi_n = 1$,

$$V_n = (-1)^n \frac{1}{n!} \frac{d}{dh_1} \frac{d}{dh_2} \cdots \frac{d}{dh_n} \left(\frac{1}{r}\right), \quad \text{........................(20')}$$

and is a solid harmonic of degrees $-(n+1)$. For, performing the differentiations transforms the equation into

$$V_n = r^{-(n+1)} S_n, \quad \text{......................................(21)}$$

where S_n is a function of the n cosines of the angles between the axes, and the line from the origin to (x, y, z) and of the $n(n-1)/2$ cosines between the different pairs of the axes. Also V_n obviously satisfies the definition of a spherical harmonic given above.

The value of S_n can be found by successive applications of (14). Thus

$$\left.\begin{aligned} &S_0 = 1, \quad S_1 = \mu_1, \quad S_2 = \tfrac{3}{2}\mu_1\mu_2 - \tfrac{1}{2}\lambda_{12}, \\ &S_3 = \tfrac{5}{2}\mu_1\mu_2\mu_3 - \tfrac{1}{2}(\mu_1\lambda_{23} + \mu_2\lambda_{31} + \mu_3\lambda_{12}), \\ &S_4 = \frac{7 \,.\, 5}{2 \,.\, 4}\mu_1\mu_2\mu_3\mu_4 - \frac{5}{2 \,.\, 4}(\mu_1\mu_2\lambda_{34} + \mu_2\mu_3\lambda_{41} + \mu_3\mu_4\lambda_{12} \\ &\qquad + \mu_4\mu_1\lambda_{23} + \mu_1\mu_3\lambda_{24} + \mu_2\mu_4\lambda_{13}) + \frac{1}{2 \,.\, 4}(\lambda_{12}\lambda_{34} + \lambda_{23}\lambda_{14} + \lambda_{31}\lambda_{24}). \end{aligned}\right\} \text{..(22)}$$

The general surface harmonic has the expression (Maxwell, *El. and Mag.* vol. i. p. 188, 2nd ed.)

$$S_n = \sum \left[(-1)^s \frac{(2n-2s)!}{2^{n-s} n! (n-s)!} \sum (\mu^{n-2s} \lambda^s) \right], \text{ ....(23)}$$

in which $\Sigma(\mu^{n-2s}\lambda^s)$ denotes the sum of all products of terms of which s of the factors are different cosines λ with double suffixes and $n-2s$ factors are different cosines μ with single suffixes, and the external Σ denotes summation for all values of s from 0 to $\frac{1}{2}n$. It is clear, since the suffix of each axis appears once and once only in each term, being brought in by the differentiation with respect to that axis, that if there be s factors with double suffixes in any term there must be $n-2s$ factors in the same term with single suffixes.

If all the axes coincide, say with the axis of z, the harmonic becomes a zonal solid harmonic and S_n degenerates into a surface harmonic of order n. Thus the solid harmonic is

$$r^{-(n+1)} Z_n = (-1)^n \frac{1}{n!} \frac{\partial^n}{\partial z^n} \left(\frac{1}{r} \right) \text{ ....(24)}$$

and

$$Z_n = (-1)^n \frac{r^{n+1}}{n!} \frac{\partial^n}{\partial z^n} \left(\frac{1}{r} \right). \text{ ....(25)}$$

It may be verified by expansion that this agrees with (11) and (12).

Useful fundamental relations of zonal harmonics can be deduced from equations (8) and (12). They may be used for example in establishing the zonal harmonic formulae of VI. and VII. above.

$$\begin{aligned} 2^n n! Z'_n &= \frac{d^{n+1}}{d\mu^{n+1}} \{(\mu^2-1)^n\} \\ &= 2n \frac{d^n}{d\mu^n} \{\mu(\mu^2-1)^{n-1}\}. \text{ ....(26)} \end{aligned}$$

But if u denote any function of μ we have by successive differentiation

$$\frac{d^n}{d\mu^n}(\mu u) = n \frac{d^{n-1} u}{d\mu^{n-1}} + \mu \frac{d^n u}{d\mu^n}. \text{ ....(27)}$$

Putting $u = (\mu^2-1)^{n-1}$, and using this result in (26) multiplied by $\mu^2 - 1$, we get

$$\begin{aligned} 2^{n-1}(n-1)!(\mu^2-1)Z'_n &= n\mu^2 \frac{d_{n-1}}{d\mu^{n-1}} \{(\mu^2-1)^{n-1}\} \\ &+ \mu(\mu^2-1) \frac{d^n}{d\mu^n} \{(\mu^2-1)^{n-1}\} - n \frac{d^{n-1}}{d\mu^{n-1}} \{(\mu^2-1)^{n-1}\}. \text{ ....(28)} \end{aligned}$$

But by (8),

$$\begin{aligned} \mu(\mu^2-1) \frac{d^n}{d\mu^n} \{(\mu^2-1)^{n-1}\} &= -\mu n(n-1) \int_\mu^1 \frac{d^{n-1}}{d\mu^{n-1}} \{(\mu^2-1)^{n-1}\} d\mu \\ &= \mu n(n-1) \frac{d^{n-2}}{d\mu^{n-2}} \{(\mu^2-1)^{n-1}\}, \end{aligned}$$

since the integral vanishes at the superior limit. Hence, taking the two first terms on the right of (28), we get

$$
\begin{aligned}
n\mu^2 \frac{d^{n-1}}{d\mu^{n-1}}\{(\mu^2-1)^{n-1}\} &+ \mu(\mu^2-1)\frac{d^n}{d\mu^n}\{(\mu^2-1)^{n-1}\} \\
&= n\mu\left[(n-1)\frac{d^{n-2}}{d\mu^{n-2}}\{(\mu^2-1)^{n-1}\} + \mu\frac{d^{n-1}}{d\mu^{n-1}}\{(\mu^2-1)^{n-1}\}\right] \\
&= n\mu\frac{d^{n-1}}{d\mu^{n-1}}\{\mu(\mu^2-1)^{n-1}\} \quad \text{[by (27)]} \\
&= \frac{n}{2n}\mu\frac{d^n}{d\mu^n}\{(\mu^2-1)^n\} \\
&= 2^{n-1}(n-1)!\,n\mu Z_n.
\end{aligned}
$$

Substituting in (28) and dividing by $2^{n-1}(n-1)!$ we find

$$(\mu^2-1)Z'_n = n\mu Z_n - nZ_{n-1}, \qquad\text{.........................(29)}$$

which is the first of the two relations used at p. 205 to obtain (73). We can still more easily prove the second of VI. 23 (73) directly; we have

$$
\begin{aligned}
Z_n &= \frac{1}{2^n n!}\frac{d^{n+1}}{d\mu^{n+1}}\{(\mu^2-1)^n\} \\
&= \frac{2n}{2^n n!}\frac{d^n}{d\mu^n}\{\mu(\mu^2-1)^{n-1}\} \\
&= \frac{1}{2^{n-1}(n-1)!}\left[n\frac{d^{n-1}}{d\mu^{n-1}}\{(\mu^2-1)^{n-1}\} + \mu\frac{d_n}{d\mu^n}\{(\mu^2-1)^{n-1}\}\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{[by (27)]} \\
&= nZ_{n-1} + \mu Z'_{n-1}. \qquad\text{..(30)}
\end{aligned}
$$

Hence $$Z_{n-1} = \frac{1}{n}(Z'_n - \mu Z'_{n-1}),$$

which is the second of VI. (73) above.

The other relations may be established by similar processes.

The following theorem is of great importance: If Z_m, Z_n, be two zonal surface harmonics of orders, m, n,

$$\int_\mu^1 Z_m Z_n d\mu = \frac{(1-\mu^2)(Z_m Z'_n - Z_n Z'_m)}{(n-m)(m+n+1)}. \qquad\text{....................(31)}$$

To prove it we have by (8)

$$\frac{d}{d\mu}\{(1-\mu^2)Z'_m\} + m(m+1)Z_m = 0,$$

$$\frac{d}{d\mu}\{(1-\mu^2)Z'_n\} + n(n+1)Z_n = 0.$$

Multiplying the first of these by Z_n, the second by Z_m and subtracting, observing that $n(n+1)-m(m+1)=(n-m)(m+n+1)$, we find

$$(n-m)(m+n+1)Z_mZ_n = Z_n\frac{d}{d\mu}\{(1-\mu^2)Z'_m\} - Z_m\frac{d}{d\mu}\{(1-\mu^2)Z'_n\}$$

$$=\frac{d}{d\mu}\{(1-\mu^2)(Z_nZ'_m - Z_mZ'_n)\},$$

which gives (31) at once by integration.

If the integral in (31) be taken from -1 to $+1$, then $1-\mu^2=0$, at both limits, and the expression on the right vanishes unless either $n=m$ or $n=-(m+1)$. Hence, if neither of these conditions is fulfilled,

$$\int_{-1}^{+1} Z_mZ_n d\mu = 0. \qquad \text{.....................................(32)}$$

We shall now give some examples of the use of spherical harmonics in expansions. First we shall take the expansion of $1/PP'$, where PP' is the distance of a point P from another point P'. Let r, r', be the distances of the points from the origin, μ the cosine of the angle POP'; then we have

$$\frac{1}{PP'} = (r^2 - 2\mu rr' + r'^2)^{-\frac{1}{2}}.$$

If we write h for r'/r, and if $h<1$, we can expand this in a convergent series of ascending powers of h. But we have seen that Z_n is the coefficient of h^n in the expansion of $(1-2\mu h+h^2)^{-\frac{1}{2}}$. Hence

$$\frac{1}{PP'} = \frac{1}{r}\{Z_0 + Z_1h + Z_2h^2 + \dots\}. \qquad \text{.........................(33)}$$

If $r'/r>1$ we have only to put $h=r/r'$, and we get

$$\frac{1}{PP'} = \frac{1}{r'}\{Z_0 + Z_1h + Z_2h^2 + \dots\}. \qquad \text{.........................(33')}$$

By means of this result the potential of any distribution, whether of attracting matter, or of electricity or magnetism, can be expressed in a series of zonal harmonics.

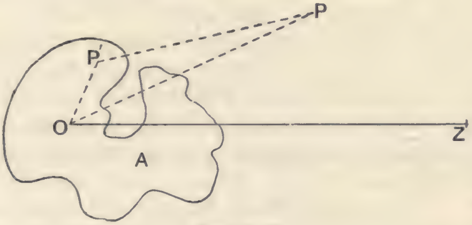

FIG. 260.

For let A be the distribution, P' the position of an element, P the point at which the potential is to be found. Then taking coordinates from an origin O, r, r', are the distances OP, OP', and μ the cosine of the angle POP'. Hence, if $d\varpi$ is an element of the distribution, its potential is

$$\frac{d\varpi}{PP'} = \frac{d\varpi}{r}(Z_0 + Z_1 h + Z_2 h^2 + \ldots) \qquad (34)$$

if $r > r'$, and

$$\frac{d\varpi}{PP'} = \frac{d\varpi}{r'}(Z_0 + Z_1 h + Z_2 h^2 + \ldots) \qquad (34')$$

if $r' > r$.

The total potential is thus

$$\left.\begin{aligned} V &= \int \frac{d\varpi}{r'}(Z_0 + Z_1 h + Z_2 h^2 + \ldots) \\ \text{or} \quad V &= \int \frac{d\varpi}{r}(Z_0 + Z_1 h + Z_2 h^2 + \ldots), \end{aligned}\right\} \qquad (35)$$

the integral being taken throughout the distribution.

If for one part of the distribution $r > r'$, and for another part $r < r'$, the integration must be divided into two corresponding parts, one for which $h = r/r'$, and the other for which $h = r'/r$.

If ZOP' be denoted by θ', ZOP by θ, and the angle which the plane of P' and the axis OZ makes with a fixed plane through the axis by ϕ', then if ρ be the density of the distribution at P',

$$d\varpi = \rho r^2 \sin\theta' \, d\theta' \, d\phi' \, dr',$$

and the integral must be taken between limits 0 and π for θ', 0 and 2π for ϕ', and 0 and r'_1 for r', where r'_1 is the superior limit of r for given values of θ and ϕ'.

An important theorem due to Legendre greatly facilitates calculations of potentials, forces, etc., for the case of symmetry round an axis. Let it be possible to express the quantity (supposed to satisfy Laplace's equation), which it is desired to calculate, for points along the axis in a series of ascending or descending powers of z, according as may be necessary for convergence. Thus for points on the axis let the quantity sought be v_a; then by hypothesis

$$\left.\begin{aligned} v_a &= a + \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_2}{z^3} + \ldots \\ \text{or} \quad v_a &= a'_0 + a'_1 z + a'_2 z^2 + a'_3 z^3 + \ldots . \end{aligned}\right\} \qquad (36)$$

We can from these expressions find the value of v for any point not on the axis, say at a distance ζ from it. If $r^2 = \sqrt{z^2 + \zeta^2}$ we have

$$\left.\begin{aligned} v &= a + a_0 \frac{Z_0}{r} + a_1 \frac{Z_1}{r^2} + \ldots \\ \text{or} \quad v &= a'_0 Z_0 + a'_1 Z_1 r + a'_2 Z_2 r^2 + \ldots , \end{aligned}\right\} \qquad (37)$$

that is we have only to substitute r for z, and multiply the terms of coefficients a_0, a_1, etc., by the zonal surface harmonics of orders indicated by the suffixes. It is to be observed that the zonal surface harmonics are chosen for the terms in the two series, so that in each case the terms are the successive zonal solid harmonics, in the first series of degrees -1, -2, -3, etc., in the second of degrees 0, 1, 2, 3, etc. These involve in both cases the same successive harmonics of orders 0, 1, 2, 3, etc., according to the theorem proved above that to every solid harmonic $r^n S_n$, of degree n, there corresponds another $r^{-(n+1)} S_n$ of degree $-(n+1)$.

As an example take the case of a wire bent into a circle of radius a, and carrying a current γ. The magnetic potential at a point on the axis of the circle at distance z from the centre is

$$V_a = 2\pi\gamma\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right).$$

We may write $1 - z/\sqrt{z^2+a^2}$ in the form $1-(1+a^2/z^2)^{-\frac{1}{2}}$, and if $a < z$ expand in descending powers of z. Thus we find

$$V_a = 2\pi\gamma\left(\frac{1}{2}\frac{a^2}{z^2} - \frac{3}{8}\frac{a^4}{z^4} + \frac{5}{16}\frac{a^6}{z^6} - \frac{35}{128}\frac{a^8}{z^8} + \ldots\right). \qquad \text{.................(38)}$$

In like manner if $a > z$, we obtain

$$V_a = 2\pi\gamma\left(1 - \frac{z}{a} + \frac{1}{2}\frac{z^3}{a} - \frac{3}{8}\frac{z^5}{a^5} + \frac{5}{16}\frac{z^7}{a^7} - \ldots\right). \qquad \text{.................(39)}$$

Thus for points taken anywhere we get from (38) and (39)

$$\left.\begin{aligned} V &= 2\pi\gamma\left(\frac{1}{2}a^2\frac{Z_1}{r^2} - \frac{3}{8}a^4\frac{Z_3}{r^4} + \frac{5}{16}a^6\frac{Z_5}{r^6} - \ldots\right) \\ \text{or}\qquad V &= 2\pi\gamma\left(1 - \frac{r}{a}Z_1 + \frac{1}{2}\frac{r^3}{a^3}Z_3 - \frac{3}{8}\frac{r^5}{a^5}Z_5 + \ldots\right), \end{aligned}\right\} \qquad \text{..............(40)}$$

according as $a <$ or $> r$.

An example is the problem treated at VI. 23 above. Another example is given by the problem of two shells discussed in VI. 21 above.

The theorem used in equations (37) and (40) may be regarded as a limiting case of Green's theorem, that if a function of x, y, z is found to satisfy Laplace's equation throughout space external to a closed surface, and to give specified values for points on the surface, that function is the only one fulfilling these conditions. In the present case the closed surface is shrunk into a line, and in strictness the theorem requires special demonstration. Legendre's own proof will be found in Minchin's *Statics*, vol. ii. p. 341 (2nd ed.). The following proof given by Minchin, p. 324, *loc. cit.*, is simpler.

For the case of symmetry round an axis, if ζ be the distance of the point considered from the axis Laplace's equation takes the form

$$\zeta\left(\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial \zeta^2}\right) + \frac{\partial V}{\partial \zeta} = 0, \qquad \text{..............................(41)}$$

which it is to be noted gives

$$\frac{\partial V}{\partial \zeta}=0, \quad \frac{\partial^2 V}{\partial \zeta^2}=0 \quad \text{..................................}(42)$$

at all points on the axis in the space throughout which it is supposed that the equation holds.

If then we know a function U which satisfies (41), and gives the specified values at points on the axis, let if possible V be *another* function which does the same thing. Then $V-U$ $(=\Phi$ say) must fulfil (41), and be zero at points on the axis. Hence at all such points

$$\frac{\partial \Phi}{\partial z}=0, \quad \frac{\partial^2 \Phi}{\partial z^2}=0, \ \ldots .$$

We can now show that for any point on the axis

$$\frac{\partial^{m+n}\Phi}{\partial z^m d\zeta^n}=0.$$

For at any point on the axis we have seen [(42)] that $\partial\Phi/\partial\zeta=0$, $\partial^2\Phi/\partial\zeta^2=0$, and by (27) above

$$\frac{\partial^n}{\partial\zeta^n}\left\{\zeta\left(\frac{\partial^2\Phi}{\partial z^2}+\frac{\partial^2\Phi}{\partial\zeta^2}\right)+\frac{\partial\Phi}{\partial\zeta}\right\}$$
$$=n\frac{\partial^{n+1}\Phi}{\partial z^2\partial\zeta^{n-1}}+(n+1)\frac{\partial^{n+1}\Phi}{\partial\zeta^{n+1}}+\zeta\left(\frac{\partial^{n+2}\Phi}{\partial z^2\partial\zeta^n}+\frac{\partial^{n+2}\Phi}{\partial\zeta^{n+2}}\right)=0.$$

Hence, for points on the axis,

$$n\frac{\partial^2}{\partial z^2}\left(\frac{\partial^{n-1}\Phi}{\partial\zeta^{n-1}}\right)+(n+1)\frac{\partial^{n+1}\Phi}{\partial\zeta^{n+1}}=0.$$

If therefore $\partial^{n-1}\Phi/\partial\zeta^{n-1}=0$ for points on the axis, $\partial^{n+1}\Phi/\partial\zeta^{n+1}=0$. But $\partial\Phi/\partial\zeta=0$ and $\partial^2\Phi/\partial\zeta^2=0$, and therefore $\partial^3\Phi/\partial\zeta^3=0$, and so on. Hence it follows, since the differentiations are commutative, that $\partial^{m+n}\Phi/\partial z^m\partial\zeta^n=0$.

Expressing then Φ as $f(z, \zeta)$ and expanding by Maclaurin's theorem, denoting values of Φ, $\partial\Phi/\partial z$, etc., for points at the origin by the suffix 0, we get

$$\Phi=\Phi_0+z\frac{\partial\Phi}{\partial z_0}+\zeta\frac{\partial\Phi}{\partial\zeta_0}+\frac{1}{1\,.\,2}\left(z^2\frac{\partial^2\Phi}{\partial z_0{}^2}+2z\zeta\frac{\partial^2\Phi}{\partial z\partial\zeta}+\zeta^2\frac{\partial^2\Phi}{\partial\zeta_0{}^2}\right)+\text{etc.}=0,$$

since all the differential coefficients vanish.

Hence $\Phi=0$, everywhere, which proves that U cannot differ from V.

It is shown above, p. 212, that for any integral value of i

$$(-1)^{i+1}(i-1)!\,a^2\int\frac{Z_i'}{r^{i+2}}\,dx=\frac{\partial^i A}{\partial x^i},$$

where Z_i is a zonal harmonic of order i, $x=\mu r$, and $A=\sqrt{a^2+x^2}-x$. The evaluation of these integrals is of great importance for the calculation of the inductances of coils, and by this theorem they can be obtained at once by simply finding the successive differential coefficients of A. As promised we give here the first eleven differential coefficients. It may be noted that they can be written down with great facility from the known expressions for the successive zonal harmonics by the equation

$$(-1)^{i+1}(i-1)!\,a^2\frac{Z'^{i}}{r^{i+1}}=\frac{\partial^{i+1}A}{\partial x^{i+1}}\;.^{*}$$

$$\frac{\partial A}{\partial x}=\frac{x}{r}-1,\quad -\frac{\partial^2 A}{\partial x^2}=\frac{a^2}{r^3}.$$

$$\frac{\partial^3 A}{\partial x^3}=-\frac{3a^2x}{r^5},\quad \frac{\partial^4 A}{\partial x^4}=3a^2(5x^2-r^2)\frac{1}{r^7}.$$

$$\frac{\partial^5 A}{\partial x^5}=-3\,.\,5a^2x(7x^2-3r^2)\frac{1}{r^9}.$$

$$\frac{\partial^6 A}{\partial x^6}=3^2\,.\,5a^2(21x^4-14x^2r^2+r^4)\frac{1}{r^{11}}.$$

$$\frac{\partial^7 A}{\partial x^7}=-3^2\,.\,5a^2x(231x^4-210x^2r^2+35r^4)\frac{1}{r^{13}}.$$

$$\frac{\partial^8 A}{\partial x^8}=3^2\,.\,5a^2(3003x^6-3465x^4r^2+945x^2r^4-35r^6)\frac{1}{r^{15}}.$$

$$\frac{\partial^9 A}{\partial x^9}=-3^2\,.\,5\,.\,7a^2x(6435x^6-9009x^4r^2-3465x^2r^4-315r^6)\frac{1}{r^{17}}.$$

$$\frac{\partial^{10} A}{\partial x^{10}}=3^3\,.\,5^2\,.\,7a^2(7293x^8-12012x^6r^2+6006x^4r^4-924x^2r^6+21r^8)\frac{1}{r^{19}}.$$

$$\frac{\partial^{11} A}{\partial x^{11}}=-3^2\,.\,5^2\,.\,7\,.\,9a^2x(46189x^8-87516x^6r^2+54054x^4r^4-12012x^2r^6+693r^8)\frac{1}{r^{21}}.$$

---

* It is to be remembered that in the table here given $\partial^i A_0/\partial x^i$ has the meaning of $-\partial^{i-1}A_0/\partial x^{i-1}$ in VI. 24, but agrees with $\partial^i A_0/\partial x^i$ in VII. [See the note on p. 209.]

TABLE OF ZONAL SPHERICAL HARMONICS

(Prof. Perry, *Phil. Mag.* Dec. 1891. See also p. 824 above.)

θ	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
0°	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
1	·9998	·9995	·9991	·9985	·9977	·9967	·9955
2	·9994	·9982	·9963	·9939	·9909	·9872	·9829
3	·9986	·9959	·9918	·9863	·9795	·9713	·9617
4	·9976	·9927	·9854	·9758	·9638	·9495	·9329
5	·9962	·9886	·9773	·9623	·9437	·9216	·8961
6	·9945	·9836	·9674	·9459	·9194	·8881	·8522
7	·9925	·9777	·9557	·9267	·8911	·8476	·7986
8	·9903	·9709	·9423	·9048	·8589	·8053	·7448
9	·9877	·9633	·9273	·8803	·8232	·7571	·6831
10	·9848	·9548	·9106	·8532	·7840	·7045	·6164
11	·9816	·9454	·8923	·8238	·7417	·6843	·5461
12	·9781	·9352	·8724	·7920	·6966	·5892	·4732
13	·9744	·9241	·8511	·7582	·6489	·5273	·3940
14	·9703	·9122	·8283	·7224	·5990	·4635	·3219
15	·9659	·8995	·8042	·6847	·5471	·3982	·2454
16	·9613	·8860	·7787	·6454	·4937	·3322	·1699
17	·9563	·8718	·7519	·6046	·4391	·2660	·0961
18	·9511	·8568	·7240	·5624	·3836	·2002	·0289
19	·9455	·8410	·6950	·5192	·3276	·1347	−·0443
20	·9397	·8245	·6649	·4750	·2715	·0719	−·1072
21	·9336	·8074	·6338	·4300	·2156	·0107	−·1662
22	·9272	·7895	·6019	·3845	·1602	−·0481	−·2201
23	·9205	·7710	·5692	·3386	·1057	−·1038	−·2681
24	·9135	·7518	·5357	·2926	·0525	−·1559	−·3095
25	·9063	·7321	·5016	·2465	·0009	−·2053	−·3463
26	·8988	·7117	·4670	·2007	−·0489	−·2478	−·3717
27	·8910	·6908	·4319	·1553	−·0964	−·2869	−·3921
28	·8829	·6694	·3964	·1105	−·1415	−·3211	−·4052
29	·8746	·6474	·3607	·0665	−·1839	−·3503	−·4114
30	·8660	·6250	·3248	·0234	−·2233	−·3740	−·4101
31	·8572	·6021	·2887	−·0185	−·2595	−·3924	−·4022
32	·8480	·5788	·2527	−·0591	−·2923	−·4052	−·3876
33	·8387	·5551	·2167	−·0982	−·3216	−·4126	−·3670
34	·8290	·5310	·1809	−·1357	−·3473	−·4148	−·3409
35	·8192	·5065	·1454	−·1714	−·3691	−·4115	−·3096
36	·8090	·4818	·1102	−·2052	−·3871	−·4031	−·2738
37	·7986	·4567	·0755	−·2370	−·4011	−·3898	−·2343
38	·7880	·4314	·0413	−·2666	−·4112	−·3719	−·1918
39	·7771	·4059	·0077	−·2940	−·4174	−·3497	−·1469
40	·7660	·3802	−·0252	−·3190	−·4197	−·3234	−·1003
41	·7547	·3544	−·0574	−·3416	−·4181	−·2938	−·0534
42	·7431	·3284	−·0887	−·3616	−·4128	−·2611	−·0065
43	·7314	·3023	−·1191	−·3791	−·4038	−·2255	·0398
44	·7193	·2762	−·1485	−·3940	−·3914	−·1878	·0846
45	·7071	·2500	−·1768	−·4062	−·3757	−·1485	·1270

TABLE OF ZONAL SPHERICAL HARMONICS (*continued*)

θ	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
46	·6947	·2238	−·2040	−·4158	−·3568	−·1079	·1666
47	·6820	·1977	−·2300	−·4252	−·3350	−·0645	·2054
48	·6691	·1716	−·2547	−·4270	−·3105	−·0251	·2349
49	·6561	·1456	−·2781	−·4286	−·2836	+·0161	·2627
50	·6428	·1198	−·3002	−·4275	−·2545	+·0563	·2854
51	·6293	·0941	−·3209	−·4239	−·2235	+·0954	·3031
52	·6157	·0686	−·3401	−·4178	−·1910	+·1326	·3153
53	·6018	·0433	−·3578	−·4093	−·1571	+·1677	·3221
54	·5878	·0182	−·3740	−·3984	−·1223	+·2002	·3234
55	·5736	−·0065	−·3886	−·3852	−·0868	+·2297	·3191
56	·5592	−·0310	−·4016	−·3698	−·0510	+·2559	·3095
57	·5446	−·0551	−·4131	−·3524	−·0150	+·2787	·2949
58	·5299	−·0788	−·4229	−·3331	·0206	+·2976	·2752
59	·5150	−·1021	−·4310	−·3119	·0557	+·3125	·2511
60	·5000	−·1250	−·4375	−·2891	·0898	+·3232	·2231
61	·4848	−·1474	−·4423	−·2647	·1229	+·3298	·1916
62	·4695	−·1694	−·4455	−·2390	·1545	+·3321	·1571
63	·4540	−·1908	−·4471	−·2121	·1844	+·3302	·1203
64	·4384	−·2117	−·4470	−·1841	·2123	+·3240	·0818
65	·4226	−·2321	−·4452	−·1552	·2381	+·3138	·0422
66	·4067	−·2518	−·4419	−·1256	·2615	+·2996	·0021
67	·3907	−·2710	−·4370	−·0955	·2824	+·2819	−·0375
68	·3746	−·2896	−·4305	−·0650	·3005	+·2605	−·0763
69	·3584	−·3074	−·4225	−·0344	·3158	+·2361	−·1135
70	·3420	−·3245	−·4130	·0038	·3281	+·2089	−·1485
71	·3256	−·3410	−·4021	·0267	·3373	+·1786	−·1811
72	·3090	−·3568	−·3898	·0568	·3434	+·1472	−·2099
73	·2924	−·3718	−·3761	·0864	·3463	+·1144	−·2347
74	·2756	−·3860	−·3611	·1153	·3461	+·0795	−·2559
75	·2588	−·3995	−·3449	·1434	·3427	+·0431	−·2730
76	·2419	−·4112	−·3275	·1705	·3362	+·0076	−·2848
77	·2250	−·4241	−·3090	·1964	·3267	−·0284	−·2919
78	·2079	−·4352	−·2894	·2211	·3143	−·0644	−·2943
79	·1908	−·4454	−·2688	·2443	·2990	−·0989	−·2913
80	·1736	−·4548	−·2474	·2659	·2810	−·1321	−·2835
81	·1564	−·4633	−·2251	·2859	·2606	−·1635	−·2709
82	·1392	−·4709	−·2020	·3040	·2378	−·1926	−·2536
83	·1219	−·4777	−·1783	·3203	·2129	−·2193	−·2321
84	·1045	−·4836	−·1539	·3345	·1861	−·2431	−·2067
85	·0872	−·4886	−·1291	·3468	·1577	−·2638	−·1779
86	·0698	−·4927	−·1038	·3569	·1278	−·2811	−·1460
87	·0523	−·4959	−·0781	·3648	·0969	−·2947	−·1117
88	·0349	−·4982	−·0522	·3704	·0651	−·3045	−·0735
89	·0175	−·4995	−·0262	·3739	·0327	−·3105	−·0381
90	·0000	−·5000	−·0000	·3750	·0000	−·3125	·0000

APPENDIX III.

TABLE FOR THE CALCULATION OF THE MUTUAL INDUCTANCE M OF TWO COAXIAL CIRCLES OF RADII a, a', AND DISTANCE APART b.

Calculated for intervals of 6′ in the value of

$$\cos^{-1}\{\sqrt{(a-a')^2+b^2}/\sqrt{(a+a')^2+b^2}\} \text{ from } 60° \text{ to } 90°.$$

[This table is taken from Maxwell's *Electricity and Magnetism.* It was lately recalculated by the Bureau of Standards at Washington, and had been set up before this was noticed. Some of the terminating figures of the logarithms were found to be in error, and some corrections have been made so as to render the largest discrepance in the last figures not greater than 3 or 4.]

		$\log_{10}\frac{M}{4\pi\sqrt{aa'}}$			$\log_{10}\frac{M}{4\pi\sqrt{aa'}}$			$\log_{10}\frac{M}{4\pi\sqrt{aa'}}$
60°	0′	$\bar{1}{\cdot}4994783$	64°	0′	$\bar{1}{\cdot}6101472$	68°	0′	$\bar{1}{\cdot}7203000$
	6′	$\bar{1}{\cdot}5022651$		6′	$\bar{1}{\cdot}6128998$		6′	$\bar{1}{\cdot}7230635$
	12′	$\bar{1}{\cdot}5050505$		12′	$\bar{1}{\cdot}6156522$		12′	$\bar{1}{\cdot}7258281$
	18′	$\bar{1}{\cdot}5078345$		18′	$\bar{1}{\cdot}6184042$		18′	$\bar{1}{\cdot}7285940$
	24′	$\bar{1}{\cdot}5106173$		24′	$\bar{1}{\cdot}6211560$		24′	$\bar{1}{\cdot}7313604$
	30′	$\bar{1}{\cdot}5133989$		30′	$\bar{1}{\cdot}6239076$		30′	$\bar{1}{\cdot}7341283$
	36′	$\bar{1}{\cdot}5161791$		36′	$\bar{1}{\cdot}6266589$		36′	$\bar{1}{\cdot}7368975$
	42′	$\bar{1}{\cdot}5189582$		42′	$\bar{1}{\cdot}6294101$		42′	$\bar{1}{\cdot}7396675$
	48′	$\bar{1}{\cdot}5217361$		48′	$\bar{1}{\cdot}6321612$		48′	$\bar{1}{\cdot}7424387$
	54′	$\bar{1}{\cdot}5245124$		54′	$\bar{1}{\cdot}6349121$		54′	$\bar{1}{\cdot}7452111$
61°	0′	$\bar{1}{\cdot}5272880$	65°	0′	$\bar{1}{\cdot}6376629$	69°	0′	$\bar{1}{\cdot}7479848$
	6′	$\bar{1}{\cdot}5300620$		6′	$\bar{1}{\cdot}6404137$		6′	$\bar{1}{\cdot}7507597$
	12′	$\bar{1}{\cdot}5328351$		12′	$\bar{1}{\cdot}6431645$		12′	$\bar{1}{\cdot}7535361$
	18′	$\bar{1}{\cdot}5356080$		18′	$\bar{1}{\cdot}6459153$		18′	$\bar{1}{\cdot}7563138$
	24′	$\bar{1}{\cdot}5383796$		24′	$\bar{1}{\cdot}6486660$		24′	$\bar{1}{\cdot}7590929$
	30′	$\bar{1}{\cdot}5411498$		30′	$\bar{1}{\cdot}6514169$		30′	$\bar{1}{\cdot}7618735$
	36′	$\bar{1}{\cdot}5439190$		36′	$\bar{1}{\cdot}6541678$		36′	$\bar{1}{\cdot}7646556$
	42′	$\bar{1}{\cdot}5466872$		42′	$\bar{1}{\cdot}6569189$		42′	$\bar{1}{\cdot}7674392$
	48′	$\bar{1}{\cdot}5494545$		48′	$\bar{1}{\cdot}6596701$		48′	$\bar{1}{\cdot}7702245$
	54′	$\bar{1}{\cdot}5522209$		54′	$\bar{1}{\cdot}6624215$		54′	$\bar{1}{\cdot}7730114$
62°	0′	$\bar{1}{\cdot}5549864$	66°	0′	$\bar{1}{\cdot}6651732$	70°	0′	$\bar{1}{\cdot}7758000$
	6′	$\bar{1}{\cdot}5577510$		6′	$\bar{1}{\cdot}6679250$		6′	$\bar{1}{\cdot}7785903$
	12′	$\bar{1}{\cdot}5605147$		12′	$\bar{1}{\cdot}6706772$		12′	$\bar{1}{\cdot}7813823$
	18′	$\bar{1}{\cdot}5632776$		18′	$\bar{1}{\cdot}6734296$		18′	$\bar{1}{\cdot}7841762$
	24′	$\bar{1}{\cdot}5660398$		24′	$\bar{1}{\cdot}6761824$		24′	$\bar{1}{\cdot}7869720$
	30′	$\bar{1}{\cdot}5688011$		30′	$\bar{1}{\cdot}6789356$		30′	$\bar{1}{\cdot}7897696$
	36′	$\bar{1}{\cdot}5715618$		36′	$\bar{1}{\cdot}6816891$		36′	$\bar{1}{\cdot}7925692$
	42′	$\bar{1}{\cdot}5743217$		42′	$\bar{1}{\cdot}6844431$		42′	$\bar{1}{\cdot}7953709$
	48′	$\bar{1}{\cdot}5770809$		48′	$\bar{1}{\cdot}6871976$		48′	$\bar{1}{\cdot}7981745$
	54′	$\bar{1}{\cdot}5798390$		54′	$\bar{1}{\cdot}6899526$		54′	$\bar{1}{\cdot}8009803$
63°	0′	$\bar{1}{\cdot}5825963$	67°	0′	$\bar{1}{\cdot}6927074$	71°	0′	$\bar{1}{\cdot}8037882$
	6′	$\bar{1}{\cdot}5853536$		6′	$\bar{1}{\cdot}6954635$		6′	$\bar{1}{\cdot}8065983$
	12′	$\bar{1}{\cdot}5881103$		12′	$\bar{1}{\cdot}6982202$		12′	$\bar{1}{\cdot}8094107$
	18′	$\bar{1}{\cdot}5908665$		18′	$\bar{1}{\cdot}7009775$		18′	$\bar{1}{\cdot}8122253$
	24′	$\bar{1}{\cdot}5936221$		24′	$\bar{1}{\cdot}7037355$		24′	$\bar{1}{\cdot}8150423$
	30′	$\bar{1}{\cdot}5963780$		30′	$\bar{1}{\cdot}7064942$		30′	$\bar{1}{\cdot}8178617$
	36′	$\bar{1}{\cdot}5991322$		36′	$\bar{1}{\cdot}7092540$		36′	$\bar{1}{\cdot}8206836$
	42′	$\bar{1}{\cdot}6018871$		42′	$\bar{1}{\cdot}7120140$		42′	$\bar{1}{\cdot}8235080$
	48′	$\bar{1}{\cdot}6046408$		48′	$\bar{1}{\cdot}7147750$		48′	$\bar{1}{\cdot}8263349$
	54′	$\bar{1}{\cdot}6073942$		54′	$\bar{1}{\cdot}7175370$		54′	$\bar{1}{\cdot}8291645$

		$\log_{10} \frac{M}{4\pi\sqrt{aa'}}$			$\log_{10} \frac{M}{4\pi\sqrt{aa'}}$			$\log_{10} \frac{M}{4\pi\sqrt{aa'}}$
72°	0′	$\bar{1}$·8319974	78°	0′	·0094959	84°	0′	·2217823
	6′	$\bar{1}$·8348323		6′	·0126385		6′	·2259728
	12′	$\bar{1}$·8376700		12′	·0157896		12′	·2301983
	18′	$\bar{1}$·8405106		18′	·0189494		18′	·2344600
	24′	$\bar{1}$·8433541		24′	·0221181		24′	·2387591
	30′	$\bar{1}$·8462005		30′	·0252959		30′	·2430970
	36′	$\bar{1}$·8490499		36′	·0284830		36′	·2474748
	42′	$\bar{1}$·8519024		42′	·0316794		42′	·2518940
	48′	$\bar{1}$·8547580		48′	·0348855		48′	·2563561
	54′	$\bar{1}$·8576164		54′	·0381014		54′	·2608626
73°	0′	$\bar{1}$·8604785	79°	0′	·0413273	85°	0′	·2654152
	6′	$\bar{1}$·8633440		6′	·0445633		6′	·2700156
	12′	$\bar{1}$·8662129		12′	·0478098		12′	·2746655
	18′	$\bar{1}$·8690852		18′	·0510668		18′	·2793670
	24′	$\bar{1}$·8719611		24′	·0543347		24′	·2841221
	30′	$\bar{1}$·8748406		30′	·0576136		30′	·2889329
	36′	$\bar{1}$·8777237		36′	·0609037		36′	·2938018
	42′	$\bar{1}$·8806106		42′	·0642054		42′	·2987312
	48′	$\bar{1}$·8835013		48′	·0675187		48′	·3037238
	54′	$\bar{1}$·8863958		54′	·0708441		54′	·3087823
74°	0′	$\bar{1}$·8892943	80°	0′	·0741816	86°	0′	·3139097
	6′	$\bar{1}$·8921969		6′	·0775316		6′	·3191092
	12′	$\bar{1}$·8951036		12′	·0808944		12′	·3243843
	18′	$\bar{1}$·8980144		18′	·0842702		18′	·3297387
	24′	$\bar{1}$·9009295		24′	·0876592		24′	·3351762
	30′	$\bar{1}$·9038489		30′	·0910619		30′	·3407012
	36′	$\bar{1}$·9067728		36′	·0944784		36′	·3463184
	42′	$\bar{1}$·9097012		42′	·0979091		42′	·3520327
	48′	$\bar{1}$·9126341		48′	·1013542		48′	·3578495
	54′	$\bar{1}$·9155717		54′	·1048142		54′	·3637749
75°	0′	$\bar{1}$·9185141	81°	0′	·1082893	87°	0′	·3698153
	6′	$\bar{1}$·9214613		6′	·1117799		6′	·3759777
	12′	$\bar{1}$·9244135		12′	·1152863		12′	·3822700
	18′	$\bar{1}$·9273707		18′	·1188089		18′	·3887006
	24′	$\bar{1}$·9303330		24′	·1223481		24′	·3952792
	30′	$\bar{1}$·9333005		30′	·1259043		30′	·4020162
	36′	$\bar{1}$·9362733		36′	·1294778		36′	·4089234
	42′	$\bar{1}$·9392515		42′	·1330691		42′	·4160138
	48′	$\bar{1}$·9422352		48′	·1366786		48′	·4233022
	54′	$\bar{1}$·9452246		54′	·1403067		54′	·4308053
76°	0′	$\bar{1}$·9482196	82°	0′	·1439539	88°	0′	·4385420
	6′	$\bar{1}$·9512205		6′	·1476207		6′	·4465341
	12′	$\bar{1}$·9542272		12′	·1513075		12′	·4548064
	18′	$\bar{1}$·9572400		18′	·1550149		18′	·4633880
	24′	$\bar{1}$·9602590		24′	·1587434		24′	·4723127
	30′	$\bar{1}$·9632841		30′	·1624935		30′	·4816206
	36′	$\bar{1}$·9663157		36′	·1662658		36′	·4913595
	42′	$\bar{1}$·9693537		42′	·1700609		42′	·5015870
	48′	$\bar{1}$·9723983		48′	·1738794		48′	·5123738
	54′	$\bar{1}$·9754497		54′	·1777219		54′	·5238079
77°	0′	$\bar{1}$·9785079	83°	0′	·1815890	89°	0′	·5360007
	6′	$\bar{1}$·9815731		6′	·1854815		6′	·5490969
	12′	$\bar{1}$·9846454		12′	·1894001		12′	·5632886
	18′	$\bar{1}$·9877249		18′	·1933455		18′	·5788406
	24′	$\bar{1}$·9908118		24′	·1973184		24′	·5961320
	30′	$\bar{1}$·9939062		30′	·2013197		30′	·6157370
	36′	$\bar{1}$·9970082		36′	·2053502		36′	·6385907
	42′	·0001181		42′	·2094108		42′	·6663883
	48′	·0032359		48′	·2135026		48′	·7027765
	54′	·0063618		54′	·2176259		54′	·7586941

APPENDIX IV.

TABLE OF ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KIND (G AND H IN THE NOTATION OF THIS BOOK, F AND E IN THE USUAL NOTATION), FROM LEGENDRE, *Traité des Fonctions Elliptiques*, Tome II.

	G or *F*.	Δ_1	*H* or *E*.	Δ_1
0°	1·570 796	120	1·570 796	− 120
1	1·570 916	359	1·570 677	− 359
2	1·571 275	599	1·570 318	− 598
3	1·571 874	839	1·569 720	− 836
4	1·572 712	1 080	1·568 884	−1 075
5	1·573 792	1 321	1·567 809	−1 312
6	1·575 114	1 564	1·566 497	−1 549
7	1·576 678	1 808	1·564 948	−1 785
8	1·578 486	2 054	1·563 162	−2 020
9	1·580 541	2 302	1·561 142	−2 255
10	1·582 843	2 551	1·558 887	−2 487
11	1·585 394	2 803	1·556 400	−2 719
12	1·588 197	3 057	1·553 681	−2 949
13	1·591 254	3 314	1·550 732	−3 177
14	1·594 568	3 574	1·547 554	−3 404
15	1·598 142	3 836	1·544 150	−3 629
16	1·601 978	4 103	1·540 521	−3 852
17	1·606 081	4 373	1·536 670	−4 073
18	1·610 454	4 647	1·532 597	−4 291
19	1·615 101	4 925	1·528 306	−4 507
20	1·620 026	5 208	1·523 799	−4 721
21	1·625 234	5 495	1·519 079	−4 932
22	1·630 729	5 788	1·514 147	−5 140
23	1·636 517	6 087	1·509 007	−5 345
24	1·642 604	6 391	1·503 662	−5 547
25	1·648 995	6 702	1·498 115	−5 746
26	1·655 697	7 019	1·492 368	−5 942
27	1·662 716	7 343	1·486 427	−6 134
28	1·670 059	7 675	1·480 293	−6 323
29	1·677 735	8 015	1·473 970	−6 508
30	1·685 750	8 364	1·467 462	−6 689
31	1·694 114	8 722	1·460 774	−6 866
32	1·702 836	9 089	1·453 908	−7 039
33	1·711 925	9 466	1·446 869	−7 207
34	1·721 391	9 854	1·439 662	−7 371
35	1·731 245	10 254	1·432 291	−7 531
36	1·741 499	10 666	1·424 760	−7 685
37	1·752 165	11 091	1·417 075	−7 835
38	1·763 256	11 530	1·409 240	−7 980
39	1·774 786	11 982	1·401 260	−8 120
40	1·786 770	12 452	1·393 140	−8 254
41	1·799 222	12 938	1·384 886	−8 382
42	1·812 160	13 442	1·376 504	−8 305
43	1·825 602	13 965	1·367 999	−8 622
44	1·839 567	14 508	1·359 377	−8 733
45	1·854 075	15 073	1·350 644	−8 838

TABLE OF ELLIPTIC INTEGRALS (*continued*)

°	*G* or *F*.	Δ_1	*H* or *E*.	Δ_1
45	1·854 075	15 073	1·350 644	−8 838
46	1·869 148	15 661	1·341 806	−8 936
47	1·884 809	16 274	1·332 870	−9 028
48	1·901 083	16 914	1·323 842	−9 113
49	1·917 997	17 584	1·314 729	−9 190
50	1·935 581	18 284	1·305 539	−9 261
51	1·953 865	19 017	1·296 278	−9 324
52	1·972 882	19 787	1·286 954	−9 380
53	1·992 670	20 597	1·277 574	−9 427
54	2·013 266	21 449	1·268 147	−9 467
55	2·034 715	22 347	1·258 680	−9 498
56	2·057 062	23 296	1·249 182	−9 520
57	2·080 358	24 300	1·239 661	−9 534
58	2·104 658	25 364	1·230 127	−9 538
59	2·130 021	26 494	1·220 589	−9 533
60	2·156 516	27 698	1·211 056	−9 518
61	2·184 213	28 982	1·201 538	−9 492
62	2·213 195	30 355	1·192 046	−9 457
63	2·243 549	31 827	1·182 589	−9 410
64	2·275 376	33 410	1·173 180	−9 351
65	2·308 787	35 118	1·163 828	−9 281
66	2·343 905	36 965	1·154 547	−9 199
67	2·380 870	38 971	1·145 348	−9 104
68	2·419 842	41 158	1·136 244	−8 995
69	2·460 999	43 551	1·127 250	−8 872
70	2·504 550	46 181	1·118 378	−8 734
71	2·550 731	49 088	1·109 643	−8 581
72	2·599 820	52 318	1·101 062	−8 412
73	2·652 138	55 930	1·092 650	−8 225
74	2·708 068	59 996	1·084 425	−8 020
75	2·768 063	64 609	1·076 415	−7 796
76	2·832 673	69 892	1·068 610	−7 550
77	2·902 565	76 004	1·061 059	−7 282
78	2·978 569	83 160	1·053 777	−6 990
79	3·061 729	91 657	1·046 786	−6 672
80	3·153 385	101 918	1·040 114	−6 325
81	3·255 303	114 565	1·033 789	−5 946
82	3·369 868	130 554	1·027 844	−5 431
83	3·500 422	151 433	1·022 313	−5 076
84	3·651 856	179 886	1·017 237	−4 573
85	3·831 742	221 016	1·012 664	−4 016
86	4·052 758	285 896	1·008 648	−3 389
87	4·338 654	404 063	1·005 259	−2 675
88	4·742 717	692 193	1·002 584	−1 832
89	5·434 910		1·000 752	− 752
90			1·000 000	

APPENDIX V.

MUTUAL INDUCTANCE OF NON-COAXIAL CIRCLES.

[From a paper by S. Butterworth, *Phil. Mag.* xxxi. May, 1916.]

THE formulae for inductances given above are for the most part confined to the cases of coaxial coils. But by placing the coils so that the axes are at different distances r apart, while the condition of parallelism is still fulfilled, the mutual inductance of two coils can be varied through a wide range of values. We give here formulae for two current carrying circles which are useful in themselves, and from which results for other and more complex cases can be deduced.

I. EQUAL CIRCLES.

We take x as the distance of the planes of the circles apart, and θ as the angle $\cos^{-1}\{x/\sqrt{r^2+x^2}\}$.

When x is large and the circles are coaxial, the mutual inductance is

$$M_0 = \frac{2\pi^2}{x^3}\left(1 - \frac{3}{4x^2} + \frac{25}{24x^4} - \frac{245}{128x^6} + \dots\right). \quad \dots\dots\dots\dots\dots(1)$$

The n^{th} term is here obtained from the $(n-1)^{\text{th}}$ by multiplying by

$$-\frac{1}{x^2}\left(\frac{2n-1}{n}\right)^2 \frac{n-1}{n+1}.$$

If P be the zonal harmonic of order n, we have

$$M = \frac{2\pi^2}{r^3}\left(P_2 - \frac{3}{4}\frac{P_4}{r^2} + \frac{25}{24}\frac{P_6}{r^4} - \frac{245}{128}\frac{P_8}{r^6} + \dots\right). \quad \dots\dots\dots\dots(2)$$

For coplanar circles this becomes (since $\theta = \frac{1}{2}\pi$)

$$M_1 = -\frac{\pi^2}{r^3}\left(1 + \frac{9}{16r^2} + \frac{125}{192r^4} + \frac{8575}{8192r^6} + \dots\right). \quad \dots\dots\dots\dots(3)$$

In this last formula the multiplying factor for successive terms is

$$\frac{1}{2r^2}\left(\frac{2n-1}{n}\right)^2 \frac{n-1}{n+1}.$$

For convergence of the preceding series it is necessary that $r > 2$. For small values of x and coaxial circles the mutual inductance is (see p. 199 above) for radius A unity,

$$M_0 = 4\pi\left\{\lambda_0 - 2 + \frac{3}{16}x^2\left(\lambda_0 - \frac{1}{3}\right) - \frac{15x^4}{1024}\left(\lambda_0 - \frac{31}{30}\right) + \frac{35}{(128)^2}x^6\left(\lambda_0 - \frac{247}{210}\right) - \dots\right\}, \quad \dots\dots\dots\dots\dots(4)$$

where λ_0 is written for $\log(8/x)$. For circles which are not coaxial this formula gives the value of the mutual inductance if x^n is replaced by $r^n P_n$ and $x^n \log x$ by

$$\frac{\partial}{\partial n}(r^n P_n) = r^n\left(P_n \log r + \frac{\partial P}{\partial n}\right);$$

since these satisfy Laplace's equation, and since they reduce to x^n and $x^n \log x$ respectively, when $\theta = 0$, that is, when the circles are coaxial.

In applying this transformation we write

$$\frac{\partial P_n}{\partial n} = P_n \log\{\tfrac{1}{2}(1+\mu)\} + \psi_n \quad [\mu = \cos\theta], \quad \dots\dots\dots\dots(5)$$

where $$\psi_n = 2\left\{\frac{2n-1}{1 \cdot 2n}(P_n - P_{n-1}) - \frac{2n-3}{2(2n-1)}(P_n - P_{n-2}) + \dots + (-1)^{n+1}\frac{1}{n(n+1)}(P_n - P_0)\right\}. \quad \dots\dots(6)$$

In particular,

$$\left.\begin{aligned} &\psi_0 = 0, \\ &\psi_2 = -\tfrac{1}{4}(1-\mu)(1+7\mu), \\ &\psi_4 = \tfrac{1}{96}(1-\mu)(21 + 241\mu - 113\mu^2 - 533\mu^3), \\ &\psi_6 = -\tfrac{1}{960}(1-\mu)(185 - 2957\mu + 3728\mu^2 \\ &\qquad\qquad + 18008\mu^3 - 3247\mu^4 - 18107\mu^5). \end{aligned}\right\} \dots(6a)$$

This gives

$$\frac{M}{4\pi} = \lambda - 2 + \frac{3}{16}r^2\left\{P_2\left(\lambda - \frac{1}{3}\right) - \psi_2\right\} - \frac{15}{1024}r^4\left\{P_4\left(\lambda - \frac{31}{30}\right) - \psi_4\right\} + \frac{35}{(128)^2}r^6\left\{P_6\left(\lambda - \frac{247}{210}\right) - \psi_6\right\} - \dots, \quad \dots\dots(7)$$

in which $\lambda = \log_e \dfrac{16}{r(1+\mu)}$.

It is convenient to write

$$\frac{M}{4\pi} = \alpha_0 + \alpha_1 r^2 + \alpha_2 r^4 + \alpha_3 r^6 + \dots - \log_e r\,(1 + \beta_1 r^2 + \beta_2 r^4 + \beta_3 r^6 + \dots), \quad \dots(8)$$

and to tabulate $\alpha_0, \alpha_1, \dots, \beta_1 \dots$ as functions of μ. This is done in Table I.

When $\mu=0$ the circles are coplanar and (7) reduces to

$$\frac{M_1}{4\pi}=\lambda_1-2-\frac{3}{32}r^2\left(\lambda_1-\frac{5}{6}\right)-\frac{45}{8192}r^4\left(\lambda_1-\frac{97}{60}\right)-\frac{175}{(512)^2}r^6\left(\lambda_1-\frac{251}{140}\right)-\dots,\qquad(9)$$

in which $\lambda_1=\log_e\frac{16}{r}$.

Formulae (4), (7) and (9) converge fairly rapidly so long as r is less than unity. They will give a rough approximation up to $r=1\cdot6$, but fail for larger values.

TABLE I.

VALUES OF COEFFICIENTS IN FORMULA (8).

μ	a_0	a_1	a_2	a_3	β_1	β_2	β_3
0·0	0·7726	−0·1817	−0·00487	−0·00066	−0·0938	−0·00549	−0·00067
0·1	0·6773	−0·1414	−0·00217	−0·00009	−0·0909	−0·00495	−0·00053
0·2	0·5903	−0·0962	+0·00208	+0·00071	−0·0825	−0·00340	−0·00017
0·3	0·5102	−0·0473	+0·00577	+0·00107	−0·0684	−0·00107	+0·00028
0·4	0·4361	+0·0044	+0·00829	+0·00098	−0·0488	+0·00166	+0·00063
0·5	0·3671	+0·0578	+0·00921	+0·00045	−0·0234	+0·00424	+0·00069
0·6	0·3026	+0·1123	+0·00818	−0·00021	+0·0075	+0·00597	+0·00037
0·7	0·2420	+0·1671	+0·00507	−0·00090	+0·0441	+0·00604	−0·00027
0·8	0·1848	+0·2216	−0·00006	−0·00098	+0·0862	+0·00341	−0·00084
0·9	0·1308	+0·2752	−0·00701	−0·00019	+0·1341	−0·00305	−0·00052
1·0	0·0795	+0·3274	−0·01531	+0·0017	+0·1875	−0·01462	+0·00214

Table II. gives values of M from the coaxial position to the position where M is zero.

TABLE II.

MUTUAL INDUCTION BETWEEN EQUAL PARALLEL CIRCLES.

Radii of circles = unit of length.

x = distance of planes ; ρ = distance of axes.

$x=0$			$x=0\cdot1$		$x=0\cdot25$		$x=0\cdot5$	
ρ	$M/4\pi$	μ	ρ	$M/4\pi$	ρ	$M/4\pi$	ρ	$M/4\pi$
0	infinity	1·0	0	2·39	0	1·50	0	0·88
0·2	2·37	0·9	0·05	2·34	0·121	1·45	0·24	0·83
0·4	1·65	0·8	0·075	2·27	0·188	1·38	0·37	0·76
0·6	1·20	0·7	0·102	2·19	0·256	1·30	0·51	0·67
0·8	0·86	0·6	0·133	2·10	0·334	1·20	0·67	0·57
1·0	0·58	0·5	0·173	1·98	0·433	1·07	0·87	0·43
1·2	0·33	0·4	0·229	1·82	0·573	0·90	1·15	0·26
1·4	0·12	0·3	0·318	1·60	0·796	0·65	1·59	0·03
1·6	−0·17	0·25	...	...	...	...	1·94	−0·10
1·8	...	0·2	0·49	1·24	1·225	0·26	...	...
2·0	−0·43	0·125	...	...	1·98	−0·24	...	...
		0·1	1·00	0·53	...	...	...	...
		0·05	2·00	−0·36	...	...	...	...

II. Unequal Circles.

Let the radii of the circles be a and A, and let x be the distance of their planes. Then if x is large and the circles are coaxial, the mutual induction is

$$M_0 = \frac{2\pi^2 a^2 A^2}{x^3}\left(1 - \frac{3}{2}\frac{A^2}{x^2}K_1 + \frac{3.5}{2.4}\frac{A^4}{x^4}K_2 - \frac{3.5.7}{2.4.6}\frac{A^6}{x^6}K_3 + \dots\right), \quad \dots(10)$$

in which

$$K_1 = 1 + \frac{a^2}{A^2},$$

$$K_2 = 1 + 3\frac{a^2}{A^2} + \frac{a^4}{A^4},$$

$$K_3 = 1 + 6\frac{a^2}{A^2} + 6\frac{a^4}{A^4} + \frac{a^6}{A^6},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$K_n = F\left(-n-1,\ -n,\ 2,\ \frac{a^2}{A^2}\right),$$

where $F(\alpha, \beta, \gamma, z)$ denotes the hypergeometric series

$$1 + \frac{\alpha\beta}{1.\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2\gamma(\gamma+1)}z^2 + \dots .$$

Hence for non-coaxial circles the mutual induction is

$$M = \frac{2\pi^2 a^2 A^2}{r^3}\left(P_2 - \frac{3}{2}\frac{A^2}{r^2}K_1 P_4 + \frac{3.5}{2.4}\frac{A^4}{r^4}K_2 P_6 - \frac{3.5.7}{2.4.6}\frac{A^6}{r^6}K_3 P_8 + \dots\right). \quad \dots\dots\dots\dots(11)$$

For coplanar circles this reduces to

$$M_1 = -\frac{\pi^2 a^2 A^2}{r^3}\left(1 + \frac{3^2}{2.4}\frac{A^2}{r^2}K_1 + \frac{3^2.5^2}{2.4^2.6}\frac{A^4}{r^4}K_2 + \frac{3^2.5^2.7^2}{2.4^2.6^2.8}\frac{A^6}{r^6}K_3 + \dots\right). \quad \dots\dots\dots(12)$$

Formulae (10), (11), and (12) converge if $r > A + a$, *i.e.* in formula (12) if one circle is entirely outside the other.

When x is small it is convenient to choose the difference in radii of the two circles as the unit of length. Then, when the circles are coaxial,

$$M_0 = 4\pi\sqrt{Aa}\left\{\lambda_0 - 2 + \frac{3}{16}\frac{(1+x^2)}{Aa}\left(\lambda_0 - \frac{1}{3}\right) - \frac{15}{1024}\frac{(1+x^2)^2}{A^2a^2}\left(\lambda_0 - \frac{31}{30}\right) + \dots\right\}, \quad \dots\dots(13)$$

in which

$$\lambda = \log_e \frac{8\sqrt{Aa}}{\sqrt{1+x^2}}.$$

The transformation of (13) to the formula for the non-coaxial case requires the determination of a solution of Laplace's equation, which will reduce to

$$(1+x^2)^m \log_e \sqrt{1+x^2}$$

at all points on the axis of x. This solution will now be found.

In cylindrical coordinates (x, ρ, ϕ), and with $\frac{\partial^2 V}{\partial \phi^2}=0$, Laplace's equation is

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial \rho}=0. \quad \text{......................}(14)$$

Transforming to spheroidal coordinates by putting

$$x=\mu\nu, \quad \rho=(1-\mu^2)^{\frac{1}{2}}(1+\nu^2)^{\frac{1}{2}}, \quad \text{..................}(14a)$$

(14) becomes $$\frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial V}{\partial \mu}\right\}+\frac{\partial}{\partial \nu}\left\{(1+\nu^2)\frac{\partial V}{\partial \nu}\right\}=0, \quad \text{............}(15)$$

a possible solution of which is

$$V=\frac{\partial}{\partial n}\{P_n(\mu)P_n(i\nu)\}-P_n(\mu)Q_n(i\nu)-\Sigma A_s P_s(\mu)P_s(i\nu), \quad \text{......}(16)$$

in which $i=\sqrt{-1}$.

When $\mu=1$, that is, when $\rho=0$, $\nu=x$, (16) reduces to

$$V=\frac{\partial P_n(i\nu)}{\partial n}-Q_n(i\nu)-\Sigma A_s P_s(i\nu). \quad \text{...............}(17)$$

Since

$$Q_n(z)=\tfrac{1}{2}P_n\log\frac{1+z}{1-z}-\frac{2n-1}{1\,.\,n}P_{n-1}+\frac{2n-5}{3(n-1)}P_{n-3}-\frac{2n-9}{5(n-2)}P_{n-5}+\dots$$

and

$$\frac{\partial P_n(z)}{\partial z}=P_n\log\tfrac{1}{2}(1+z)+2\left\{\frac{2n-1}{1\,.\,2n}(P_n-P_{n-1})-\frac{2n-3}{2(2n-1)}(P_n-P_{n-2})\right.$$
$$\left.+\frac{2n-5}{3(2n-2)}(P_n-P_{n-3})-\dots\right\},$$

the logarithmic term in (17) is

$$P_n(i\nu)\log\tfrac{1}{2}\sqrt{1+\nu^2}.$$

Also, if s takes the values n, $n-2$, $n-4$, ... and

$$\left.\begin{aligned}A_n&=\log\tfrac{1}{2}+2\left\{\frac{2n-1}{1\,.\,2n}-\frac{2n-3}{2(2n-1)}+\frac{2n-5}{3(2n-2)}-\dots\right\},\\ A_{n-2}&=\frac{2n-3}{2n-1}, \quad A_{n-4}=\frac{2n-7}{2n-3}, \quad \dots,\end{aligned}\right\} \quad \text{......}(18)$$

the value of V along the axis of symmetry becomes

$$P_n(ix)\log\sqrt{1+x^2}.$$

Therefore to obtain the solution sought it only remains to expand $(1+x^2)^n$ in a series involving $P_n(ix)$ and to apply the results (16) and (18) to each term of this series.

The series in question is

$$(1+x^2)^m = (-)^m 2^m \underline{|m}\, \frac{1\,.\,3\,\ldots\,2m-1}{1\,.\,3\,\ldots\,4m+1}\left\{(4m+1)P_{2m}(ix) - (4m-3)\frac{m}{\underline{|1}}\cdot\frac{4m+1}{2m-1}P_{2m-2}(ix) + (4m-7)\frac{m(m-1)}{\underline{|2}}\cdot\frac{(4m+1)(4m-1)}{(2m-1)(2m-3)}P_{2m-4}(ix) - \ldots\right\}, \quad \ldots(19)$$

from which

$$\left.\begin{aligned} 1+x^2 &= -\tfrac{2}{3}P_2(ix) + \tfrac{2}{3}P_0(ix), \\ (1+x^2)^2 &= \tfrac{8}{35}P_4(ix) - \tfrac{16}{21}P_2(ix) + \tfrac{8}{15}P_0(ix). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(20)$$

On applying these results to (13) and inserting the values of $P_n(\mu)$, $P_n(i\nu)$, the mutual induction between non-coaxial circles is found to be

$$M = 4\pi\sqrt{Aa}\left[\lambda - 2 + \frac{3}{16}\frac{1}{Aa}\left\{\left(\lambda - \frac{1}{3}\right)\phi_2 - \chi_2\right\} - \frac{15}{1024}\frac{1}{A^2a^2}\left\{\left(\lambda - \frac{31}{30}\right)\phi - \chi_4\right\} + \ldots\right], \quad \ldots\ldots\ (21)$$

in which

$$\lambda = \log_e \frac{16\sqrt{Aa}}{(1+\mu)\sqrt{1+\nu^2}},$$

$$\begin{aligned} \phi_2 &= \tfrac{2}{3} - \tfrac{2}{3}P_2(\mu)P_2(i\nu) \\ &= \tfrac{1}{2}\{(1+\mu^2) - \nu^2(1-3\mu^2)\}, \\ \phi_4 &= \tfrac{8}{15} - \tfrac{16}{21}P_2(\mu)P_2(i\nu) + \tfrac{8}{35}P_4(\mu)P_4(i\nu) \\ &= \tfrac{1}{8}\{(3+2\mu^2+3\mu^4) - 2\nu^2(1+6\mu^2-15\mu^4) + \nu^4(3-30\mu^2+35\mu^4)\}, \\ \chi_2 &= \tfrac{1}{4}(1-\mu)\{(1-\mu) - \nu^2(1+7\mu)\}, \\ \chi_4 &= \tfrac{1}{96}(1-\mu)\{3(1-\mu)(7+2\mu+7\mu^2) - 6\nu^2(5-\mu-\mu^2+59\mu^3) \\ &\qquad + \nu^4(21+241\mu-113\mu^2-533\mu^3)\}\ ; \end{aligned}$$

and from (14a), μ^2, $-\nu^2$ are the roots of

$$t^2 - t(1-\rho^2-x^2) - x^2 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

In (21) the difference of the radii of the two circles is unity. If $A \sim a = c$, then replace in (21) $1/Aa$ by c^2/Aa, multiply by c, and make μ^2, $-\nu^2$ the roots of

$$c^2t^2 - t(c^2-\rho^2-x^2) - x^2 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

To test the formula, let c approach zero. The limiting value of μ is $x/\sqrt{x^2+\rho^2} \equiv x/r$, and that of $c\nu$ is r. Using these in (21), we obtain formula (6a).

When $x=0$ the circles are coplanar.

If $r > c$, $\mu = 0$, $\nu^2 = \frac{r^2}{c^2} - 1$, and the mutual induction is

$$M_1 = 4\pi\sqrt{Aa}\left[\lambda_1 - 2 - \frac{3}{32}\frac{r^2}{Aa}\left(\lambda_1 - \frac{5}{6}\right)\left(1 - \frac{2c^2}{r^2}\right) - \frac{45}{8192}\frac{r^4}{A^2a^2}\right.$$
$$\left. \times\left\{\lambda_1\left(1 - \frac{8}{3}\frac{c^2}{r^2} + \frac{8}{3}\frac{c^4}{r^4}\right) - \left(\frac{97}{60} - \frac{214}{45}\frac{c^2}{r^2} + \frac{214}{45}\frac{c^4}{r^4}\right)\right\} - \dots\right], \quad \dots(24)$$

in which $\lambda_1 = \log_e \frac{16\sqrt{Aa}}{r}$.

This formula holds good when the two circles intersect.

If $r < c$, $\nu = 0$, $\mu^2 = 1 - \frac{r^2}{c^2}$, and the mutual induction is

$$M'_1 = 4\pi\sqrt{Aa}\left[\lambda'_1 - 2 + \frac{3}{32}\frac{c^2}{Aa}\left\{\left(\lambda'_1 - \frac{1}{3}\right)(1 + \mu^2) - \frac{1}{2}(1 - \mu)^2\right\}\right.$$
$$- \frac{15}{8192}\frac{c^4}{A^2a^2}\left\{\left(\lambda'_1 - \frac{31}{30}\right)(3 + 2\mu^2 + 3\mu^4)\right.$$
$$\left.\left. - \frac{1}{4}(1 - \mu)^2(7 + 2\mu + 7\mu^2)\right\} + \dots\right], \quad \dots\dots(25)$$

in which $\lambda'_1 = \log \frac{16\sqrt{Aa}}{c(1 + \mu)}$.

This formula holds when one circle is entirely inside the other.

If $r = c$, the two circles touch internally and the mutual induction is

$$M''_1 = 4\pi\sqrt{Aa}\left\{\lambda''_1 - 2 + \frac{3}{32}\frac{c^2}{Aa}\left(\lambda''_1 - \frac{5}{6}\right)\right.$$
$$\left. - \frac{45}{8192}\frac{c^4}{A^2a^2}\left(\lambda''_1 - \frac{97}{60}\right) + \dots\right\}, \quad \dots\dots(26)$$

in which $\lambda''_1 = \log_e \frac{16\sqrt{Aa}}{c}$.

It is interesting to notice the similarity between formulae (25) and (9).

TABLE III.

MUTUAL INDUCTION BETWEEN UNEQUAL CIRCLES.

$A=2a$.

$x/a=0$

$\frac{\rho}{a}$	1st Term	2nd Term	3rd Term	$\frac{M}{4\pi\sqrt{Aa}}$
0·00	0·426	0·196	−0·005	0·617
0·60	0·521	0·167	−0·003	0·685
0·80	0·649	0·144	−0·003	0·790
0·92	0·783	0·125	−0·002	0·906
0·98	0·937	0·112	−0·002	1·047
1·00	1·119	0·107	−0·002	1·224
1·41	0·773	0·000	−0·003	0·770
1·73	0·570	−0·085	−0·006	0·479
2·00	0·426	−0·149	−0·011	0·266
2·24	0·314	−0·208	−0·016	0·090
2·45	0·223	−0·260	−0·022	−0·059

$x/a=0·2$

$\frac{\rho}{a}$	1st Term	2nd Term	3rd Term	$\frac{M}{4\pi\sqrt{Aa}}$
0·00	0·406	0·202	−0·005	0·603
0·45	0·453	0·187	−0·005	0·635
0·62	0·501	0·174	−0·004	0·671
0·74	0·548	0·160	−0·003	0·705
0·84	0·597	0·147	−0·003	0·741
0·93	0·640	0·133	−0·002	0·771
1·02	0·671	0·118	−0·002	0·787
1·15	0·673	0·074	−0·001	0·746
1·39	0·591	0·047	−0·000	0·638
2·22	0·220	−0·129	+0·000	0·091
2·69	0·052	−0·225	+0·001	−0·172

$x/a=0·5$

$\frac{\rho}{a}$	1st Term	2nd Term	3rd Term	$\frac{M}{4\pi\sqrt{Aa}}$
0·00	0·314	0·232	−0·007	0·539
0·50	0·342	0·216	−0·007	0·551
0·71	0·366	0·202	−0·005	0·563
0·88	0·382	0·185	−0·003	0·564
1·04	0·385	0·165	−0·001	0·549
1·22	0·367	0·142	+0·001	0·510
1·47	0·313	0·107	+0·004	0·424
1·86	0·192	0·045	+0·012	0·249
2·64	−0·053	−0·070	+0·042	−0·081

$x/a=1·0$

$\frac{\rho}{a}$	1st Term	2nd Term	3rd Term	$\frac{M}{4\pi\sqrt{Aa}}$
0·00	0·089	0·330	−0·015	0·404
0·65	0·075	0·313	−0·008	0·380
0·96	0·061	0·297	−0·003	0·355
1·24	0·032	0·280	−0·001	0·311
1·53	−0·002	0·250	+0·010	0·258
1·94	−0·091	0·213	+0·028	0·150
2·47	−0·207	0·184	+0·068	0·045
3·32	−0·392	0·133	+0·200	−0·059

APPENDIX VI.

RECOMMENDATIONS OF INTERNATIONAL CONFERENCE ON ELECTRICAL UNITS HELD IN LONDON IN OCTOBER, 1908, AND ORDER IN COUNCIL OF DATE JANUARY, 1910, RELATIVE THERETO.

In the First Edition of this book there was given a report of the Board of Trade Committee on Electrical Standards, containing certain resolutions which it was proposed should be adopted by the Board of Trade with the view of obtaining international agreement as to such standards. An International Conference was held in London in October, 1908, when Delegates were present from twenty-two countries and from the principal British Dependencies. The Conference, as a result of its deliberations, adopted the following resolutions and specifications, to be laid by the Delegates before their respective Governments with the view to obtaining uniformity in legislation with regard to Electrical Units and Standards.

SCHEDULE B.

Resolutions.

I. The Conference agrees that, as heretofore, the magnitudes of the fundamental electric units shall be determined on the electro-magnetic system of measurement with reference to the centimetre as the unit of length, the gramme as the unit of mass, and the second as the unit of time.

These fundamental units are (1) the ohm, the unit of electrical resistance which has the value of 1,000,000,000 in terms of the centimetre and second ; (2) the ampere, the unit of electric current which has the value of one-tenth (0·1) in terms of the centimetre, gramme, and the second ; (3) the volt, the unit of electromotive force, which has the value 100,000,000 in terms of the centimetre, the gramme, and the second ; (4) the watt, the unit of power which has the value 10,000,000 in terms of the centimetre, the gramme, and the second.

II. As a system of units representing the above, and sufficiently near to them to be adopted for the purpose of electrical measurements and as a basis for legislation, the Conference recommends the adoption of the international ohm, the international ampere, and the international volt defined according to the following definitions :

III. The ohm is the first primary unit.

IV. The international ohm is defined as the resistance of a specified column of mercury.

V. The international ohm is the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14·4521 grammes in mass, of a constant cross-sectional area and of a length of 106·300 centimetres.

To determine the resistance of a column of mercury in terms of the international ohm, the procedure to be followed shall be that set out in Specification I. attached to these Resolutions.

VI. The ampere is the second primary unit.

VII. The international ampere is the unvarying electric current which, when passed through a solution of nitrate of silver in water, in accordance with Specification II. attached to these Resolutions, deposits silver at the rate of 0·00111800 of a gramme per second.

VIII. The international volt is the electrical pressure [difference of potential], which, when steadily applied to a conductor whose resistance is one international ohm, will produce a current of one international ampere.

IX. The international watt is the energy expended per second by an unvarying electric current of one international ampere under an electric pressure of one international volt.

SPECIFICATION I.

Specification relating to Mercury Standards of Resistance.

The glass tubes used for mercury standards of resistance must be made of glass such that the dimensions may remain as constant as possible. The tubes must be well annealed and straight. The bore must be as nearly as possible uniform and circular, and the area of cross-section of the bore must be approximately one square millimetre. The mercury must have a resistance of approximately one ohm.

Each of the tubes must be accurately calibrated. The correction to be applied to allow for the area of the cross-section of the bore not being exactly the same at all parts of the tube must not exceed 5 parts in 10,000.

The mercury filling the tube must be considered as bounded by plane surfaces placed in contact with the ends of the tube.

The length of the axis of the tube, the mass of mercury the tube contains, and the electrical resistance of the mercury are to be determined at a temperature as near to 0° C. as possible. The measurements are to be corrected to 0° C.

For the purpose of the electrical measurements, end vessels carrying connections for the current and potential terminals are to be fitted to the tube. These end vessels are to be spherical in shape (of a diameter of approximately four centimetres) and should have cylindrical pieces attached to make connections with the tubes. The outside edge of each end of the tube is to be coincident with the inner surface of the corresponding spherical end vessel. The leads which make contact with the mercury are to be of thin platinum wire fused into glass. The point of entry of the current lead and the end of the tube are to be at opposite ends of a diameter of the bulb ; the potential lead is to be midway between these two points. All the leads must be so thin that no error in the resistance is introduced through conduction of heat to the mercury. The filling of the tube with mercury for the purpose of the resistance measurements must be carried out under the same conditions as the filling for the determination of the mass.

The resistance which has to be added to the resistance of the tube to allow for the effect of the end vessels is to be calculated by the formula—

$$A = \frac{0{\cdot}80}{1063\pi}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\text{ohm},$$

where r_1 and r_2 are the radii in millimetres of the end sections of the bore of the tube.

The mean of the calculated resistances of at least five tubes shall be taken to determine the value of the unit of resistance.

For the purpose of the comparison of resistances with a mercury tube the measurements shall be made with at least three separate fillings of the tube.

SPECIFICATION II.

Specification relating to the Deposition of Silver.

The electrolyte shall consist of a solution of from 15 to 20 parts by weight of silver nitrate in 100 parts of distilled water. The solution must only be used once, and only for so long that not more than 30 per cent. of the silver in the solution is deposited.

The anode shall be of silver, and the kathode of platinum. The current density at the anode shall not exceed 1/5 ampere per square centimetre and at the kathode 1/50 ampere per square centimetre.

Not less than 100 cubic centimetres of electrolyte shall be used in a voltameter.

Care must be taken that no particles which may become mechanically detached from the anode shall reach the kathode.

Before weighing, any traces of solution adhering to the kathode must be removed, and the kathode dried.

APPENDIX VII.

THE WESTON NORMAL CELL.

THE Conference agreed on the following Schedule (C) with respect to the Weston normal cell, which describes the formation of the cell except as regards the preparation of the mercurous sulphate used in the depolarizing paste of the cell.

The Weston normal cell may be conveniently employed as a standard of electric pressure for the measurement both of e.m.f. and of current, and, when set up in accordance with the following specification, may be taken, provisionally, as having, at a temperature of 20° C., an e.m.f. of 1·0184 volt.

The Weston normal cell is a voltaic cell which has a saturated aqueous solution of cadmium sulphate ($CdSO_4 . 8/3\, H_2O$) as its electrolyte.

The electrolyte must be neutral to congo red.

The positive electrode of the cell is mercury.

The negative electrode of the cell is cadmium amalgam, consisting of 12·5 parts by weight of cadmium in 100 parts of amalgam.

The depolariser, which is placed in contact with the positive electrode, is a paste made by mixing mercurous sulphate with powdered crystals of cadmium sulphate and a saturated aqueous solution of cadmium sulphate.

The different methods of preparing the mercurous sulphate paste are described in the notes. One of the methods there specified must be carried out [a method used at the N.P.L. is here appended].

For setting up the cell, the H form is the most suitable. The leads passing through the glass to the electrodes must be of platinum wire, which must not be allowed to come into contact with the electrolyte. The amalgam is placed in one limb, the mercury in the other.

The depolariser is placed above the mercury and a layer of cadmium sulphate crystals is introduced into each limb. The entire cell is filled with a saturated solution of cadmium sulphate and then hermetically sealed.

The following formula is recommended for the e.m.f. of the cell in terms of the temperature between the limits 0° C. and 40° C. :

$$E_t = E_{20} - 0{\cdot}0000406\,(t - 20^\circ) - 0{\cdot}00000095\,(t - 20^\circ)^2 + 0{\cdot}00000001\,(t - 20^\circ)^3.$$

[The value of E_{20} is given above.]

PREPARATION OF THE WESTON CADMIUM STANDARD CELL.

The cell has mercury for its positive electrode, and an amalgam consisting of from 12 to 12·5 parts by weight of cadmium in 100 parts of the amalgam for its negative electrode. The electrolyte consists of a saturated solution of cadmium sulphate, and solid cadmium sulphate is contained within the cell. A paste, consisting of solid mercurous sulphate, mercury, and solid cadmium sulphate, rests on the positive electrode.

For the positive electrode, pure distilled mercury should be used.

The amalgam may be made either by electro-deposition or by mechanical mixing. It should be fused and freed from oxide by washing with dilute sulphuric acid.

For the preparation of the cadmium sulphate crystals and solution, commercially pure recrystallised cadmium sulphate should be dissolved in pure distilled water so as to form a clear saturated solution. Evaporation at about 35° C. is then allowed to proceed, when crystals separate from the solution. The crystals are washed with successive small quantities of distilled water, and part of them is dissolved in distilled water to form a saturated solution. The solution should be neutral to congo red.

The mercurous sulphate should be quite pure, and its crystals should not be so small as to have an abnormal solubility or so large as to be inefficient as a depolariser. The following is an example of a method for preparing the salt satisfactorily:

Add 15 cubic centimetres of pure strong nitric acid to 100 grammes of pure mercury, and place on one side until the action is over or nearly over. Transfer the mercurous nitrate thus formed, together with the excess of mercury, to a beaker containing about 200 cubic centimetres of dilute nitric acid (1 volume of acid in about 40 volumes of water); a clear solution should result. Prepare about 1 litre of dilute sulphuric acid (1 volume of acid to 3 of water), and while the mixture is hot add the acid mercurous nitrate solution to it. The solution should be added as a very fine stream from the narrow orifice of a pipette, and the mixture violently agitated during the mixing. Mercurous sulphate is precipitated. Decant the hot clear liquid and wash the precipitate twice by decantation with dilute sulphuric acid (1 volume of acid to 6 of water). The precipitate should then be filtered and washed three times with dilute sulphuric acid (1 to 6), and afterwards 6 or 7 times with saturated cadmium sulphate solution to remove the acid. The mercurous sulphate should then be flooded with saturated cadmium sulphate solution and left for one hour, after which the solution is tested with congo-red paper. In general no acid will be detected, and if so the mercurous sulphate is ready for use.

To set up the cell the H form of vessel is the most convenient. The platinum wires inside the vessel should be amalgamated by passing an electric current to each in turn through an acid solution of mercurous nitrate. The vessel must afterwards be washed out twice with dilute nitric acid and several times with distilled water; it must be free from stains and scrupulously clean; it is dried by the application of heat. The amalgam is fused and its surface flooded with very dilute sulphuric acid; sufficient of it to cover completely the amalgamated platinum wire should then be introduced into one of the limbs of the **H** vessel. To free from acid the amalgam may be remelted and washed with distilled water. Into the other limb of the vessel sufficient mercury is introduced to cover completely the amalgamated platinum wire. Then the paste, finely powdered crystals of cadmium sulphate, and saturated cadmium sulphate solution are added in the order named and the cell sealed.

Its electromotive force at 20° C. is 1·018 volt.

The electromotive force at any temperature (t) may be obtained from the equation:

$$E_t = 1{\cdot}018_4 - 0{\cdot}0000406\,(t-20) - 0{\cdot}00000095\,(t-20)^2 \\ + 0{\cdot}00000001\,(t-20^\circ)^3,$$

the limits of temperature being—(these have not yet been fixed).

In cases in which it is not desired to set up the standard provided in the Resolutions of Schedule B, the Conference recommends the following as working methods for the realization of the international ohm, the ampere, and the volt:

1. *For the International Ohm.*

The use of copies, constructed of suitable material and of suitable form verified from time to time, of the international ohm, its multiples and sub-multiples.

2. *For the International Ampere.*

(*a*) The measurement of current by the aid of a current balance standardised by comparison with a silver voltameter; or

(*b*) The use of a Weston normal cell whose electromotive force has been determined in terms of the international ohm and international ampere, and of a resistance of known value in international ohms.

3. *For the International Volt.*

(*a*) A comparison with the difference of electrical potential between the ends of a coil of resistance of known value in international ohms when carrying a current of known value in international amperes; or

(*b*) The use of a Weston normal cell whose electromotive force has been determined in terms of the international ohm and the international ampere.

Following on these recommendations of the Conference, an Order in Council was made on January 10, 1910, which provided definitions of standards as follows :

The International Ohm is the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice 14·4521 grammes in mass of a constant cross-sectional area and of a length of 106·300 centimetres.

The International Ampere is the unvarying electric current which, when passed through a solution of nitrate of silver in water, deposits silver at the rate of 0·00111800 of a gramme per second.

The International Volt is the electrical pressure [difference of potential] which, when steadily applied to a conductor whose resistance is one International Ohm, will produce a current of one International Ampere.

The limits of accuracy obtainable in these standards were stated as follows :

For the ohm within one-hundredth part of one per cent.
For the ampere within one-tenth part of one per cent.
For the volt within one-tenth part of one per cent.

The following Schedule was appended to this Order :

I. *Standard of Electrical Resistance.*

A standard of electrical resistance denominated one ohm agreeing in value within the limits of accuracy aforesaid with that of the International Ohm and being the resistance between the copper terminals of the instrument marked " Board of Trade Ohm Standard Verified, 1894 and 1909," to the passage of an unvarying electrical current when the coil of insulated wire forming part of the aforesaid instrument and connected to the aforesaid terminals is in all parts at a temperature of 16·4 C.

II. *Standard of Electrical Current.*

A standard of electrical current denominated one ampere agreeing in value within the limits of accuracy aforesaid with that of the International Ampere and being the current which is passing in and through the coils of wire forming part of the instrument marked " Board of Trade Standard Verified, 1894 and 1909," when on reversing the current in the fixed coils the change in the forces acting upon the suspended coil in its sighted position is exactly balanced by the force exerted by gravity in Westminster upon the iridio-platinum weight marked A and forming part of the said instrument.

III. *Standard of Electrical Pressure.*

A standard of electrical pressure denominated one volt agreeing in value within the limits of accuracy aforesaid with that of the International Volt and being one-hundredth part of the pressure which when applied between the terminals forming part of the instrument marked "Board of Trade Volt Standard Verified, 1894 and 1909," causes that rotation of the suspended portion of the instrument which is exactly measured by the coincidence of the sighting wire with the image of the fiducial mark A before and after application of the pressure and with that of the fiducial mark B during the application of the pressure, these images being produced by the suspended mirror and observed by means of the eyepiece.

The coils and instruments referred to in this Schedule are deposited at the Board of Trade Standardizing Laboratory, 8 Richmond Terrace, Whitehall, London.

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